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Complete Convergence and Strong Laws of Large Numbers for Weighted Sums of Sequences of Independent B-Valued Random Elements

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Abstract In this paper, we obtain theorems of complete convergence and strong laws of large numbers for weighted sums of sequences of independent random elements in a Banach space of type $p\,(1 \le p \le 2)$. The results improve and extend the corresponding results on real random variables obtained by [1] and [2].

Keywords B-valued random elements; weighted sums; complete convergence; strong laws of large numbers; space of type p.

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1. Introduction

When $\{X, X_n, n \ge 1\}$ is a sequence of i.i.d. random variables and $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real constants, the convergence of weighted sums $\sum_{i=1}^n a_{ni}X_i$ was studied by many authors[1−5]. We recommend the paper of Rosalsky and Sreehari[3] for more information. Recently, Bai and Cheng[1] proved the strong law of large numbers

$$n^{-1/q}\sum_{i=1}^n a_{ni}X_i \to 0 \quad \text{a.s.} \tag{1}$$

where $\{X, X_n, n \ge 1\}$ is a sequence of i.i.d. random variables satisfying $EX = 0, E|X|^\beta < \infty$, and $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real constants satisfying

$$A_{\alpha,n}^\alpha = \frac{1}{n}\sum_{i=1}^n |a_{ni}|^\alpha, \quad A_\alpha = \limsup_{n\to\infty} A_{\alpha,n} < \infty, \tag{2}$$

for some $1 < \alpha, \beta < \infty, 1 < q < 2$, and $1/q = 1/\alpha + 1/\beta$.

In this paper, we let $\{\Omega, \Im, P\}$ be a complete probability space and B be a real separable Banach space with norm $\|\cdot\|$. The Banach space B is called type $p\,(1 \le p \le 2)$ if there exists a $C = C_p > 0$ such that

$$E\|\sum_{i=1}^n X_i\|^p \le C\sum_{i=1}^n E\|X_i\|^p,$$

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where the independent B-valued random elements $X_1, X_2, \dots, X_n$ have mean zero and $E\|X_i\|^p < \infty, i = 1, 2, \dots, n$.

Hereinafter in this paper, C always stands for a positive constant which may differ from one place to another; $\{X_n, n \ge 1\} \prec X$ means $\sup_{n\ge1} P(\|X_n\| \ge x) \le CP(X \ge x)$, where $x > 0$ and X is a non-negative real-valued random variable.

In this paper, we shall extend the result of Bai and Cheng[1] and Cuzick[2] to independent B-valued random elements, and remove the identical distribution condition. The method of proof in this paper differs from that used by Bai and Cheng[1] and it is simpler than Bai and Cheng's[1]. In addition, the complete convergence of independent B-valued random elements is studied.

In order to prove our main results, we need the following lemmas. By Corollary 2.4 of Shao[6], we immediately have the following:

Lemma 1 *Let Banach space B be of type $p\,(1 \le p \le 2)$ and $\{X_n, n \ge 1\}$ be a sequence of independent B-valued random elements with mean zero. Then for $q \ge p$,*

$$E \max_{1\le k\le n} \|\sum_{i=1}^{k} X_i\|^q \le C[\sum_{i=1}^{n} E\|X_i\|^q + (\sum_{i=1}^{n} E\|X_i\|^p)^{q/p}], \quad n \ge 1.$$

Where C is a constant independent of n.

Lemma 2[7] *Let $\{X_n, n \ge 1\}$ be a sequence of independent symmetric B-valued random elements. Then for every integer $j \ge 1$ and every $t > 0$,*

$$P(\|\sum_{i=1}^{n} X_i\| > 3^j t) \le C_j P(\max_{1\le i\le n} \|X_i\| > t) + D_j (P(\|\sum_{i=1}^{n} X_i\| > t))^{2^j},$$

where C_j and D_j are positive depending only on j.

Lemma 3[8] *Suppose that X_0 is a B-valued random element, and*

$$P(\|X_0\| \ge x)) \le CP(X \ge x), \quad \forall x > 0,$$

where X is a non-negative real-valued random variable. Then $\forall t > 0, x > 0$,

$$E\|X_0\|^t I(\|X_0\| \le x) \le Cx^t P(X > x) + CEX^t I(X \le x),$$

$$E\|X_0\|^t I(\|X_0\| > x) \le CEX^t I(X > x).$$

2. Main results and proofs

Theorem 1 *Assume $1 \le p \le 2$, $0 < \alpha, \beta, v < \infty$, $\alpha < v\beta$, $0 < q < p$ and $1/q = 1/\alpha + 1/\beta$. Let $h(x) > 0$ be a slowly varying function as $x \to +\infty$, Banach space B be of type p, $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of constants satisfying (2) and $\{X_n, n \ge 1\}$ be a sequence of independent B-valued random elements with $\{X_n, n \ge 1\} \prec X$, $EX^{v\beta} h(X^\beta) < \infty$. If $\alpha > 1$, moreover we* *assume that* $EX_n = 0, n \geq 1$. *Then for any* $\epsilon > 0$,

$$\sum_{n=1}^{\infty} n^{v-1} h(n) P(\max_{1\leq j\leq n} \|\sum_{i=1}^{j} a_{ni} X_i\| > \epsilon n^{1/q}) < \infty. \tag{3}$$

Proof For any $t : 0 < t < \alpha$, by (2) and the Hölder's inequality, we get

$$\sum_{i=1}^{n} |a_{ni}|^t \leq (\sum_{i=1}^{n} |a_{ni}|^{\alpha})^{t/\alpha} (\sum_{i=1}^{n} 1)^{1-t/\alpha} \leq Cn. \tag{4}$$

For any $t : t \geq \alpha$, by (2) and the C_r-inequality, we get

$$\sum_{i=1}^{n} |a_{ni}|^t = \sum_{i=1}^{n} (|a_{ni}|^{\alpha})^{t/\alpha} \leq (\sum_{i=1}^{n} |a_{ni}|^{\alpha})^{t/\alpha} \leq Cn^{t/\alpha}. \tag{5}$$

Putting $\gamma = \min\{\alpha, p\}$, since $\alpha < v\beta$, we have $EX^{\gamma} < \infty$.

When $0 < \gamma \leq 1$, by the C_r-inequality and (4), we have

$$P(\|\sum_{i=1}^{n} a_{ni} X_i\| > \epsilon n^{1/q}) \leq Cn^{-\gamma/q} \sum_{i=1}^{n} |a_{ni}|^{\gamma} E\|X_i\|^{\gamma} \leq Cn^{(1-\gamma/q)}. \tag{6}$$

When $\gamma > 1$, since Banach space B is of type p, it is of type γ. So, it follows from Lemma 1 and (4),

$$P(\|\sum_{i=1}^{n} a_{ni} X_i\| > \epsilon n^{1/q}) \leq Cn^{-\gamma/q} \sum_{i=1}^{n} |a_{ni}|^{\gamma} E\|X_i\|^{\gamma} \leq Cn^{(1-\gamma/q)}. \tag{7}$$

By (6) and (7) and $q < \gamma$, we have

$$n^{-1/q} \sum_{i=1}^{n} a_{ni} X_i \xrightarrow{p} 0.$$

Hence by the Ottaviani inequality[10] we have

$$P(\max_{1\leq j\leq n} \|\sum_{i=1}^{j} a_{ni} X_i\| \geq \epsilon n^{1/q}) \leq CP(\|\sum_{i=1}^{n} a_{ni} X_i\| > \frac{\epsilon}{2} n^{1/q}).$$

By symmetrization inequality[10], in order to prove (3), it suffices to prove that

$$\sum_{n=1}^{\infty} n^{v-1} h(n) P(\|\sum_{i=1}^{n} a_{ni} X_i^s\| > \frac{\epsilon}{4} n^{1/q}) < \infty, \tag{8}$$

where $\{X_n^s, n \geq 1\}$ is a symmetrized version of $\{X_n, n \geq 1\}$. So we assume $\{X_n, n \geq 1\}$ is a sequence of independent symmetric B-valued random elements.

Define $X_{ni} = X_i I(\|X_i\| \leq n^{1/\beta})$ and $Y_{ni} = X_i I(\|X_i\| > n^{1/\beta})$ for $1 \leq i \leq n$ and $n \geq 1$.

Case 1 $\alpha \leq p$.

Taking $t > \max(p, v\beta)$, by Lemma 1, Lemma 3, (5) and the C_r-inequality we have

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{v-1} h(n) P(\|\sum_{i=1}^{n} a_{ni} X_{ni}\| > \frac{\epsilon}{8} n^{1/q}) \\ &\quad \leq C \sum_{n=1}^{\infty} n^{(v-1-t/q)} h(n) [(\sum_{i=1}^{n} E\|a_{ni} X_{ni}\|^p)^{t/p} + \sum_{i=1}^{n} E\|a_{ni} X_{ni}\|^t] \end{aligned}$$

$$\leq C\sum_{n=1}^{\infty} n^{(v-1-t/\beta)}h(n)[EX^tI(X\leq n^{1/\beta})+n^{t/\beta}P(X\geq n^{1/\beta})]$$

$$= C\sum_{n=1}^{\infty} n^{(v-1-t/\beta)}h(n)EX^tI(X\leq n^{1/\beta})+$$

$$C\sum_{n=1}^{\infty} n^{v-1}h(n)P(X\geq n^{1/\beta}) \triangleq A+B. \tag{9}$$

Since $t > v\beta$, by the property of slowly varying function and Lemma 1 of Bai and Su[9], we have

$$\begin{aligned}
A &= C\sum_{i=0}^{\infty}\sum_{2^i\leq n<2^{i+1}} n^{(v-1-t/\beta)}h(n)EX^tI(X\leq n^{1/\beta})\\
&\leq C+C\sum_{i=0}^{\infty} 2^{i(v-t/\beta)}h(2^i)EX^tI(X\leq 2^{(i+1)/\beta})\\
&\leq C+C\sum_{i=0}^{\infty} 2^{i(v-t/\beta)}h(2^i)\sum_{j=0}^{i} EX^tI(2^{j/\beta}<X\leq 2^{(j+1)/\beta})\\
&\leq C+C\sum_{j=0}^{\infty} EX^tI(2^{j/\beta}<X\leq 2^{(j+1)/\beta})\sum_{i=j}^{\infty} 2^{i(v-t/\beta)}h(2^i)\\
&\leq C+C\sum_{j=0}^{\infty} 2^{j(v-t/\beta)}h(2^j)EX^tI(2^{j/\beta}<X\leq 2^{(j+1)/\beta})\\
&\leq C+C\sum_{j=0}^{\infty} 2^{jv}h(2^j)P(2^{j/\beta}<X\leq 2^{(j+1)/\beta})\\
&\leq C+CEX^{v\beta}h(X^\beta)<\infty.
\end{aligned} \tag{10}$$

Similar to the proof of (10), we have

$$B\leq C+CEX^{v\beta}h(X^\beta)<\infty. \tag{11}$$

Since $\alpha < v\beta$, we have $\alpha < (v+1)q$. When $1 < \alpha \leq p$ by Lemma 1 or $0 < \alpha \leq 1$ by the C_r-inequality, the property of slowly varying function and Lemma 1 of Bai and Su[9], similar to the proof of (10), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{v-1}h(n)P(\|\sum_{i=1}^{n} a_{ni}Y_{ni}\|>\frac{\epsilon}{8}n^{1/q})\\
&\leq C\sum_{n=1}^{\infty} n^{(v-1-\alpha/q)}h(n)\sum_{i=1}^{n} E\|a_{ni}Y_{ni}\|^\alpha \leq C+CEX^{v\beta}h(X^\beta)<\infty.
\end{aligned} \tag{12}$$

Therefore, if $\alpha \leq p$, by (9)–(12), (8) holds.

Case 2 $\alpha > p$.

By Lemma 2, we have

$$P(\|\sum_{i=1}^{n} a_{ni}X_i\|>\epsilon n^{1/q})\leq C_jP(\max_{1\leq i\leq n}\|a_{ni}X_i\|>3^{-j}\epsilon n^{1/q})+D_j(P(\|\sum_{i=1}^{n} a_{ni}X_i\|>3^{-j}\epsilon n^{1/q}))^{2^j},$$

where C_j and D_j are positive depending only on j. We can choose integer j such that $v+(1-\gamma/q)2^j<0$. So by (7), in order to prove (8), it is enough to show that

$$\sum_{n=1}^{\infty} n^{v-1}h(n)\sum_{i=1}^{n}P(\|a_{ni}X_i\|>3^{-j}\epsilon n^{1/q})<\infty. \tag{13}$$

Define $\epsilon_1=3^{-j}\epsilon/2$, by Lemma 3 and the Markov's inequality, similar to the proof of (10), we have

$$\begin{aligned}\sum_{n=1}^{\infty} n^{v-1}h(n)\sum_{i=1}^{n}P(\|a_{ni}Y_{ni}\|>\epsilon_1 n^{1/q})&\le C\sum_{n=1}^{\infty} n^{(v-1-\alpha/q)}h(n)\sum_{i=1}^{n}E\|a_{ni}Y_{ni}\|^{\alpha}\\ \le C\sum_{n=1}^{\infty} n^{(v-\alpha/q)}h(n)EX^{\alpha}I(X>n^{1/\beta})&\le C+CEX^{v\beta}h(X^{\beta})<\infty. \end{aligned}\tag{14}$$

Choosing $t>\max(\alpha, v\beta)$, by Lemma 3 and the Markov's inequality, similar to the proof of (10), we have

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{v-1}h(n)\sum_{i=1}^{n}P(\|a_{ni}X_{ni}\|>\epsilon_1 n^{1/q})\le C\sum_{n=1}^{\infty} n^{(v-1-t/q)}h(n)\sum_{i=1}^{n}E\|a_{ni}X_{ni}\|^{t}\\ &\le C\sum_{n=1}^{\infty} n^{(v-1-t/q)}h(n)(\sum_{i=1}^{n}|a_{ni}|^t)[EX^tI(X\le n^{1/\beta})+n^{t/\beta}P(X>n^{1/\beta})]\\ &\le C\sum_{n=1}^{\infty} n^{(v-1-t/\beta)}h(n)[EX^tI(X\le n^{1/\beta})+n^{t/\beta}P(X>n^{1/\beta}]\\ &\le C+CEX^{v\beta}h(X^{\beta})<\infty. \end{aligned}\tag{15}$$

Therefore by (14)–(15), (13) holds, so (3) holds.

Theorem 2 *Assume* $1\le p\le 2, 0<\alpha,\beta<\infty, 0<q<p$ *and* $1/q=1/\alpha+1/\beta$. *Let Banach space* B *be of type* p, $\{a_{ni}, 1\le i\le n, n\ge 1\}$ *be an array of constants satisfying (2) and* $\{X_n, n\ge 1\}$ *be a sequence of independent* B*-valued random elements with* $\{X_n, n\ge 1\}\prec X$, $EX^{\beta}<\infty$. *If* $\min\{\alpha,\beta\}>1$, *moreover we assume that* $EX_n=0,\ n\ge 1$. *Then*

$$\lim_{n\to\infty} n^{-1/q}\max_{1\le j\le n}\|\sum_{i=1}^{j}a_{ni}X_i\|=0, \quad \text{a. s.} \tag{16}$$

Proof Define $X_{ni}=X_iI(\|X_i\|\le i^{1/\beta})$, $Y_{ni}=X_iI(\|X_i\|>i^{1/\beta})$, $1\le i\le n, n\ge 1$. To prove (16), we only need to prove that

$$\lim_{n\to\infty} n^{-1/q}\max_{1\le j\le n}\|\sum_{i=1}^{j}a_{ni}X_{ni}\|=0, \quad \text{a. s.}, \tag{17}$$

$$\lim_{n\to\infty} n^{-1/q}\max_{1\le j\le n}\|\sum_{i=1}^{j}a_{ni}Y_{ni}\|=0, \quad \text{a. s.}. \tag{18}$$

Noting that $EX^{\beta}<\infty \Longleftrightarrow \sum_{n=1}^{\infty}P(X>n^{1/\beta})<\infty$, we have $\sum_{n=1}^{\infty}P(\|X_n\|>n^{1/\beta})<\infty$.

Hence by the Borel-Cantelli lemma, we can get $P(\|X_n\| > n^{1/\beta}, i.o.) = 0$. It follows that

$$n^{-1/q} \max_{1\le j\le n} \|\sum_{i=1}^{j} a_{ni}Y_{ni}\| \le n^{-1/q} \max_{1\le i\le n} |a_{ni}| \sum_{i=1}^{n} \|Y_{ni}\| \le A_{\alpha,n} n^{-1/\beta} \sum_{i=1}^{n} \|Y_{ni}\| \to 0 \quad \text{a.s.} \quad (19)$$

Therefore, (18) holds.

Putting $\gamma = \min\{\alpha, \beta, p\}$, as in Theorem 1, we have

$$P(\|\sum_{i=1}^{n} a_{ni}X_{ni}\| \ge \epsilon 3^{-j} n^{1/q}) \le Cn^{(1-\gamma/q)}. \quad (20)$$

Therefore, we can get

$$n^{-1/q} \sum_{i=1}^{n} a_{ni}X_i \xrightarrow{p} 0.$$

To prove (17), by the Ottaviani inequality[10], the symmetrization inequality[10] and the Borel-Cantelli lemma, it is enough to show that

$$\sum_{n=1}^{\infty} P(\|\sum_{i=1}^{n} a_{ni}X_{ni}^{s}\| > \epsilon n^{1/q}) < \infty, \quad \forall \epsilon > 0, \quad (21)$$

where $\{X_{ni}^{s}, 1 \le i \le n, n \ge 1\}$ is a symmetrized version of $\{X_{ni}, 1 \le i \le n,\ n \ge 1\}$. So we assume $\{X_{ni}, 1 \le i \le n,\ n \ge 1\}$ is symmetric B-valued random elements. By Lemma 2, we have

$$\begin{aligned} &P(\|\sum_{i=1}^{n} a_{ni}X_{ni}\| > \epsilon n^{1/q}) \\ &\le C_j P(\max_{1\le i\le n} \|a_{ni}X_{ni}\| > \epsilon 3^{-j} n^{1/q}) + D_j (P(\|\sum_{i=1}^{n} a_{ni}X_{ni}\| \ge \epsilon 3^{-j} n^{1/q}))^{2^j}. \end{aligned} \quad (22)$$

We can choose integer j such that $2^j(\gamma/q - 1) > 1$. So by (20) and (22), in order to prove (21), it is enough to show that

$$\sum_{n=1}^{\infty} P(\max_{1\le i\le n} \|a_{ni}X_{ni}\| > \epsilon 3^{-j} n^{1/q}) < \infty. \quad (23)$$

In fact, choosing $t > \max(\alpha, \beta)$, by the Markov's inequality, the C_r-inequality and (5), we can get

$$\begin{aligned} &\sum_{n=1}^{\infty} P(\max_{1\le i\le n} \|a_{ni}X_{ni}\| > \epsilon 3^{-j} n^{1/q}) \le \sum_{n=1}^{\infty} \sum_{i=1}^{n} P(\|a_{ni}X_{ni}\| > \epsilon 3^{-j} n^{1/q}) \\ &\le C \sum_{n=1}^{\infty} n^{-t/q} \sum_{i=1}^{n} |a_{ni}|^t E\|X_i\|^t I(\|X_i\| \le n^{1/\beta}) \\ &\le C \sum_{n=1}^{\infty} n^{-t/q} \sum_{i=1}^{n} |a_{ni}|^t [EX^t I(X \le n^{1/\beta}) + n^{t/\beta} P(X \ge n^{1/\beta})] \\ &\le C + CEX^\beta < \infty. \end{aligned}$$

So (23) holds, therefore (16) holds.

Remark Noting that the real space R is of type 2, we see that Theorems 1 and 2 improve and extend the corresponding results of Bai and Cheng[1] and Cuzick[2] to B-valued random elements, and remove the identical distribution condition. The method of proof differs from Bai and Cheng's[1] and it is simpler than Bai and Cheng's[1].

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