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Error Estimates of Fitting for Bivariate Fractal Interpolation

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Abstract A given bivariate continuous function is fitted by using a bivariate fractal interpolation function, and the error of fitting is studied in this paper. The results of error estimates are obtained in two metric cases. This provides a theoretical basis for the algorithms of fractal surface reconstruction.

Keywords fractal interpolation; fitting; error estimate.

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1. Introduction

The fractal interpolation method[1]developed from the theory of iterated function system (IFS) has become a new tool for fitting experimental data. Especially, in the process of fitting non-smooth and irregular rough curves and surfaces, fractal interpolation has exhibited more advantages[2−4] as compared with the traditional polynomial interpolation and spline interpolation. The error analysis of fractal interpolation, as a principal problem of interpolation theory, has attracted a great deal of attention in recent years. As we know, a fractal interpolation function (FIF) is essentially the attractor of an IFS, and usually it has no explicit function expression that is convenient to analyzing. Therefore, it is very difficult to investigate the error of fitting for fractal interpolation. In [5, 6], Sha, Liu and Ruan studied the interpolation error of fitting for the univariate affine FIF in the case where the knots of interpolation are non-equidistant and in the case of block fractal interpolation, respectively. They presented the expression of error estimates for the fitting of univariate fractal interpolation. So far, many of the researches on the error of fractal interpolation have mostly been restricted within univariate fractal interpolation. There are few results about the error analysis of fitting for bivariate fractal interpolation. However, in many practical problems, such as the three-dimensional fractal reconstruction of terrain surfaces, the simulation of rock fracture surfaces etc., the bivariate fractal interpolation is often applied

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to establish mathematical models of fractal objects. Thus, it is very significant in theory and applications to study the error of fitting for bivariate fractal interpolation.

In this paper, the methods of error analysis in [5, 6] will be extended to three-dimensional situation. We will use a bivariate FIF to fit a given bivariate continuous function and study the error of fitting. The results of error estimates are obtained in two metric cases.

2. Main results

Let $I = [0,1]$ and $D = [0,1]^2$. And let $N > 1, M > 1$ be two given positive integers. Set $x_i = \frac{i}{N}, i = 0,1,\ldots,N$, $y_j = \frac{j}{M}, j = 0,1,\ldots,M$. We obtain a set of the rectangular grid points $\{(x_i, y_j) : i = 0,1,\ldots,N, j = 0,1,\ldots,M\}$ of D. Let $\phi(x,y) \in C(D)$ be a given continuous function defined on D, and $z_{i,j} = \phi(x_i, y_j)$. Then

$$\Delta_{ij} = \{(x_i, y_j, z_{i,j}) : i = 0,1,\ldots,N, j = 0,1,\ldots,M\} \tag{1}$$

is a set of data points (interpolation points) in R^3.

Define mappings $w_{ij} : D \times \mathrm{R} \to \mathrm{R}^3$, $i = 0,1,\ldots,N-1, j = 0,1,\ldots,M-1$, as follows:

$$w_{ij}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} u_i(x) \\ v_j(y) \\ F_{ij}(x,y,z) \end{pmatrix} = \begin{pmatrix} \frac{(-1)^{\delta(i)}}{N}x + \frac{i+1-\delta(i+1)}{N} \\ \frac{(-1)^{\delta(j)}}{M}y + \frac{j+1-\delta(j+1)}{M} \\ a_{ij}x + b_{ij}y + c_{ij}xy + dz + e_{ij} \end{pmatrix}, \tag{2}$$

where $\delta(k) = k \bmod 2$, and the $a_{ij}, b_{ij}, c_{ij}, e_{ij}$ and d are real constants. Clearly, if $\delta(i) = 0$, then $u_i : I \to [x_i, x_{i+1}]$ satisfies $u_i(0) = x_i$ and $u_i(1) = x_{i+1}$; if $\delta(i) = 1$, then $u_i : I \to [x_i, x_{i+1}]$ satisfies $u_i(0) = x_{i+1}$ and $u_i(1) = x_i$. The mappings $F_{ij} : D \times \mathrm{R} \to \mathrm{R}$ must obey the following conditions

$$F_{ij}(\delta(k), \delta(l), z_{\Delta(k,l)}) = z_{k,l},\ (k,l) \in \{i, i+1\} \times \{j, j+1\}, \tag{3}$$

where $\Delta(k,l) = (N\delta(k), M\delta(l))$. Take d to be a free parameter, which is called vertical scaling factor. Obviously, we can see from (3) that the coefficients $a_{ij}, b_{ij}, c_{ij}, e_{ij}$ depend on not only the data set Δ_{ij} but also the vertical scaling factor d. From [7], we know that the mappings F_{ij} may be written as the following form

$$F_{ij}(x,y,z) = dz + \sum_{(k,l)\in\{i,i+1\}\times\{j,j+1\}} (z_{k,l} - dz_{\Delta(k,l)})\Phi_{\Delta(k,l)}(x,y), \tag{4}$$

where the bivariate functions $\Phi_{N,M}, \Phi_{0,0}, \Phi_{0,M}, \Phi_{N,0} : D \to I$ are defined as

$$\Phi_{N,M}(x,y) = xy, \Phi_{0,0}(x,y) = (1-x)(1-y), \Phi_{0,M}(x,y) = (1-x)y, \Phi_{N,0}(x,y) = x(1-y).$$

In [7], Malysz proved that when $0 < |d| < 1$, the IFS $\{\mathrm{R}^3, w_{ij} : i = 0,1,\ldots,N-1, j = 0,1,\ldots,M-1\}$ formed by (2) generates a bivariate FIF whose graph is a continuous fractal interpolation surface (FIS) passing through the data set (1). Moreover, the Minkowski dimension of the FIS is also discussed in [7]. In this paper, we will apply the above FIS to fit the given bivariate continuous surface $\phi(x,y)$ and study the corresponding error of fitting. For this purpose, we need to represent the mappings F_{ij} by another equivalent expression.

Let $p_\phi(x,y)$ be a segmented bilinear interpolation function defined on D, such that the graph of $p_\phi(x,y)$ on each small rectangle $D_{ij}=[x_i,x_{i+1}]\times[y_j,y_{j+1}], i=0,1,\ldots,N-1, j=0,1,\ldots,M-1$, is a bilinear interpolation surface passing through the four points $(x_i,y_j,z_{i,j})$, $(x_{i+1},y_j,z_{i+1,j})$, $(x_i,y_{j+1},z_{i,j+1})$ and $(x_{i+1},y_{j+1},z_{i+1,j+1})$. And let $h_\phi(x,y)$ be a bilinear interpolation function passing through the four points $(x_0,y_0,z_{0,0})$, $(x_0,y_M,z_{0,M})$, $(x_N,y_0,z_{N,0})$ and $(x_N,y_M,z_{N,M})$. By using (4), we are able to obtain that

$$F_{ij}(x,y,z)=p_\phi(u_i(x),v_j(y))+d\cdot[z-h_\phi(x,y)],\ \ (x,y,z)\in D\times \mathrm{R}.$$

According to the above expression, we may define mappings $w_{ij}^{'}:D\times\mathrm{R}\to\mathrm{R}^3$, $i=0,1,\ldots,N-1, j=0,1,\ldots,M-1$, by

$$w_{ij}^{'}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}u_i(x)\\ v_j(y)\\ p_\phi(u_i(x),v_j(y))+d\cdot[z-h_\phi(x,y)]\end{pmatrix},\tag{5}$$

where $u_i(x)$ and $v_j(y)$ are the same as those in (2). So, we can declare that the IFSs $\{\mathrm{R}^3,w_{ij}^{'}: i=0,1,\ldots,N-1, j=0,1,\ldots,M-1\}$ and $\{\mathrm{R}^3,w_{ij}: i=0,1,\ldots,N-1, j=0,1,\ldots,M-1\}$ generate the same FISs.

Set $C_0(D)=\{g\in C(D): g(x_i,y_j)=z_{i,j}, i=0,1,\ldots,N, j=0,1,\ldots,M\}$. Define an operator $T:C_0(D)\to C_0(D)$ by

$$(Tg)(x,y)=p_\phi(x,y)+d\cdot[g(u_i^{-1}(x),v_j^{-1}(y))-h_\phi(u_i^{-1}(x),v_j^{-1}(y))],$$

for any $g\in C_0(D)$ and $(x,y)\in D_{ij}$. Obviously, when $0<|d|<1$, the operator T is contractive and must have a unique fixed function $f(x,y)\in C_0(D)$ such that, for any $(x,y)\in D_{ij}$,

$$f(x,y)=p_\phi(x,y)+d\cdot[f(u_i^{-1}(x),v_j^{-1}(y))-h_\phi(u_i^{-1}(x),v_j^{-1}(y))].\tag{6}$$

To get a high resolution expression for $f(x,y)$, we introduce the notion of code space in the following.

Define the code space by

$$\Omega=\{\omega=(i_1,i_2,\ldots,i_k,\ldots): i_k\in\{0,1,\ldots,N-1\}\}.$$

Let $\omega=(i_1,i_2,\ldots,i_k,\ldots),\tilde\omega=(\tilde i_1,\tilde i_2,\ldots,\tilde i_k,\ldots)\in\Omega$. The distance between the ω and $\tilde\omega$ is defined by

$$|\omega-\tilde\omega|=\sum_{k=1}^{\infty}\frac{|i_k-\tilde i_k|}{N^k}.$$

So the Ω forms a compact metric space.

Define a shift operator $\sigma:\Omega\to\Omega$ by $\sigma\omega=(i_2,i_3,\ldots)\in\Omega$, where $\omega=(i_1,i_2,i_3,\ldots)\in\Omega$. Let σ^k denote the k-fold composition of σ with itself. Then we have $\sigma^k\omega=(i_{k+1},i_{k+2},\ldots)$. Let $\omega(m)=(i_1,i_2,\ldots,i_m)$ and $u_{\omega(m)}(x)=u_{i_1}\circ u_{i_2}\circ\cdots\circ u_{i_m}(x), x\in[0,1]$. From the expression of $u_i(x)$, by using the successive iteration and mathematical inductive method, we are able to show the following lemma.

Lemma 1 *Let* $\omega=(i_1,i_2,\ldots,i_k,\ldots), i_k\in\{0,1,\ldots,N-1\}$. *Then for arbitrary positive integer*

$m \geq 1$, *we have*

$$u_{\omega(m)}(x) = \frac{(-1)^{\sum_{k=1}^{m}\delta(i_k)}}{N^m}x + \sum_{r=1}^{m}\frac{(-1)^{\sum_{k=1}^{r-1}\delta(i_k)}[i_r + 1 - \delta(i_r + 1)]}{N^r}, \quad x \in [0,1], \tag{7}$$

where $\sum_{k=1}^{0}\delta(i_k) = 0$.

Define a mapping $\varphi : \Omega \to [0,1]$ by

$$\varphi(\omega) = \sum_{r=1}^{\infty}\frac{(-1)^{\sum_{k=1}^{r-1}\delta(i_k)}[i_r + 1 - \delta(i_r + 1)]}{N^r}, \tag{8}$$

for $\omega = (i_1, i_2, \ldots, i_k, \ldots) \in \Omega$. From (7), we can see that the $\varphi : \Omega \to [0,1]$ is a continuous onto function.

Similarly, we may define another code space $\Theta = \{\theta = (j_1, j_2, \ldots, j_k, \ldots) : j_k \in \{0, 1, \ldots, M-1\}\}$. Set $\theta(m) = (j_1, j_2, \ldots, j_m)$, and $v_{\theta(m)}(y) = v_{j_1} \circ v_{j_2} \circ \cdots \circ v_{j_m}(y), y \in [0,1]$. Analogous to (7), we have

$$v_{\theta(m)}(y) = \frac{(-1)^{\sum_{k=1}^{m}\delta(j_k)}}{M^m}y + \sum_{r=1}^{m}\frac{(-1)^{\sum_{k=1}^{r-1}\delta(j_k)}[j_r + 1 - \delta(j_r + 1)]}{M^r}, \quad y \in [0,1]. \tag{9}$$

Define mapping $\psi : \Theta \to [0,1]$ by

$$\psi(\theta) = \sum_{r=1}^{\infty}\frac{(-1)^{\sum_{k=1}^{r-1}\delta(j_k)}[j_r + 1 - \delta(j_r + 1)]}{M^r}, \tag{10}$$

for $\theta = (j_1, j_2, \ldots, j_k, \ldots) \in \Theta$. Then $\psi : \Theta \to [0,1]$ is also a continuous onto function.

Theorem 1 *Let $f(x,y)$ be the bivariate FIF determined by (2)–(3),* $0 < |d| < 1$. *Then we have*

$$f(\varphi(\omega), \psi(\theta)) = \sum_{k=1}^{\infty} d^k (p_\phi - h_\phi)(\varphi(\sigma^k\omega), \psi(\sigma^k\theta)) + p_\phi(\varphi(\omega), \psi(\theta)), \omega \in \Omega, \theta \in \Theta. \tag{11}$$

Proof Let $\omega = (i_1, i_2, \ldots) \in \Omega$ and $\theta = (j_1, j_2, \ldots) \in \Theta$. Define $i\omega = (i, i_1, i_2, \ldots), i \in \{0, 1, \ldots, N-1\}$, and $j\theta = (j, j_1, j_2, \ldots), j \in \{0, 1, \ldots, M-1\}$. Let $x = \varphi(\omega)$ and $y = \psi(\theta)$. Then from the definition of $u_i(x)$ and (8), we have

$$u_i(x) = u_i(\varphi(\omega)) = \frac{(-1)^{\delta(i)}}{N}\varphi(\omega) + \frac{i + 1 - \delta(i+1)}{N} = \varphi(i\omega).$$

Similarly, the equality $v_j(y) = \psi(j\theta)$ holds.

From (6), for any $(x,y) \in D$, we get that

$$f(u_i(x), v_j(y)) = d \cdot [f(x,y) - h_\phi(x,y)] + p_\phi(u_i(x), v_j(y)),$$

which can be written as

$$\begin{aligned} f(\varphi(i\omega), \psi(j\theta)) =& d \cdot [f(\varphi(\omega), \psi(\theta)) - h_\phi(\varphi(\omega), \psi(\theta))] + p_\phi(\varphi(i\omega), \psi(j\theta)) \\ =& d \cdot [f(\varphi(\sigma(i\omega)), \psi(\sigma(j\theta))) - h_\phi(\varphi(\sigma(i\omega)), \psi(\sigma(j\theta)))] + \\ & p_\phi(\varphi(i\omega), \psi(j\theta)). \end{aligned} \tag{12}$$

For arbitrary $\omega \in \Omega, \theta \in \Theta$, rewrite (12) as

$$f(\varphi(\omega), \psi(\theta)) = d \cdot [f(\varphi(\sigma\omega), \psi(\sigma\theta)) - h_\phi(\varphi(\sigma\omega), \psi(\sigma\theta))] + p_\phi(\varphi(\omega), \psi(\theta)). \tag{13}$$

The Eq. (13) provides a recursion formula from which we obtain that for any positive integer $m \geq 1$,

$$f(\varphi(\omega),\psi(\theta)) = d^m(f-h_\phi)(\varphi(\sigma^m\omega),\psi(\sigma^m\theta)) + \sum_{k=1}^{m-1} d^k(p_\phi - h_\phi)(\varphi(\sigma^k\omega),\psi(\sigma^k\theta)) + p_\phi(\varphi(\omega),\psi(\theta)).$$

Since $0<|d|<1$, (11) follows by letting $m\to\infty$. The proof is completed.

We now express (11) as another form. When $x=\varphi(\omega), y=\psi(\theta)$, we have

$$f(x,y) = p_\phi(x,y) + \sum_{k=1}^{\infty} d^k(p_\phi - h_\phi)(u_{i_k}^{-1}\circ u_{i_{k-1}}^{-1}\circ\cdots\circ u_{i_1}^{-1}(x), v_{j_k}^{-1}\circ v_{j_{k-1}}^{-1}\circ\cdots\circ v_{j_1}^{-1}(y)). \quad (14)$$

Let $I_{\omega(k)} = u_{i_1}\circ u_{i_2}\circ\cdots\circ u_{i_k}(I)$ and $I_{\theta(k)} = v_{j_1}\circ v_{j_2}\circ\cdots\circ v_{j_k}(I)$. For $m\geq 1, \forall q\in C(D)$ and $\forall (x,y)\in I_{\omega(m)}\times I_{\theta(m)}$, we define an operator $P_m: C(D)\to C(D)$ by

$$P_m(q)(x,y) = p_q(x,y) + \sum_{k=1}^{m} d^k(p_q - h_q)(u_{i_k}^{-1}\circ u_{i_{k-1}}^{-1}\circ\cdots\circ u_{i_1}^{-1}(x), v_{j_k}^{-1}\circ v_{j_{k-1}}^{-1}\circ\cdots\circ v_{j_1}^{-1}(y)), \quad (15)$$

where the methods of the construction of two functions p_q and h_q are the same as those of p_ϕ and h_ϕ in (5), respectively. Obviously, P_m is a continuous mapping on $C(D)$ and has the following properties, for any $q\in C(D)$,

$$P_m(q)(x_i,y_j) = p_q(x_i,y_j) = q(x_i,y_j), i=0,1,\ldots,N, j=0,1,\ldots,M.$$

Let $|\cdot|_\infty$ and $|\cdot|_2$ denote the ∞-norm and L^2-norm on $C(D)$, respectively, i.e.,

$$|q|_\infty = \max_{(x,y)\in D}|q(x,y)|,\ |q|_2 = \Big(\iint_D [q(x,y)]^2 \mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{2}}, \quad \forall q\in C(D).$$

Theorem 2 *Let $\phi(x,y)\in C(D)$ be a given function, and $f(x,y)$ be the bivariate FIF determined by (2)–(3), and $0<|d|<1$. For any positive integer m, let P_m be the operator defined by (15). Then*

$$|\phi - f|_r \leq |\phi - P_m(\phi)|_r + \frac{|d|^{m+1}}{1-|d|}|p_\phi - h_\phi|_r, \quad r=2,\infty. \quad (16)$$

Proof Set $s_k(x,y) = d^k(p_\phi - h_\phi)(u_{i_k}^{-1}\circ u_{i_{k-1}}^{-1}\circ\cdots\circ u_{i_1}^{-1}(x), v_{j_k}^{-1}\circ v_{j_{k-1}}^{-1}\circ\cdots\circ v_{j_1}^{-1}(y))$, for $(x,y)\in I_{\omega(k)}\times I_{\theta(k)}$. From (14) and (15), we have $(P_m(\phi)-f)(x,y) = -\sum_{k=m+1}^{\infty} s_k(x,y)$. Hence,

$$|s_k|_2^2 = \iint_D [s_k(x,y)]^2 \mathrm{d}x\mathrm{d}y = \sum_{\substack{0\leq i_1,\ldots,i_k\leq N-1\\ 0\leq j_1,\ldots,j_k\leq M-1}} \iint_{I_{\omega(k)}\times I_{\theta(k)}} [s_k(x,y)]^2\mathrm{d}x\mathrm{d}y. \quad (17)$$

Performing transformation of variables by $x=u_{i_1}(s)$ and $y=v_{j_1}(t)$ in the integral on the right hand side in (17), we get

$$|s_k|_2^2 \leq d^{2k}\sum_{\substack{0\leq i_1,\ldots,i_k\leq N-1\\ 0\leq j_1,\ldots,j_k\leq M-1}} \frac{1}{NM}\iint_{I_{i_2\cdots i_k}\times I_{j_2\cdots j_k}} [(p_\phi - h_\phi)(u_{i_k}^{-1}\circ\cdots\circ u_{i_2}^{-1}(s), v_{j_k}^{-1}\circ\cdots\circ v_{j_2}^{-1}(t))]^2 \mathrm{d}s\mathrm{d}t, \quad (18)$$

where $I_{i_2\cdots i_k} = u_{i_2}\circ\cdots\circ u_{i_k}(I)$ and $I_{j_2\cdots j_k} = v_{j_2}\circ\cdots\circ v_{j_k}(I)$. Transforming (18) $k-1$ times successively, we obtain

$$|s_k|_2^2 \le d^{2k} \sum_{\substack{0\le i_1,\dots,i_k\le N-1\\ 0\le j_1,\dots,j_k\le M-1}} \frac{1}{(NM)^k} \iint\limits_D [(p_\phi - h_\phi)(x,y)]^2 \mathrm{d}x\mathrm{d}y = d^{2k}|p_\phi - h_\phi|_2^2.$$

Thus,

$$\begin{aligned}|\phi - f|_2 &\le |\phi - P_m(\phi)|_2 + |P_m(\phi) - f|_2 \le |\phi - P_m(\phi)|_2 + \sum_{k=m+1}^{\infty} |s_k|_2 \\ &\le |\phi - P_m(\phi)|_2 + \frac{|d|^{m+1}}{1-|d|}|p_\phi - h_\phi|_2.\end{aligned}$$

This means that (16) holds when $r = 2$. For the case $r = \infty$, we obtain from (11) and (15) that

$$(P_m(\phi) - f)(\varphi(\omega), \psi(\theta)) = -\sum_{k=m+1}^{\infty} d^k (p_\phi - h_\phi)(\varphi(\sigma^k\omega), \psi(\sigma^k\omega)).$$

Consequently,

$$\begin{aligned}|\phi - f|_\infty &\le |\phi - P_m(\phi)|_\infty + |P_m(\phi) - f|_\infty \\ &\le |\phi - P_m(\phi)|_\infty + \frac{|d|^{m+1}}{1-|d|}|p_\phi - h_\phi|_\infty.\end{aligned}$$

The proof of Theorem 2 is completed. □

Remark 1 Since $0 < |d| < 1$, from Theorem 2, we can approximately replace the deviation between f and ϕ by the deviation between $P_m(\phi)$ and ϕ when m is sufficiently large. Thus, the fitting for fractal surfaces will be performed effectively provided that d is selected properly such that the deviation between $P_m(\phi)$ and ϕ is as small as possible. This provides a theoretical basis for the algorithms of fitting of fractal surfaces.

Remark 2 The FIS used in this paper is generated by the iterated mappings (2). The corresponding set of interpolation points is the set of arbitrary data without the collinear requirement[7] on the boundary of the domain D. But the vertical scaling factor d of each of the iterated mappings (2) is identical. This means that the FIS generated by the iteration lacks the flexibility. In [8], Dalla constructed a class of FISs on rectangular grids. In Dalla's construction, the interpolation points on the boundary of the rectangular domain are required to be collinear, but the vertical scaling factors d_{ij} $(0 < |d_{ij}| < 1)$ may not be the same. Using the similar method introduced in this paper, we are able to discuss the error of fitting for this class of FISs. However, it will be a very complicated nonlinear optimization problem how to choose the best d_{ij} such that the deviation $|\phi - P_m(\phi)|$ reaches to the minimum in a metric sense.

Recently, Bouboulis, Dalla and Drakopoulos[9] extended the concept of affine FIS and proposed a constructive method for the recurrent bivariate FISs. The resulting fractal surfaces are neither self-affine nor self-similar. These fractal surfaces can be used to approximate arbitrary natural surfaces. It is a considerable problem how to utilize this class of recurrent FISs to fit some given functions or discrete data.

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