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# The Equivalence of Two Convergent Sequence of Bounded Sequences in Normed Space

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**Abstract** Two kinds of convergent sequences on the real vector space $\mathbf{m}$ of all bounded sequences in a real normed space $X$ were discussed in this paper, and we prove that they are equivalent, which improved the results of [1].

**Keywords** almost convergence; quasi-almost convergence.

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## 1. Introduction

On the real vector space $\mathbf{m}$ of all bounded sequences in a real normed space $X$, in [1] the almost convergence $(x_i) \in \mathbf{m}$ was defined, and it was gotten that $(x_i)$ almost converges to $s \in X$ iff

$$\|\frac{1}{p}\sum_{i=0}^{p-1} x_{k+i} - s\| \to 0 \text{ as } p \to \infty$$

uniformly in $k = 0, 1, \ldots$.

In [2], the quasi almost convergence $(x_i) \in \mathbf{m}$ was defined, and it was shown that $(x_i)$ quasi almost converges to $s \in X$ iff

$$\|\frac{1}{p}\sum_{i=np}^{(n+1)p-1} x_i - s\| \to 0 \text{ as } p \to \infty$$

uniformly in $n = 0, 1, \ldots$.

And [2] gave the following theorem:

**Theorem 1** *If a sequence $(x_i) \in \mathbf{m}$ almost converges to $s \in X$, then it quasi almost converges to $s$.*

In this paper we show that the converse of the above theorem is also true.

## 2 Main result

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First we give some notations and facts.

Let $(x_i) \in \mathbf{m}$. We denote $a_n = \frac{1}{n}\sum_{i=0}^{n-1} x_i$. Since $(x_i) \in \mathbf{m}$, $m = \sup_{i\in\mathbf{Z}}\{\|x_i\|\}$ is finite. And we can easily see that $\|a_n\| \le m$, for any $n \in \mathbf{Z}$.

Now we give our main theorem.

**Theorem 2** *Let $\{x_i\}_{i=0}^{\infty} \in \mathbf{m}$. Then the following aconditions are equivalent.*

*(I)* $\|\frac{1}{p}\sum\limits_{i=np}^{(n+1)p-1} x_i - s\| \to 0$ *as* $p \to \infty$, *uniformly in* $n = 0, 1, \ldots$.

*(II)* $\|\frac{1}{p}\sum\limits_{i=0}^{p-1} x_{k+i} - s\| \to 0$ *as* $p \to \infty$, *uniformly in* $k = 0, 1, \ldots$.

**Proof** If (II) is true, by Theorem 1, we get (I) is also true.

For the converse, if (II) does not hold, then there exits $\varepsilon_0 > 0$ such that for any $p > 0$, there is a $p' > p$ and $k' \in \mathbf{Z}$, such that $\|\frac{1}{p'}\sum_{i=0}^{p'-1} x_{k'+i} - s\| > \varepsilon_0$.

For the $\varepsilon_0$ above, by (I), there are $K$ and $p_0 \in \mathbf{Z}$ satisfying:

(i) $\|\frac{1}{k}\sum_{i=nk}^{(n+1)k-1} x_i - s\| < \frac{\varepsilon_0}{6}$, as $k \ge K$, uniformly in $n = 0, 1, \ldots$.

(ii) $p_0 > K$, $\frac{K}{p_0}m < \frac{\varepsilon_0}{6}$ and $\frac{K}{p_0}\|s\| < \frac{\varepsilon_0}{6}$, where $m = \sup_{i\in\mathbf{Z}}\{\|x_i\|\}$.

Since (II) does not hold, for the $p_0$ above, $p_1 > p_0$ and $k_0 \in \mathbf{Z}$ exist such that $\|\frac{1}{p_1}\sum_{i=0}^{p_1-1} x_{k_0+i} - s\| > \varepsilon_0$.

1) If $k_0 \le K$, then

$$\begin{aligned}
\|\frac{1}{p_1}\sum_{i=0}^{p_1-1} x_{k_0+i} - s\| &\le \|\frac{1}{p_1+k_0}\sum_{i=0}^{p_1+k_0-1} x_i - s\| + \|\frac{1}{p_1+k_0}\sum_{i=0}^{p_1+k_0-1} x_i - \frac{1}{p_1}\sum_{i=0}^{p_1-1} x_{k_0+i}\| \\
&\le \frac{\varepsilon_0}{6} + \|a_{p_1+k_0} - \frac{(p_1+k_0)a_{p_1+k_0} - k_0 a_{k_0}}{p_1}\| \\
&\le \frac{\varepsilon_0}{6} + \|\frac{k_0}{p_1}a_{p_1+k_0}\| + \|\frac{k_0}{p_1}a_{k_0}\| \\
&\le \frac{\varepsilon_0}{2} < \varepsilon_0.
\end{aligned}$$

2) If $k_0 > K$, then there are two cases:

(a) If $k_0 \le p_1$, then

$$\begin{aligned}
\|\frac{1}{p_1}\sum_{i=0}^{p_1-1} x_{k_0+i} - s\| &= \|\frac{(p_1+k_0)a_{p_1+k_0} - k_0 a_{k_0}}{p_1} - s\| \\
&\le \|a_{k_0+p_1} - s\| + \frac{k_0}{p_1}\|a_{k_0+p_1} - s\| + \frac{k_0}{p_1}\|a_{k_0} - s\| \\
&\le \frac{\varepsilon_0}{6} + \frac{\varepsilon_0}{6} + \frac{\varepsilon_0}{6} < \varepsilon_0.
\end{aligned}$$

(b) If $k_0 > p_1$, since $p_1 > p_0 > K$, there exist $n_0, n_1 \in \mathbf{N}$ and $k_1, k_2 \in \mathbf{Z}, k_1, k_2 < K$ such that $n_0K + k_1 = k_0$ and $n_1K - k_2 = p_1 + k_1$.

Let $a = \frac{1}{k_1}\sum_{i=n_0K}^{k_0-1} x_i$, $b = \frac{1}{k_2}\sum_{i=k_0+p_1}^{k_0+p_1+k_2-1} x_i$ and $a_i = \frac{1}{K}\sum_{j=(n_0+i-1)K}^{(n_0+i-1)K-1} x_j$, where $i =$

$1, \ldots, n_1$. Then

$$\|\frac{1}{p_1}\sum_{i=0}^{p_1-1} x_{k_0+i} - s\| = \|\frac{K(a_1 + \cdots + a_{n_1}) - k_1 a - k_2 b}{p_1} - s\|$$
$$\leq \sum_{i=1}^{n_1} \frac{K\|a_i - s\|}{p_1} + \|\frac{n_1 K - p_1}{p_1} s\| + \|\frac{k_1 a}{p_1}\| + \|\frac{k_2 b}{p_1}\|$$
$$\leq 2\frac{\varepsilon_0}{6} + 2\frac{\varepsilon_0}{6} + \frac{\varepsilon_0}{6} + \frac{\varepsilon_0}{6} = \varepsilon_0,$$

contradicting the assumption that we made before. □

Similarly with the proof above, we can get a theorem for multi sequences on the real vector space $\mathbf{T}$ of all bounded multi sequences in a real normed space $X$.

**Theorem 3** *Let* $\{x_{i,j}\}_{i,j=0}^{\infty} \in \mathbf{T}$. *Then the following conditions are equivalent.*

*(I)* $\|\frac{1}{pq}\sum_{i=np}^{(n+1)p-1}\sum_{i=mq}^{(m+1)q-1} x_{i,j} - s\| \to 0$ *as* $p, q \to \infty$, *uniformly in* $n, m = 0, 1, \ldots$.

*(II)* $\|\frac{1}{pq}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1} x_{k+i,l+j} - s\| \to 0$ *as* $p, q \to \infty$, *uniformly in* $k, l = 0, 1, \ldots$.

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