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Viscosity Approximation Methods for Equilibrium Problems in Hilbert Spaces Involving the New Iterative Process with Error

RAO Ruo Feng, *ZHANG Shi Sheng*
(Department of Mathematics, Yibin College, Sichuan 644000, China)
(E-mail: rrf2@163.com)

Abstract In this paper, we introduce an iterative scheme with error by the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. A strong convergence theorem is given, which generalizes all the results obtained by S.Takahashi and W.Takahashi in 2007. In addition, some of the methods applied in this paper improve those of S.Takahashi and W.Takahashi.

Keywords viscosity approximation method; equilibrium problem; nonexpansive mapping.

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1. Introduction and preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, C be a nonempty closed convex subset of H, and $\mathbb{N}=\{1,2,3,\ldots\}$ be the set of natural numbers. Assume, F is a bifunction of $C\times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F: C\times C\to\mathbb{R}$ is to find $x\in C$ such that

$$F(x,y)\geq 0,\quad \forall y\in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{EP}(F)$. Given a mapping $T: C\to H$, let $F(x,y)=\langle Tx, y-x\rangle$ for all $x,y\in C$. Then, $z\in\mathrm{EP}(F)$ if and only if $\langle Tz, y-z\rangle\geq 0$ for all $y\in C$, i.e., z is a solution of the variational inequality. Numerous problems in Physics, optimization, and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem. Very recently, S.Takahashi and W.Takahashi[1] studied an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(F)$ is nonempty. In addition, there are several papers studying the same problem, such as [2], [3], [10] and so on.

Definition 1.1 *A mapping S of C into H is called nonexpansive if*

$$\|Sx-Sy\|\leq\|x-y\|,\quad \forall\, x,\, y\in C.$$

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Let $\mathrm{Fix}(S)$ *be the set of fixed points of the mapping* S. *If* $C \subset H$ *is bounded, closed and convex and* S *is a nonexpansive mapping of* C *into itself, then* $\mathrm{Fix}(S)$ *is nonempty*[4].

Definition 1.2 *A mapping* P_C *is called the metric projection of* H *onto* C, *if for any* $x \in H$, *there exists a unique nearest point in* C, *denoted by* $P_C(x)$, *such that*

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall\, y \in C.$$

We know that P_C *is nonexpansive. Further, for* $x \in H$ *and* $z \in C$,

$$z = P_C x \Leftrightarrow \langle x - z, z - y \rangle \geq 0, \quad \forall\, y \in C.$$

We also know that for any sequence $\{x_n\} \subset H$ *with* $x_n \rightharpoonup x$, *the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\ |$$

holds for every $y \in H$ *with* $x \neq y$, *see [4,5] for more detail. Here we denote weak convergence by* $\rightharpoonup$.

To solve the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions

(A_1) $F(x, x) = 0$ for all $x \in C$;

(A_2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A_3) for each $x, y, z \in C$,

$$\lim_{0 \leq t \to 0} F(tz + (1-t)x, y) \leq F(x, y);$$

(A_4) for each $x \in C$, $y \to F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [6].

Lemma 1.1[6] *Let* C *be a nonempty closed convex subset of* H *and let* F *be a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A_1)–(A_4). *Let* $r > 0$ *and* $x \in H$. *Then, there exists* $z \in C$ *such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \quad \forall\, y \in C.$$

Lemma 1.2[2] *Assume that* $F : C \times C \to \mathbb{R}$ *satisfies* (A_1)–(A_4). *For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \ \ \forall\, y \in C\}$$

for all $x \in H$. *Then, the following holds:*

(1) T_r *is single-valued,*

(2) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle,$$

(3) $F(T_r) = \mathrm{EP}(F)$,

(4) $\mathrm{EP}(F)$ *is closed and convex.*

In 2007, S.Takahashi and W.Takahashi [1] proved the following strong convergence theorem:

Theorem A[1] *Let* C *be a nonempty closed convex subset of* H. *Let* F *be a bifunction from*

$C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4) and let S be a nonexpansive mapping of C into H such that $\mathrm{Fix}(S) \cap \mathrm{EP}(F) \neq \emptyset$*. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

$$\liminf_{n\to\infty} r_n > 0, \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$$

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in \mathrm{Fix}(S) \cap \mathrm{EP}(F)$, where $z = P_{\mathrm{Fix}(S)\cap \mathrm{EP}(F)} f(z)$.

In this paper, we introduce the new iterative process with error, which is as follows

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall\, y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n-\gamma_n)Su_n + \gamma_n v_n, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}, \{\gamma_n\}$ are two real sequences in [0,1] with $\alpha_n + \gamma_n \leq 1$, $\{r_n\} \subset (0,\infty)$, both $\{x_n\}$ and $\{u_n\}$ are sequences generated by $x_1 \in H$ and the iteration (1.2), v_n is a bounded sequence in H.

In this paper, we will give a strong convergence theorem of the iteration (1.2). Thereby, we also need the following lemma, which can be proved similarly as the proof of [7, P.171] or [8, Lemma 1] or [9].

Lemma 1.3[9] *Let $\{a_n\} \subset [0,\infty)$, $\{b_n\} \subset [0,\infty)$, $\{c_n\} \subset [0,\infty)$ be sequences of real numbers such that*

$$a_{n+1} \leq (1-h_n)a_n + b_n + c_n, \quad \forall\, n \geq n_0,$$

where $h_n \subset [0,1]$ is a nonnegative real sequence with $\sum_{n=1}^{\infty} h_n = \infty$, $b_n = o(h_n)$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

2. Main results

Now we give the main results of this paper.

Theorem 2.1 *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4) and let S be a nonexpansive mapping of C into H such that* $\mathrm{Fix}(S) \cap \mathrm{EP}(F) \neq \emptyset$*. Let f be a contraction of H into itself with the contractive constant $\alpha \in (0,1)$, and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and the iteration (1.2). $\{v_n\}$ is a bounded sequence in H, $\{\alpha_n\}$, $\{\gamma_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

$$\liminf_{n\to\infty} r_n > 0, \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty, \quad \sum_{n=1}^{\infty} \gamma_n < \infty.$$

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in \mathrm{Fix}(S) \cap \mathrm{EP}(F)$, where $z = P_{\mathrm{Fix}(S)\cap \mathrm{EP}(F)} f(z)$.

Proof For any given $v \in \mathrm{Fix}(S) \cap \mathrm{EP}(F)$, by the assumptions of Theorem 2.1 we know that there exists a corresponding constant $M > 0$ such that

$$\|x_1\| + \|f(v)\| + \|v\| + \|v_n\| \le M, \quad \forall\, n \in \mathbb{N}. \tag{2.1}$$

Let $Q = P_{\mathrm{Fix}(S)\cap \mathrm{EP}(F)}$. Then Qf is a contraction of H into itself. Since H is complete, there exists a unique element $z = Qf(z)$.

It is obvious that $\|x_1 - v\| \le M \le \frac{M}{1-\alpha}$. Assuming that $\|x_n - v\| \le \frac{M}{1-\alpha}$, we will prove that $\|x_{n+1} - v\| \le \frac{M}{1-\alpha}$. i.e., $\{x_n\}$ is bounded.

Indeed, by (1.2),(2.1) and Lemma 1.2, we have

$$\begin{aligned}
\|x_{n+1} - v\| &= \|\alpha_n f(x_n) + (1-\alpha_n-\gamma_n) S u_n + \gamma_n v_n - v\| \\
&\le \alpha_n \|f(x_n) - v\| + (1-\alpha_n-\gamma_n)\|Su_n - v\| + \gamma_n \|v_n - v\| \\
&\le \alpha_n(\alpha\|x_n - v\| + \|f(v) - v\|) + (1-\alpha_n-\gamma_n)\|u_n - v\| + \gamma_n M \\
&\le \alpha\alpha_n\|x_n - v\| + \alpha_n\|f(v) - v\| + (1-\alpha_n-\gamma_n)\|T_{r_n}x_n - v\| + \gamma_n M \\
&\le \alpha\alpha_n\|x_n - v\| + \alpha_n M + (1-\alpha_n-\gamma_n)\|x_n - v\| + \gamma_n M \\
&\le (1-(\alpha_n+\gamma_n)(1-\alpha))\|x_n - v\| + (\alpha_n+\gamma_n)M \le \frac{M}{1-\alpha}.
\end{aligned}$$

This implies that $\{x_n\}$ is bounded. i.e., we have

$$\|x_n - v\| \le \frac{M}{1-\alpha}, \quad \forall\, n \in \mathbb{N}. \tag{2.2}$$

By (2.2) we know that $\{x_n\}$ is bounded, which implies that $\{u_n\}$, $\{Su_n\}$ and $\{f(x_n)\}$ all are bounded. i.e., for any given $v \in \mathrm{Fix}(S) \cap \mathrm{EP}(F)$, there exists a corresponding constant $M_0 > 0$ such that

$$M + \|z\| + \|x_n\| + \|u_n\| + \|v_n\| + \|f(x_n)\| + \|Su_n\| \le M_0, \quad n \in \mathbb{N}. \tag{2.3}$$

Next, we want to prove

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.4}$$

Indeed, by (1.2) and (2.3) we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad = \|\alpha_n f(x_n) - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) - \alpha_{n-1} f(x_{n-1}) + (1-\alpha_n-\gamma_n) S u_n \\
&\quad - (1-\alpha_n-\gamma_n) S u_{n-1} + (1-\alpha_n-\gamma_n) S u_{n-1} - (1-\alpha_{n-1}-\gamma_{n-1}) S u_{n-1} + \\
&\qquad \gamma_n v_n - \gamma_n v_{n-1} + \gamma_n v_{n-1} - \gamma_{n-1} v_{n-1}\| \\
&\quad \le \alpha_n \|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}| \cdot \|f(x_{n-1})\| + (1-\alpha_n-\gamma_n)\|Su_n - Su_{n-1}\| + \\
&\qquad |(1-\alpha_n-\gamma_n) - (1-\alpha_{n-1}-\gamma_{n-1})| \cdot \|Su_{n-1}\| + \\
&\qquad \gamma_n \|v_n - v_{n-1}\| + |\gamma_n - \gamma_{n-1}| \cdot \|v_{n-1}\|
\end{aligned}$$

$$\leq \alpha_n\alpha\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_0 + (1-\alpha_n-\gamma_n)\|u_n - u_{n-1}\|+ (|\alpha_n - \alpha_{n-1}| + \gamma_n + \gamma_{n-1})M_0 + 2\gamma_n M_0 + (\gamma_n+\gamma_{n-1})M_0. \tag{2.5}$$

In view of (1.2) and $u_n = T_{r_n}x_n$, we can get

$$F(u_n, u_{n+1}) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0. \tag{2.6}$$

Similarly, we can also have

$$F(u_{n+1}, u_n) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0. \tag{2.7}$$

By (A_2), (2.6) and (2.7), we have

$$\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1}-x_{n+1}}{r_{n+1}}\rangle \geq 0,$$

i.e.,

$$\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\rangle \geq 0,$$

i.e.,

$$\begin{aligned}\|u_{n+1} - u_n\|^2 \leq &\ \langle u_{n+1} - u_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1} - x_{n+1})\rangle \\ \leq &\ \|u_{n+1} - u_n\| \cdot \{\|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_{n+1}}\|u_{n+1} - x_{n+1}\|\}.\end{aligned} \tag{2.8}$$

By the assumption on r_n we can assume that there exists a real number b with $0 < b < r_n$ for all $n \in \mathbb{N}$. Then by (2.3) and (2.8), we have

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{b}|r_{n+1} - r_n|M_0. \tag{2.9}$$

Hence, by (2.5) and (2.9) we have

$$\begin{aligned}\|x_{n+1} - x_n\| \leq &(1 - \alpha_n + \alpha_n\alpha - \gamma_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_0 + \\ &(1-\alpha_n-\gamma_n)\frac{M_0}{b}|r_{n+1} - r_n| + (|\alpha_n - \alpha_{n-1}| + \gamma_n + \gamma_{n-1})M_0 + \\ &2\gamma_n M_0 + (\gamma_n + \gamma_{n-1})M_0,\end{aligned}$$

i.e.,

$$\begin{aligned}\|x_{n+1} - x_n\| \leq &(1 - (1-\alpha)\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_0 + \frac{M_0}{b}|r_{n+1} - r_n| + \\ &(|\alpha_n - \alpha_{n-1}| + \gamma_n + \gamma_{n-1})M_0 + 2\gamma_n M_0 + (\gamma_n + \gamma_{n-1})M_0.\end{aligned} \tag{2.10}$$

Let $h_n = (1-\alpha)\alpha_n$, $b_n = 0$ and

$$c_n = |\alpha_n - \alpha_{n-1}|M_0 + \frac{M_0}{b}|r_{n+1} - r_n| + (|\alpha_n - \alpha_{n-1}| + \gamma_n + \gamma_{n-1})M_0 + 2\gamma_n M_0 + (\gamma_n + \gamma_{n-1})M_0.$$

Then by (2.10), the assumptions of Theorem 2.1 and Lemma 1.3 we know that (2.4) holds, which together with (2.9) imply that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{2.11}$$

By (1.2), (2.3) and (2.11), we have

$$\|x_n - Su_n\| \leq \|x_n - Su_{n-1}\| + \|Su_{n-1} - Su_n\|$$

$$\leq \alpha_{n-1}\|f(x_{n-1}) - Su_{n-1}\| + \gamma_{n-1}\|v_{n-1} - Su_{n-1}\| + \|u_{n-1} - u_n\|$$
$$\leq \alpha_{n-1}M_0 + \gamma_{n-1}M_0 + \|u_{n-1} - u_n\| \to 0. \tag{2.12}$$

For $v \in \mathrm{Fix}(S)\bigcap \mathrm{EP}(F)$, we have

$$\begin{aligned}\|u_n - v\|^2 &= \|T_{r_n}x_n - T_{r_n}v\|^2 \leq \langle T_{r_n}x_n - T_{r_n}v, x_n - v\rangle = \langle u_n - v, x_n - v\rangle \\ &= \frac{1}{2}(\langle u_n - v, x_n - u_n + u_n - v\rangle + \langle u_n - x_n + x_n - v, x_n - v\rangle) \\ &= \frac{1}{2}(\|u_n - v\|^2 + \|x_n - v\|^2 - \|x_n - u_n\|^2),\end{aligned}$$

i.e.,

$$\|u_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - u_n\|^2. \tag{2.13}$$

Therefore, by (1.2), (2.3), (2.13) and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}\|x_{n+1} - v\|^2 &\leq \alpha_n\|f(x_n) - v\|^2 + (1 - \alpha_n - \gamma_n)\|Su_n - v\|^2 + \gamma_n\|v_n - v\|^2 \\ &\leq \alpha_n\|f(x_n) - v\|^2 + (1 - \alpha_n)\|u_n - v\|^2 + \gamma_n M_0^2 \\ &\leq \alpha_n(\|f(x_n)\| + \|v\|)^2 + (1 - \alpha_n)(\|x_n - v\|^2 - \|x_n - u_n\|^2) + \gamma_n M_0^2 \\ &\leq \alpha_n M_0^2 + \|x_n - v\|^2 - (1 - \alpha_n)\|x_n - u_n\|^2 + \gamma_n M_0^2,\end{aligned}$$

i.e.,

$$\begin{aligned}(1 - \alpha_n)\|x_n - u_n\|^2 &\leq (\alpha_n + \gamma_n)M_0^2 + \|x_n - v\|^2 - \|x_{n+1} - v\|^2, \\ &\leq (\alpha_n + \gamma_n)M_0^2 + \|x_n - x_{n+1}\|(\|x_n - v\| + \|x_{n+1} - v\|),\end{aligned}$$

which together with (2.4) imply

$$\lim_{n\to\infty}\|x_n - u_n\|^2 = 0 = \lim_{n\to\infty}\|x_n - u_n\|. \tag{2.14}$$

By (2.12) and (2.14), we have

$$\|Su_n - u_n\| \leq \|Su_n - x_n\| + \|x_n - u_n\| \to 0. \tag{2.15}$$

Next, we want to prove that

$$\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle \leq 0, \tag{2.16}$$

where $z = P_{\mathrm{Fix}(S)\cap\mathrm{EP}(F)}f(z)$, i.e., we want to prove that there exists a subsequence $\{u_{n_i}\} \subset \{u_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z) - z, x_n - z\rangle = \lim_{i\to\infty}\langle f(z) - z, x_{n_i} - z\rangle \leq 0. \tag{2.17}$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{ij}}\} \subset \{u_{n_i}\}$, which converges weakly to w. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup w$. Then by (2.15), we know that $Su_{n_i} \rightharpoonup w$. We will show that $w \in \mathrm{EP}(F)$. Considering that $u_n = T_{r_n}x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in C.$$

Then by (A_2), we know

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n),$$

which implies that

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \geq F(y, u_{n_i}).$$

Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$, from (A_4), similarly as in [1], we also have

$$0 \geq F(y, w), \quad \forall\, y \in C.$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$ and hence $F(y_t, w) \leq 0$. So, from (A_1) and (A_4), we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, w) \leq tF(y_t, y),$$

i.e., $0 \leq F(y_t, y)$, which together with (A_3) imply

$$0 \leq F(w, y), \quad \forall\, y \in C,$$

i.e., we have $w \in \mathrm{EP}(F)$.

Next, we predict $w \in \mathrm{Fix}(S)$.

We assume that $w \notin \mathrm{Fix}(S)$. Since $u_{n_i} \rightharpoonup w$ and $w \neq Sw$, from Opial's theorem[5] and (2.15), we have

$$\liminf_{i\to\infty} \|u_{n_i} - w\| < \liminf_{i\to\infty} \|u_{n_i} - Sw\| \leq \liminf_{i\to\infty} \{\|u_{n_i} - Su_{n_i}\| + \|Su_{n_i} - Sw\|\} \leq \liminf_{i\to\infty} \|u_{n_i} - w\|.$$

This is a contradiction.

Of course, we can also prove $w \in \mathrm{Fix}(S)$ immediately by $u_{n_i} \rightharpoonup w$, (2.15) and the Demi-Closed Theory (e.g., [12, Lemma 1.1] or [11]).

Thereby, $w \in \mathrm{Fix}(S) \cap \mathrm{EP}(F)$.

Since $z = P_{\mathrm{Fix}(S)\cap \mathrm{EP}(F)} f(z)$, we have

$$\limsup_{n\to\infty} \langle f(z) - z, x_n - z\rangle = \lim_{i\to\infty} \langle f(z) - z, x_{n_i} - z\rangle = \langle f(z) - z, w - z\rangle \leq 0,$$

which verifies (2.16) and (2.17).

Finally, we will complete the proof by way of the method, different from that of [1].

By (1.2), (2.3) and (2.13), we have

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\leq (1 - \alpha_n - \gamma_n)^2 \|Su_n - z\|^2 + 2\alpha_n \langle f(x_n) - z, x_{n+1} - z\rangle + 2\gamma_n \langle v_n - z, x_{n+1} - z\rangle \\
&\leq (1 - \alpha_n)^2 \|u_n - z\|^2 + 2\alpha_n \langle f(x_n) - f(z) + f(z) - z, x_{n+1} - z\rangle + 2\gamma_n \|v_n - z\| \cdot \|x_{n+1} - z\| \\
&\leq (1 - \alpha_n)^2 \|x_n - z\|^2 + \alpha\alpha_n (\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \\
&\quad 2\alpha_n \langle f(z) - z, x_{n+1} - z\rangle + 2\gamma_n M_0^2. \qquad (2.18)
\end{aligned}$$

We know, there exists $n_0 \in \mathbb{N}$ such that $1 - \alpha\alpha_n \geq \frac{1}{2}$ if $n \geq n_0$. Then by (2.18), we have

$$\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\leq \frac{(1 - \alpha_n)^2 + \alpha\alpha_n}{1 - \alpha\alpha_n} \|x_n - z\|^2 + \frac{2\alpha_n}{1 - \alpha\alpha_n} \langle f(z) - z, x_{n+1} - z\rangle + \frac{2\gamma_n M_0^2}{1 - \alpha\alpha_n} \\
&\leq (1 - \frac{2(1 - \alpha)\alpha_n}{1 - \alpha\alpha_n}) \|x_n - z\|^2 + \frac{\alpha_n^2 \|x_n - z\|^2}{1 - \alpha\alpha_n} + 4\alpha_n \langle f(z) - z, x_{n+1} - z\rangle + 4\gamma_n M_0^2
\end{aligned}$$

$$\leq (1-\frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n})\|x_n-z\|^2+2\alpha_n^2M_0^2+4\alpha_n\eta_n+4\gamma_nM_0^2, \quad \forall\, n\geq n_0, \tag{2.19}$$

where $\eta_n=\max\{\langle f(z)-z,x_{n+1}-z\rangle,0\}$. We claim

$$\lim_{n\to\infty}\eta_n=0. \tag{2.20}$$

Indeed, by (2.16) we know, for any $\varepsilon>0$, there exists a natural number $n_1>n_0$ such that $\langle f(z)-z,x_{n+1}-z\rangle<\varepsilon$ if $n\geq n_1$, i.e., $0\leq\eta_n<\varepsilon$, $\forall\, n\geq n_1$. By the arbitrariness of ε we know that (2.20) holds.

Let $h_n=\frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}$. Then $\sum_{n=n_0}^{\infty}h_n\geq 2(1-\alpha)\sum_{n=n_0}^{\infty}\alpha_n=\infty$. Let $b_n=2\alpha_n^2M_0^2+4\alpha_n\eta_n$ and let $c_n=4\gamma_nM_0^2$. Then by Lemma 1.3, we know

$$\lim_{n\to\infty}\|x_n-z\|^2=0,$$

which implies the completeness of the proof of Theorem 2.1. □

Corollary 2.2 *Let C be a nonempty closed convex subset of H. Let S be a nonexpansive mapping of C into H such that* $\mathrm{Fix}(S)\neq\emptyset$. *Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be a sequence generated by $x_1\in H$ and*

$$x_{n+1}=\alpha_nf(x_n)+(1-\alpha_n-\gamma_n)SP_Cx_n+\gamma_nv_n, \tag{2.21}$$

where $\{v_n\}$ is a bounded sequence in H, $\{\alpha_n\}$, $\{\gamma_n\}\subset[0,1]$ and $\{r_n\}\subset(0,\infty)$ satisfy

$$\lim_{n\to\infty}\alpha_n=0,\quad \sum_{n=1}^{\infty}\alpha_n=\infty\quad \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty,\quad \sum_{n=1}^{\infty}\gamma_n<\infty.$$

Then, $\{x_n\}$ converges strongly to $z\in\mathrm{Fix}(S)$, where $z=P_{\mathrm{Fix}(S)}f(z)$.

Proof Put $F(x,y)=0$ for all $x,y\in C$ and $r_n=1$ for all $n\in\mathbb{N}$ in Theorem 2.1. Then we have $u_n=P_Cx_n$. So, from Theorem 2.1, the sequence $\{x_n\}$ generated by $x_1\in H$ and (2.21) for all $n\in\mathbb{N}$ converges strongly to $z\in\mathrm{Fix}(S)$, where $z=P_{\mathrm{Fix}(S)}f(z)$.

Corollary 2.3 *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4) such that* $\mathrm{EP}(F)\neq\emptyset$. *Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1\in H$ and*

$$\begin{cases} F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\geq 0, & \forall\, y\in C,\\ x_{n+1}=\alpha_nf(x_n)+(1-\alpha_n-\gamma_n)u_n+\gamma_nv_n, & n\geq 0\end{cases}$$

where $\{v_n\}$ is a bounded sequence in H, $\{\alpha_n\}$, $\{\gamma_n\}\subset[0,1]$ and $\{r_n\}\subset(0,\infty)$ satisfy

$$\lim_{n\to\infty}\alpha_n=0,\quad \sum_{n=1}^{\infty}\alpha_n=\infty,\quad \sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty,$$

$$\liminf_{n\to\infty}r_n>0,\quad \sum_{n=1}^{\infty}|r_{n+1}-r_n|<\infty,\quad \sum_{n=1}^{\infty}\gamma_n<\infty.$$

Then, $\{x_n\}$ and $\{u_n\}$ converge strongly to $z\in\mathrm{EP}(F)$, where $z=P_{\mathrm{EP}(F)}f(z)$.

Proof Put $Sx=x$ for all $x\in C$ and $r_n=1$ in Theorem 2.1. Then, from Theorem 2.1 the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in \mathrm{EP}(F)$, where $z = P_{\mathrm{EP}(F)}f(z)$.

Remark (1) Theorem A (i.e.,[1, Theorem 3.2]) is Theorem 2.1 in the case of $\gamma_n \equiv 0$. Moreover, some of the methods used in the proof of Theorem 2.1 improve that of [1, Theorem 3.2].

(2) In the case of $\gamma_n \equiv 0$, Corollary 2.2 and Corollary 2.3 become [1, Corollary 3.3] and [1, Corollary 3.4], respectively. Hence, the main results of this paper generalize all of results of [1].

(3) In the case of $\gamma_n \equiv 0$, we obtain Wittmann's theorem [10] in the case when $f(y) = x_1 \in C$ for all $y \in H$ and S is a nonexpansive mapping of C into itself in Corollary 2.2. We also obtain Combettes and Hirstoaga's theorem[2] in the case when $f(y) = x_1 \in H$ for all $y \in H$ in Corollary 2.3.

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