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Growth of Solutions of a Class of Higher Order Linear Differential Equations with Coefficients Being Gap Series

TU Jin[1], *LIU Jie*[2]
(1. College of Mathematics and Information Science, Jiangxi Normal University, Jiangxi 330022, China;
2. Department of Natural Sciences, Nanchang Teachers College, Jiangxi 330029, China)
(E-mail: tujin2008@sina.com)

Abstract In this paper, we investigate the growth of solutions of a class of higher order linear differential equations with coefficients being gap series. In this case, we remove the condition that the order of coefficients in equations is less than $\frac{1}{2}$, and obtain some results which improve the previous results.

Keywords linear differential equations; hyper order; gap series.

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1. Introduction and main results

We shall assume that reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions[1,2]. In addition, we will use the notation $\sigma(f)$ to denote the order of growth of meromorphic function $f(z)$ and $\sigma_2(f)$ to denote the hyper order of $f(z)$[3], which is defined to be

$$\sigma_2(f)=\overline{\lim_{r\to\infty}}\frac{\log\log T(r,f)}{\log r}.$$

For higher order linear differential equation

$$f^{(k)}+A_{k-1}f^{(k-1)}+\cdots+A_0f=0, \tag{1.1}$$

and

$$f^{(k)}+A_{k-1}f^{(k-1)}+\cdots+A_0f=F, \tag{1.2}$$

it is well known that every solution of (1.1) and (1.2) is entire function when the coefficients $A_j(z)\,(j=0,\ldots,k-1)$ and $F(z)$ are entire functions. If the coefficients satisfy $\sigma(A_j)<\frac{1}{2}\,(j=0,\ldots,k-1)$, many authors have studied the growth of the solutions of (1.1) and (1.2) and obtained the following results.

Theorem A[4] *Let* $A_0,\ldots,A_{k-1}$ *be entire functions satisfying* $\max\{\sigma(A_j),j=1,\ldots,k-1\}<$

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$\sigma(A_0)<\infty$. Then every solution of (1.1) satisfies $\sigma_2(f)=\sigma(A_0)$.

Theorem B[5] *Let A_0 and A_1 be entire functions satisfying $\sigma(A_0)<\sigma(A_1)<\frac{1}{2}$, or A_1 be transcendental, and A_0 be a polynomial. Then every non-trivial solution of*

$$f''+A_1f'+A_0f=0 \tag{1.3}$$

satisfies $\sigma_2(f)=\sigma(A_1)$.

Remark 1 The same result as Theorem B also holds for higher order linear differential equation (1.1) when $\max\{\sigma(A_j),j\neq 1\}<\sigma(A_1)<\frac{1}{2}$.

Theorem C[4] *Let $A_0,\dots,A_{k-1}$ be entire functions. Suppose that there exists some $d\in\{0,\dots,k-1\}$ such that transcendental entire function A_d satisfies $\max\{\sigma(A_j),j\neq d\}\leq\sigma(A_d)$, then at least one of the solution of (1.1) satisfies $\sigma_2(f)=\sigma(A_d)$.*

Theorem D[4] *Let $A_0,\dots,A_{k-1}$ and F be entire functions. Suppose that there exists some $d\in\{0,\dots,k-1\}$ such that transcendental entire function A_d satisfies*

$$\max\{\sigma(F),\sigma(A_j),j\neq d\}<\sigma(A_d)<\frac{1}{2},$$

then every transcendental solution of (1.2) satisfies $\sigma_2(f)=\sigma(A_d)$. Furthermore, if $F\not\equiv 0$, then every transcendental solution of (1.2) satisfies $\overline{\lambda}_2(f)=\sigma(A_d)$.

Then a natural question is: Can we get the similar result to Theorem B when we get rid of the condition $\sigma(A_0)<\sigma(A_1)<\frac{1}{2}$? In this paper, we answer the question when the coefficients in (1.1) and (1.2) are gap series, and obtain the following results. First we will introduce some notations about gap series.

Let $f(z)=\sum_{n=1}^{\infty}c_{\lambda_n}z^{\lambda_n}$ be an entire function of finite order[6], where $\Lambda=\{\lambda_0,\lambda_1,\lambda_2,\dots,\}$ is an increasing sequence of nonnegative integers satisfying the gap condition

$$\frac{\lambda_n}{n}\to\infty. \tag{1.4}$$

Theorem 1 *Let $A_0,\dots,A_d,\dots,A_{k-1}\,(k\geq 2, 1\leq d\leq k-1)$ be entire functions satisfying $\max\{\sigma(A_j),j\neq d\}<\sigma(A_d)$. Suppose that $A_d(z)=\sum_{n=1}^{\infty}c_{\lambda_n}z^{\lambda_n}$ is an entire function of regular growth such that the sequence of exponents $\{\lambda_n\}$ satisfies (1.4), then every transcendental solution f of (1.1) satisfies $\sigma_2(f)=\sigma(A_d)$.*

Corollary 1 *Let $A_0\not\equiv 0, A_1$ be entire functions satisfying $\sigma(A_0)<\sigma(A_1)$. Suppose that $A_1(z)=\sum_{n=1}^{\infty}c_{\lambda_n}z^{\lambda_n}$ is an entire function of regular growth such that the sequence of exponents $\{\lambda_n\}$ satisfies (1.4), then every non-trivial solution f of (1.3) satisfies $\sigma_2(f)=\sigma(A_1)$.*

Theorem 2 *Let $F\not\equiv 0, A_0,\dots,A_d,\dots,A_{k-1}\,(k\geq 2, 1\leq d\leq k-1)$ be entire functions satisfying $\max\{\sigma(F),\sigma(A_j),j\neq d\}<\sigma(A_d)$. Suppose that $A_d(z)=\sum_{n=1}^{\infty}c_{\lambda_n}z^{\lambda_n}$ is an entire function of regular growth such that the sequence of exponents $\{\lambda_n\}$ satisfies (1.4), then every transcendental solution f of (1.2) satisfies $\overline{\lambda}_2(f)=\lambda_2(f)=\sigma_2(f)=\sigma(A_d)$.*

Remark 2 Here we give an example to show the existence of A_d in Theorems 1 and 2. For example,

$$A_d(z) = \sum_{n=1}^{\infty} \frac{1}{n^{2n^2}} z^{n^2}$$

is a regular growing entire function and satisfies $\frac{n^2}{n} \to \infty$, and

$$\sigma(A_d) = \lim_{n\to\infty} \frac{n^2 \log n^2}{\log n^{2n^2}} = 1 > \frac{1}{2}.$$

This shows that the hypothesis on A_d is valid.

2. Lemmas

Lemma 1[7] *Let $f(z)$ be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then for any given $\varepsilon > 0$, there exists a set $E \subset [1,\infty)$ that has finite logarithmic measure, a constant $B > 0$ that depends only on α and (i,j), where i and j are integers with $0 \le i < j$, such that for all z satisfying $|z| = r \notin E$, there holds*

$$\Big|\frac{f^{(j)}(z)}{f^{(i)}(z)}\Big| \le B\Big[\frac{T(\alpha r, f)}{r}(\log^{\alpha} r)\log T(\alpha r, f)\Big]^{j-i}. \tag{2.1}$$

Lemma 2[8] *Let $f(z)$ be an entire function of order $\sigma(f) = \alpha < +\infty$. Then for any given $\varepsilon > 0$, there is a set $E \subset [1,\infty)$ that has finite linear measure and finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \bigcup E$, there holds*

$$\exp\{-r^{\alpha+\varepsilon}\} \le |f(z)| \le \exp\{r^{\alpha+\varepsilon}\}. \tag{2.2}$$

Lemma 3[9] *Let $f(z) = \sum_{n=1}^{\infty} c_{\lambda_n} z^{\lambda_n}$ be an entire function of finite lower order. If the sequence of exponents $\{\lambda_n\}$ satisfies condition (1.4), then for any given $0 < \varepsilon < 1$, there is a set $E \subset [0,\infty)$ with infinite logarithmic measure such that for all z satisfying $|z| = r \in E$, there holds*

$$\log L(r,f) > (1-\varepsilon)\log M(r,f), \tag{2.3}$$

where $M(r,f) = \sup_{|z|=r} |f(z)|,\ L(r,f) = \inf_{|z|=r} |f(z)|$.

Lemma 4[10] *Let $f(z)$ be an entire function. Then there exists a set $E \subset (1,\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \bigcup E$ and $|f(z)| = M(r,f)$, there holds*

$$\Big|\frac{f(z)}{f^{(s)}(z)}\Big| \le 2r^s \quad (s \in N). \tag{2.4}$$

Lemma 5[10] *Let $A_j\,(j = 0,\dots,k-1)$ be entire functions satisfying $\sigma(A_j) \le \sigma < \infty$. Then all the solutions of (1.1) satisfy $\sigma_2(f) \le \sigma$.*

Remark 3 We can get the following result by the same reasoning as in Lemma 5.

Lemma 6 *Let $A_j\,(j = 0,\dots,k-1), F$ be entire functions satisfying $\max\{\sigma(F),\ \sigma(A_j)\,(j = 0,\dots,k-1)\} \le \sigma < \infty$. Then all the solutions of (1.2) satisfy $\sigma_2(f) \le \sigma$.*

3. Proof of Theorem 1

Suppose that $f(z)$ is a transcendental solution of (1.1). By (1.1), we have

$$|A_d| \leq \left|\frac{f^{(k)}}{f^{(d)}}\right| + \cdots + |A_{d+1}|\left|\frac{f^{(d+1)}}{f^{(d)}}\right| + \left|\frac{f}{f^{(d)}}\right|\Big(|A_{d-1}|\left|\frac{f^{(d-1)}}{f}\right| + \cdots + |A_0|\Big). \tag{3.1}$$

By Lemma 1, there exists a set $E_1 \subset [1, \infty)$ with finite logarithmic measure and a constant $B > 0$ such that

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B\left(T(2r, f)\right)^{2k}, \quad 0 \leq i < j \leq k \tag{3.2}$$

holds for all $|z| = r \notin E_1$ and for sufficiently large r. Since $\max\{\sigma(A_j), j \neq d\} < \sigma(A_d)$, we choose α, β to satisfy $\max\{\sigma(A_j), j \neq d\} < \alpha < \beta < \sigma(A_d)$, by Lemma 2, there exists a set $E_2 \subset [1, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E_2$ and for sufficiently large r, we have

$$|A_j(z)| \leq \exp\{r^\alpha\}, \quad j \neq d. \tag{3.3}$$

Since A_d is regular growing, by Lemma 3, there exists a set $E_3 \subset [1, \infty)$ with infinite logarithmic measure such that for all z satisfying $|z| = r \in E_3$, we have

$$|A_d(z)| \geq \min|A_d(z)| \geq (M(r, A_d))^{\frac{1}{2}} \geq \exp\left\{\frac{1}{2}r^\beta\right\}. \tag{3.4}$$

By Lemma 4, there exists a set $E_4 \subset [0, \infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E_4$ and $|f(z)| = M(r, f)$, we have

$$\left|\frac{f(z)}{f^{(d)}(z)}\right| \leq 2r^d. \tag{3.5}$$

Hence from (3.1)–(3.5), for all z satisfying $|z| = r \in E_3 \backslash (E_1 \bigcup E_2 \bigcup E_4)$ and $|f(z)| = M(r, f)$, we have

$$\exp\left\{\frac{1}{2}r^\beta\right\} \leq kr^M \exp\{r^\alpha\} \cdot (T(2r, f))^{2k}. \tag{3.6}$$

Since β is arbitrarily close to $\sigma(A_d)$ and by (3.6), we have

$$\overline{\lim_{r\to\infty}} \frac{\log\log T(r, f)}{\log r} \geq \sigma(A_d).$$

On the other hand, by Lemma 5, we have $\sigma_2(f) \leq \sigma(A_d)$. Therefore, $\sigma_2(f) = \sigma(A_d)$.

4. Proof of Theorem 2

Suppose that $f(z)$ is a transcendental solution of (1.2). By (1.2), we have

$$|A_d| \leq \left|\frac{f^{(k)}}{f^{(d)}}\right| + \cdots + |A_{d+1}|\left|\frac{f^{(d+1)}}{f^{(d)}}\right| + \left|\frac{f}{f^{(d)}}\right|\Big(|A_{d-1}|\left|\frac{f^{(d-1)}}{f}\right| + \cdots + |A_0| + \frac{F}{f}\Big). \tag{4.1}$$

For all z satisfying $|f(z)| = M(r, f)$ and for sufficiently large r, we have $|f(z)| > 1$. Thus, we have

$$\left|\frac{F(z)}{f(z)}\right| \leq |F(z)|. \tag{4.2}$$

From (4.1)–(4.2), and by the same arguments as in Theorem 1 and Lemma 6, we can get $\sigma_2(f) = \sigma(A_d)$. Furthermore, if $F \not\equiv 0$, by the same reasoning as in Theorem 1 in [4], we can obtain that $\overline{\lambda}_2(f) = \lambda_2(f) = \sigma_2(f)$ holds for every transcendental solution f of (1.2). Thus, we complete the proof of Theorem 2. □

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