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Involutions Fixing $RP(2^m) \sqcup P(2^m,\, 2n-1)$

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Abstract Let $(M^{2^m+4n+k-2}, T)$ be a smooth closed manifold with a smooth involution T whose fixed point set is $RP(2^m) \sqcup P(2^m,\, 2n-1)$ $(m > 3,\, n > 0)$. For $2n \geq 2^m$, $(M^{2^m+4n+k-2},\, T)$ is bordant to $(P(2^m,\, RP(2n)),\, T_0)$.

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1. Introduction

Let $(M,\, T)$ be a smooth closed manifold with a smooth involution T and $F = \{x \mid T(x) = x,\, x \in M\}$ the fixed point set of T on M[1]. $RP(m)$ denotes the m-dimensional real projective space and $P(m,\, n)$ the Dold manifold of dimension $m+2n$ obtained from the product $S^m \times CP(n)$ of the m-dimensional sphere and the n-dimensional complex projective space by identifying $(x,\, z)$ with $(-x,\, \overline{z})$[2,3]. When F is $RP(\text{odd}) \sqcup P(m,\, n)$ or $RP(8) \sqcup P(8,\, 2n-1)$, the existence and the representative (up to bordism) of $(M,\, T)$ have been studied in [4,5] and [6]. The purpose of this paper is to determine the existence and the representative up to bordism of involutions fixing a disjoint union $RP(2^m) \sqcup P(2^m,\, 2n-1)$ $(m > 3,\, n > 0)$, The main result is stated as follows.

Theorem *Suppose $(M^{2^m+4n+k-2},\, T)$ is a smooth closed manifold of dimension $2^m+4n+k-2$ with a smooth involution T fixing $RP(2^m) \sqcup P(2^m,\, 2n-1)$ $(m > 3,\, n > 0)$. For $2n \geq 2^m$, then $(M^{2^m+4n+k-2},\, T)$ is bordant to $(P(2^m,\, RP(2n)),\, T_0)$.*

The T_0 is defined in Lemma 2.5.

The paper is organized as follows. In Section 2, some lemmas are stated. In Section 3, we discuss the cases in which involutions do not exist. In Section 4, we determine the cases in which involutions exist and also give representatives up to bordism of those involutions. Throughout this paper, the manifolds and involutions are smooth and the involutions are nontrivial. The coefficient group is Z_2 (integers mod 2). Let w denote the total Stiefel-Whitney class, w_i the i-th

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Stiefel-Whitney class, $\sigma_i(x)$ the i-th elementary symmetric function $\sum x_{j_1} x_{j_2} \cdots x_{j_i}$, and $\lambda \to F$ the normal bundle of F in M.

2. Preliminaries

Let (M^n, T) be a closed n-dimensional manifold with an involution T. Let F^{n-k} denote the union of $(n-k)$-dimensional components of the fixed point set F of T and λ^k the normal bundle of F^{n-k} in M^n. From [1], we know that the bordism class of the involution (M^n, T) is determined by the bordism class of the normal bundle $\{(F^{n-k}, \lambda^k)\}$. Kosniowski and Stong[3] gave a formula for the calculation of the Stiefel-Whitney numbers of M^n in terms of the fixed point data $\{(F^{n-k}, \lambda^k)\}$. That is the following

Lemma 2.1[3] *If $f(x_1, x_2, \dots, x_n)$ is a symmetric polynomial over Z_2 in n variables of degree at most n, then*

$$f(x_1, \dots, x_n)[M^n] = \sum_k \frac{f(1+y_1, \dots, 1+y_k, z_1, \dots, z_{n-k})}{\prod_i (1+y_i)}[F^{n-k}]$$

where the expressions are evaluated by replacing the elementary symmetric functions $\sigma_i(x)$, $\sigma_i(y)$ and $\sigma_i(z)$ by the Stiefel-Whitney class $w_i(M^n)$, $w_i(\lambda^k)$ and $w_i(F^{n-k})$ respectively and taking the value of the resulting cohomology class on the fundamental homology class of M^n or F^{n-k}.

Lemma 2.2[3] *If $\sigma_i(x_1, \dots, x_k, x_{k+1}, \dots, x_{k+n})$ is the i-elementary symmetric polynomial over Z_2 in $k+n$ variables, then*

$$\sigma_i(1+y_1, \dots, 1+y_k, z_1, \dots, z_n) = \sum_{p+q \le i} \binom{k-p}{i-p-q} \sigma_p(y_1, \dots, y_k)\sigma_q(z_1, \dots, z_n).$$

Lemma 2.3[1] *Let (M, T) be a closed manifold with an involution T and F the fixed point set of T on M, then $\chi(M) = \chi(F) \pmod 2$, where $\chi(\cdot)$ denotes the Euler characteristic number.*

Let $RP(m)$ denote m-dimensional real projective space and $a \in H^1(RP(m); Z_2)$ the generator. The mod 2 cohomology of $RP(m)$ is given by [7]

$$H^*(RP(m); Z_2) = Z_2[a]/(a^{m+1} = 0)$$

the total Stiefel-Whitney class of $RP(m)$ is given by $w(RP(m)) = (1+a)^{m+1}$.

Let $\mu \to RP(m)$ be a vector bundle. Then the total Stiefel-Whitney class of μ has form $w(\mu) = (1+a)^h$, where h is a nonnegative integer.

Let $P(m, n)$ denote Dold manifold. The mod 2 cohomology of $P(m, n)$ is given by [2]

$$H^*(P(m, n); Z_2) = Z_2[c, d]/(c^{m+1} = d^{n+1} = 0)$$

where $c \in H^1(P(m, n); Z_2)$ and $d \in H^2(P(m, n); Z_2)$ are generators. The total Stiefel-Whitney class of $P(m, n)$ is given by $w(P(m, n)) = (1+c)^m(1+c+d)^{n+1}$.

Let $\nu \to P(m, n)$ be a vector bundle. According to the work of Stong[8], we may write the total Stiefel-Whitney class of ν in the form

$$w(\nu) = (1+c)^s(1+c+d)^t w(\rho)^\varepsilon$$

where $\varepsilon = 0$ or 1 and $w(\rho) = 1 +$ terms of dimension at least 4 is an exotic class ($\varepsilon = 0$ except for $m = 2, 4, 5$, or 6).

Lemma 2.4[1] *Let (M^n, T) be a closed manifold with an involution T whose fixed point set is $RP(2r)$. Then $n = 4r$ and (M^n, T) is bordant to $(RP(2r) \times RP(2r), twist)$, where $twist : (x, y) \to (y, x)$.*

For our purpose, let

$$P(m, RP(n)) = \frac{S^m \times RP(n) \times RP(n)}{-1 \times twist}$$

as in [4].

Lemma 2.5 *There exists an involution T_0 on $P(m, RP(n))$ whose fixed data is $\xi_1 \to RP(m) \sqcup \xi_2 \to P(m, n-1)$, where $w(\xi_1) = (1+a)^n$ and $w(\xi_2) = 1 + c + d$.*

Proof Set $T_1 : RP(n) \longrightarrow RP(n)$ by

$$T_1([x_0, x_1, \ldots, x_n]) = [-x_0, x_1, \ldots, x_n],$$

which fixes $RP(0)$ with normal bundle $n\iota$ and $RP(n-1)$ with normal bundle ι, where ι is the nontrivial line bundle.

Then we may obtain the involution T_0 on $P(m, RP(n))$ induced by $1 \times T_1 \times T_1$. From [5, p1294], whose fixed data is $RP(m)$ with normal bundle $\xi_1^{2n} = n\iota \bigoplus n\mathrm{R}$ and $P(m, n-1)$ with normal bundle $\xi_2^2 = \eta$, where ι is the nontrivial line bundle over $RP(m)$, η a 2-plane bundle over $P(m, n-1)$ and R the trivial bundle.

Notice that $w(\iota) = (1 + a)$ and $w(\eta) = 1 + c + d$, thus we have $w(\xi_1) = (1 + a)^n$ and $w(\xi_2) = 1 + c + d$. The Lemma holds.

3. Nonexistence of the involution

Throughout the following sections, we always suppose that $(M^{2^m+4n+k-2}, T)$ is a closed $(2^m+4n+k-2)$-dimensional manifold with an involution T whose fixed point set $F = RP(2^m) \sqcup P(2^m, 2n-1)$ $(m > 3,\ n > 0)$. Let $\lambda \to F = \lambda_1 \to RP(2^m) \sqcup \lambda_2 \to P(2^m, 2n-1)$ be the normal bundle of F in $M^{2^m+4n+k-2}$. First, one has $w(RP(2^m)) = (1+a)^{2^m+1}$ and $w(\lambda_1) = (1+a)^h$ where h is a nonnegative integer and $a \in H^1(RP(2^m); Z_2)$ generator. Let $c \in H^1(P(2^m, 2n-1); Z_2)$ and $d \in H^2(P(2^m, 2n-1); Z_2)$ be generators. It follows that $w(P(2^m, 2n-1)) = (1+c)^{2^m}(1+c+d)^{2n}$ and for $m > 3$ $w(\lambda_2) = (1+c)^s(1+c+d)^t$ where s and t are nonnegative integers. By [1], we have characteristic numbers $a^{2^m}[RP(2^m)] = 1$ and $c^{2^m}d^{2n-1}[P(2^m, 2n-1)] = 1$.

Next we will express h, s and t in the 2-adic expansion and determine the cases in which the involution does not exist.

For convenience, let $2n = 2^{n_1} + \cdots + 2^{n_j}$, $n_1 > \cdots > n_j \geq 1$. Since $(1+a)^{2^{m+1}} = 1$, $(1+c)^{2^{m+1}} = 1$ and $(1+c+d)^{2^{n_1+1}} = (1+c)^{2^{n_1+1}}$, we may ignore $2^{m+1}A$ in the 2-adic expansion of h, $2^{m+1}B$ in the 2-adic expansion of s and $2^{n_1+1}C$ in the 2-adic expansion of t. Hence we suppose $0 \leq h < 2^{m+1}$, $0 \leq s < 2^{m+1}$, $0 \leq t < 2^{n_1+1}$. If $2n = 2^u$, we also suppose $0 \leq t < 2n$.

If $(M^{2^m+4n+k-2}, T)$ exists, by Lemma 2.3, we have

$$\chi(M^{2^m+4n+k-2}) = \chi(RP(2^m)) + \chi(P(2^m,\,2n-1)) = 1.$$

Since the Euler characteristic number of any odd-dimensional manifold is always zero, it immediately follows that k must be even.

Lemma 3.1 *If $k=0$, there does not exist any required involution $(M^{2^m+4n+k-2}, T)$.*

Proof If there exists an involution $(M^{2^m+4n+k-2}, T)$ with fixed point set $RP(2^m)\sqcup P(2^m,\,2n-1)$, since $k=0$, then $\lambda_2\to P(2^m,\,2n-1)$ is 0-bundle. So we could obtain an involution (N^{2^m+4n-2}, T') with fixed point set $RP(2^m)$. By Lemma 2.4 it follows that $2^m+4n-2=2^{m+1}$, i.e., $2^{m-1}=2n-1$. Since $m>3$, this is impossible. The Lemma holds.

Hence, in the following discussions, we may assume that $k>0$.

Lemma 3.2 *If t is even, then the required involution $(M^{2^m+4n+k-2}, T)$ does not exist except for m, n and k such that $2^m-4n-k+2=0$.*

Proof Noticing that t is even, $w(P(2^m,\,2n-1)) = (1+c)^{2^m}(1+c+d)^{2n}$ and $w(\lambda_2) = (1+c)^s(1+c+d)^t$, we know that the all characteristic numbers of $\lambda_2\to P(2^m,\,2n-1)$ are zero, hence the bordism class $[\lambda_2\to P(2^m,\,2n-1)]=0$. From [1, Theorem 25.2] and [3, §11, Proposition], the bordism class of $(M^{2^m+4n+k-2}, T)$ is determined by the normal bundle $\lambda_1\to RP(2^m)$ and $\lambda_1\to RP(2^m)$ is fixed data of an involution. By Lemma 2.4, $2^m+4n+k-2=2^{m+1}$, i.e., $2^m-4n-k+2=0$. The Lemma holds.

Lemma 3.3 *There does not exist any required involution $(M^{2^m+4n+k-2}, T)$ for which t, s are all odd and h is even.*

Proof From Lemma 2.2 direct computations show

$$\sigma_1(1+y,z)(\lambda_1\to RP(2^m)) = a,\quad \sigma_1(1+y,z)(\lambda_2\to P(2^m,\,2n-1)) = 0.$$

Where $\sigma_1(1+y,z)(\lambda_1\to RP(2^m))$ are evaluated by replacing the elementary symmetric functions $\sigma_1(y)$ and $\sigma_1(z)$ by the Stiefel-Whitney class $w_1(\lambda_1)$ and $w_1(RP(2^m))$, respectively, the same as $\sigma_1(1+y,z)(\lambda_2\to P(2^m,\,2n-1))$. Taking the symmetric polynomial $f(x)=\sigma_1^{2^m}(x)$ where $\deg f(x) = 2^m < 2^m+2(2n-1)+k$, by Lemma 2.1, we have

$$\begin{aligned}
0 &= f(x)[M^{2^m+4n+k-2}]\\
&= \frac{\sigma_1^{2^m}(1+y,z)}{w(\lambda_1)}[RP(2^m)] + \frac{\sigma_1^{2^m}(1+y,z)}{w(\lambda_2)}[P(2^m,\,2n-1)]\\
&= \frac{a^{2^m}}{(1+a)^h}[RP(2^m)] + \frac{0}{(1+c)^s(1+c+d)^t}[P(2^m,\,2n-1)]\\
&= a^{2^m}[RP(2^m)] + 0 = 1.
\end{aligned}$$

This is a contradiction and so $(M^{2^m+4n+k-2}, T)$ does not exist. The Lemma holds.

In the following proofs, "Taking $\mathbf{f(x)}$ leads to a contradiction" denotes "If there exists the required involution, we take the symmetric polynomial $f(x)$. By Lemma 2.1 an analogous proof to Lemma 3.3 leads to a contradiction". This will be not pointed out again.

Lemma 3.4 *There does not exist any required involution $(M^{2^m+4n+k-2}, T)$ for which t, h are odd and s is even.*

Proof From Lemma 2.2, direct computations show

$$\sigma_1(1+y,z)(\lambda_1 \to RP(2^m)) = 0, \quad \sigma_1(1+y,z)(\lambda_2 \to P(2^m,\, 2n-1)) = c;$$

$$\sigma_2(1+y,z)(\lambda_1 \to RP(2^m)) = \binom{2(2n-1)+k}{2} + a + a^2 + \binom{h}{2}a^2,$$

$$\sigma_2(1+y,z)(\lambda_2 \to P(2^m,\, 2n-1)) = \binom{k}{2} + c + \binom{s+t}{2}c^2 + d + \binom{n}{1}c^2.$$

Taking $f(x) = \sigma_1^{2^m}(x)[\sigma_2(x)+\binom{k}{2}+\sigma_1(x)+\binom{s+t}{2}\sigma_1^2(x)+\binom{n}{1}\sigma_1^2(x)]^{2n-1}$ leads to a contradiction.

Lemma 3.5 *If t is odd and h, s are even, then $(M^{2^m+4n+k-2}, T)$ does not exist except for the following cases $w(\lambda_1) = (1+a)^{2i}$, $w(\lambda_2) = 1+c+d$, $n \equiv i (\mathrm{mod}\, 2^m)$, $k = 2$.*

Proof From Lemma 2.2, direct computations show

$$\sigma_1(1+y,z)(\lambda_1 \to RP(2^m)) = a, \quad \sigma_1(1+y,z)(\lambda_2 \to P(2^m,\, 2n-1)) = c;$$

$$\sigma_2(1+y,z)(\lambda_1 \to RP(2^m)) = \binom{2(2n-1)+k}{2} + \binom{h}{2}a^2,$$

$$\sigma_2(1+y,z)(\lambda_2 \to P(2^m,\, 2n-1)) = \binom{k}{2} + c + \binom{s+t}{2}c^2 + d + \binom{n}{1}c^2.$$

If $k \geq 4$, taking $f(x) = \sigma_1^{2^m}(x)[\sigma_2(x) + \binom{k}{2}]^{2n}$ leads to a contradiction.

If $k = 2$, then $\lambda_2 \to P(2^m, 2n-1)$ is a 2-bundle and $s + 2t \leq 2$. Since s is even and t is odd, then $s = 0$, $t = 1$.

When n is odd, if $\binom{h}{2} = 0$, then $h = 4l$ where l is a nonnegative integer. Taking $f(x) = \sigma_1^{2^m-2}(x)[\sigma_2(x) + \sigma_1^2(x)]$ leads to a contradiction. If $\binom{h}{2} = 1$, then $h = 4l+2$. If $a^{2j} (j > 0)$ is the term with the highest degree of a in $\frac{(1+a)^{2n}}{(1+a)^h}$, taking $f(x) = \sigma_1^{2^m-2j}(x)[\sigma_2(x)+\sigma_1(x)+\sigma_1^2(x)+1]^{2n}$ leads to a contradiction. Thus it follows that $2n = 2^{m+1}u + 4l + 2$ where u is a nonnegative integer, $l = 0, 1, 2, \ldots, 2^{m-1} - 1$.

Hence, when n is odd, $(M^{2^m+4n+k-2}, T)$ may exist for $h = 4l+2$, $n \equiv 2l+1 (\mathrm{mod}\, 2^m)$, $s = 0$, $t = 1$, $k = 2$.

When n is even, if $\binom{h}{2} = 1$, then $h = 4l+2$, taking $f(x) = \sigma_1^{2^m-2}(x)\sigma_2(x)$ leads to a contradiction. If $\binom{h}{2} = 0$, then $h = 4l$. If $a^{2j} (j > 0)$ is the term with the highest degree of a in $\frac{(1+a)^{2n}}{(1+a)^h}$, taking $f(x) = \sigma_1^{2^m-2j}(x)[\sigma_2(x) + \sigma_1(x) + 1]^{2n}$ leads to a contradiction. Thus it follows that $2n = 2^{m+1}u + 4l$ where u is a nonnegative integer, $l = 0, 1, 2, \ldots, 2^{m-1} - 1$.

Hence, when n is even, $(M^{2^m+4n+k-2}, T)$ may exist for $h = 4l$, $n \equiv 2l (\mathrm{mod}\, 2^m)$, $s = 0$, $t = 1$, $k = 2$.

From the above arguments, $(M^{2^m+4n+k-2}, T)$ may exist for $w(\lambda_1) = (1+a)^{2i}$, $w(\lambda_2) = 1 + c + d$, $n \equiv i (\mathrm{mod}\, 2^m)$, $k = 2$. The Lemma holds.

Lemma 3.6 *If t, s and h are all odd, then $(M^{2^m+4n+k-2},\,T)$ does not exist for $2n\ge 2^m$.*

Proof From Lemma 2.2, direct computations show

$$\sigma_1(1+y,z)(\lambda_1\to RP(2^m))=0,\quad \sigma_1(1+y,z)(\lambda_2\to P(2^m,\,2n-1))=0;$$

$$\sigma_2(1+y,z)(\lambda_1\to RP(2^m))=\binom{2(2n-1)+k}{2}+a+a^2+\binom{h}{2}a^2,$$

$$\sigma_2(1+y,z)(\lambda_2\to P(2^m,\,2n-1))=\binom{k}{2}+\binom{s+t}{2}c^2+d+\binom{n}{1}c^2;$$

$$\sigma_3(1+y,z)(\lambda_1\to RP(2^m))=a+a^2,\quad \sigma_3(1+y,z)(\lambda_2\to P(2^m,\,2n-1))=cd.$$

When $2n\ge 2^m$, since s, t are odd, then $k\ge 4$, taking $f(x)=\sigma_3^{2^m}(x)[\sigma_2(x)+\binom{k}{2}]^{2n-2^m}$ leads to a contradiction.

4. Existence of the involution and the representative

Theorem 4.1 *If $(M^{2^m+4n+k-2},T)$ is a closed manifold with an involution T fixing $F=RP(2^m)\sqcup P(2^m,\,2n-1)(m>3,\,n>0)$. Let $\lambda\to F=\lambda_1\to RP(2^m)\sqcup\lambda_2\to P(2^m,\,2n-1)$ be the normal bundle of F in M, $w(\lambda_2)=(1+c)^s(1+c+d)^t$, where t is even. For m, n, k such that $2^m-4n-k+2=0$, then $(M^{2^m+4n+k-2},\,T)$ exists and is bordant to $(RP(2^m)\times RP(2^m),twist)$.*

Proof Since t is even, from the proof of Lemma 3.2, we see that bordism class $[\lambda_2\to P(2^m,\,2n-1)]=0$. From [1, Theorem 25.2], the bordism class of $(M^{2^m+4n+k-2},\,T)$ is determined by the normal bundle $\lambda_1\to RP(2^m)$.

Since $[\lambda_2\to P(2^m,\,2n-1)]=0$, from [3, §11, Proposition], there exist an involution $(M_1^{2^m+4n+k-2},\,T_1)$ with fixed data $\lambda_2\to P(2^m,\,2n-1)$ and an involution $(M_2^{2^m+4n+k-2},\,T_2)$ with fixed data $\lambda_1\to RP(2^m)$.

Setting $N^{2^m+4n+k-2}=M_1\sqcup M_2$, we define an involution T' on N such that $T'|_{M_1}=T_1$ and $T'|_{M_2}=T_2$. By [1, Theorem 25.2], $(M^{2^m+4n+k-2},\,T)$ is bordant to $(N,\,T')$ and $(N,\,T')$ is bordant to $(M_2,\,T_2)$. From Lemma 2.4, $(M_2,\,T_2)$ is bordant to $(RP(2^m)\times RP(2^m),\,twist)$. Thus $(M^{2^m+4n+k-2},\,T)$ is bordant to $(RP(2^m)\times RP(2^m),twist)$. The Theorem holds.

Theorem 4.2 *If $(M^{2^m+4n+k-2},\,T)$ is a closed manifold with an involution T fixing $F=RP(2^m)\sqcup P(2^m,\,2n-1)(m>3,\,n>0)$. Let $\lambda\to F=\lambda_1\to RP(2^m)\sqcup\lambda_2\to P(2^m,\,2n-1)$ be the normal bundle of F in M, $w(\lambda_1)=(1+a)^h$, $w(\lambda_2)=(1+c)^s(1+c+d)^t$, where t is odd. For $h\equiv s(\bmod 2)$, $2n\ge 2^m$, then $(M^{2^m+4n+k-2},\,T)$ exists and is bordant to $(P(2^m,\,RP(2n)),\,T_0)$.*

Proof By Lemmas 2.5, 3.1 and 3.3–3.6, there exists an involution $(P(2^m,\,RP(2n)),\,T_0)$, the fixed data of T_0 is $\xi_1\to RP(2^m)\sqcup\xi_2\to P(2^m,\,2n-1)$, where

$$w(\xi_1)=(1+a)^{2i},\ w(\xi_2)=1+c+d,\ n\equiv i(\bmod 2^m).$$

From [1, Theorem 25.2], it follows that $(M^{2^m+4n+k-2},\,T)$ is bordant to $(P(2^m,\,RP(2n)),\,T_0)$. The Theorem holds.

Note For $t \equiv 1 \pmod 2$, $h \equiv s \equiv 0 \pmod 2$, we can prove that $(M^{2^m+4n+k-2}, T)$ exists and is bordant to $(P(2^m, RP(2n)), T_0)$.

From Theorems 4.1 and 4.2, we obtain the main result in this paper

Theorem 4.3 *If $(M^{2^m+4n+k-2}, T)$ is a closed manifold with an involution T fixing*

$$RP(2^m) \sqcup P(2^m, 2n-1) \ (m > 3,\ n > 0).$$

Then for $2n \geq 2^m$, $(M^{2^m+4n+k-2}, T)$ is bordant to $(P(2^m, RP(2n)), T_0)$.

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