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Reproducing Kernel for $D^2(\Omega,\rho)$ and Metric Induced by Reproducing Kernel

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Abstract An important property of the reproducing kernel of $D^2(\Omega,\rho)$ is obtained and the reproducing kernels for $D^2(\Omega,\rho)$ are calculated when $\Omega = B_n \times B_n$ and ρ are some special functions. A reproducing kernel is used to construct a semi-positive definite matrix and a distance function defined on $\Omega\times\Omega$. An inequality is obtained about the distance function and the pseudo-distance induced by the matrix.

Keywords harmonic Bergman spaces; harmonic Bergman kernels.

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0. Introduction

Throughout this paper, p denotes a number satisfying $1 < p < \infty$, Ω denotes a connected open set of $R^{2n}(n > 1)$ and dV denotes Lebesgue volume measure. For any $x \in R^{2n}$, we rewrite $x = (x^{(1)}, x^{(2)})$, where $x^{(1)} = (x_1, x_2, \dots, x_n)$, $x^{(2)} = (x_{n+1}, x_{n+2}, \dots, x_{2n})$. $\rho(x)$ is a continuous function defined on the closure of Ω satisfying $\rho(x) > 0, x \in \Omega$. $\Delta_1 = \frac{\partial}{\partial x_1^2} + \frac{\partial}{\partial x_2^2} + \cdots + \frac{\partial}{\partial x_n^2}$, $\Delta_2 = \frac{\partial}{\partial x_{n+1}^2} + \frac{\partial}{\partial x_{n+2}^2} + \cdots + \frac{\partial}{\partial x_{2n}^2}$.

Definition *$D^p(\Omega,\rho)$ is the set of functions u defined on Ω satisfying the following condions: u is twice continuously differentiable, complex-valued, $\Delta_1 u = 0$, $\Delta_2 u = 0$,*

$$\|u\|_p = \Big(\int_\Omega \rho|u|^p \mathrm{d}V\Big)^{1/p} < \infty.$$

In the next section, we will prove $D^2(\Omega,\rho)$ is a Hilbert space with inner product

$$\langle u, v\rangle = \int_\Omega \rho u\overline{v}\mathrm{d}V.$$

For each $x \in \Omega$, the map $u \to u(x)$ is a bounded linear functional on the Hilbert space $D^2(\Omega,\rho)$. Thus there exists a unique function $T_\Omega(x,) \in D^2(\Omega,\rho)$ such that

$$u(x) = \int_\Omega \rho(y)u(y)\overline{T_\Omega(x,y)}\mathrm{d}V(y) \tag{1}$$

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for every $u \in D^2(\Omega,\rho)$. The function T_Ω, which can be viewed as a function on $\Omega\times\Omega$, is called the reproducing kernel of Ω or the reproducing kernel for $D^2(\Omega,\rho)$. In several complex variables, the Bergman kernel of product domains equals to the product of kernels. But harmonic Bergman kernel has no such property. What interests us is that $T_\Omega(x,y)$ has the following important property: If $\rho(x)\equiv 1, \Omega=\Omega_1\times\Omega_2$, where Ω_1,Ω_2 are the connected open sets in R^n, then $T_\Omega(x,y)=R_{\Omega_1}(x^{(1)},y^{(1)})\times R_{\Omega_2}(x^{(2)},y^{(2)})$, where $R_{\Omega_1},R_{\Omega_2}$ are the harmonic Bergman kernels of Ω_1,Ω_2, respectively.

Harmonic Bergman spaces have not been studied as extensively as their holomorphic counterparts. Holomorphic Bergman kernel functions have the following important property[1]: Let Ω_1, Ω_2 be domains in C^n, $f:\Omega_1\to\Omega_2$ be biholomorphic. Then

$$\det J_C f(z) K_{\Omega_2}(f(z),f(\zeta))\overline{\det J_C f(\zeta)} = K_{\Omega_1}(z,\zeta).$$

Harmonic Bergman kernel functions have no such good property. Furthermore, the computation of harmonic functions is more difficult than their holomorphic counterparts. For example, one must use zonal harmonic functions and Poisson kernel to compute the harmonic Bergman kernel for the unit ball in R^n[2]. And one also needs Poisson kernel to compute the harmonic Bergman kernel for the upper half-space[3]. Bergman kernel functions play an important role in several complex variables. One of the main applications is to construct Bergman metrics on bounded domains of C^n. In this paper, the author makes use of $T_\Omega(x,y)$ to construct metrics for bounded domains in R^{2n}.

Let us summarize the main results of this paper: In the first section, we proved that $D^2(\Omega,\rho)$ is complete, that $T_\Omega(x,y)$ is the product of harmonic Bergman kernels, when Ω is a product domain and $\rho\equiv 1$; In the second section, we calculated the reproducing kernel for $D^2(\Omega,\rho)$, when $\Omega=B_n\times B_n$ and ρ are some special functions; In the third section, we constructed a distance function and a semi-positive definite matrix drawing on $T_\Omega(x,y)$ and obtained an inequality about the distance function and the pseudo-distance induced by the semi-positive definite matrix.

1. Preliminaries and basic lemmas

In this paper, B denotes the unit ball of R^n and S denotes the unit sphere. Normalized surface-area measure on S is denoted by σ ($\sigma(S)=1$). Ω will denote a bounded domain in R^{2n}.

Lemma 1.1[2] *If u is a harmonic function on $B(a,r)$, then there exist $p_m\in H_m(R^n)$ such that*

$$u(x)=\sum_{m=0}^{\infty} p_m(x-a)$$

for all $x\in B(a,r)$, the series converges absolutely and uniformly on compact subsets of $B(a,r)$.

Theorem 1.2[2] *Suppose (u_m) is a sequence of harmonic functions on D such that u_m converges uniformly to a function u on each compact subset of D. Then u is harmonic on Ω. Moreover, for every multiindex α, $D^\alpha u_m$ converges uniformly to $D^\alpha u$ on each compact subset of D.*

Lemma 1.3 *If f is integral on R^n, then*

$$\frac{1}{nV(B)}\int_{R^n} f\mathrm{d}V=\int_0^\infty r^{n-1}\int_S f(r\xi)\mathrm{d}\sigma(\xi)\mathrm{d}r.$$

Theorem 1.4 *$D^p(\Omega,\rho)$ is a complete space.*

Proof Suppose that $\{u_j\}$ is a Cauchy sequence in $D^p(\Omega,\rho)$. Then $\{\rho^{1/p}u_j\}$ is a Cauchy sequence in $L^P(\Omega)$. There exists $f\in L^P(\Omega)$ such that $u_j\to f$ in $L^P(\Omega)$. Let $\mathcal{K}$ be a compact subset of Ω and $2r=\mathrm{dist}(\mathcal{K},\mathrm{R}^{2n}\backslash\Omega)$. $\mathcal{K}^*=\{x|\mathrm{dist}(x,\mathcal{K})\le r\}$. For each $x\in\mathcal{K}, B(x,r)\subset\mathcal{K}^*$. If $u\in D^p(\Omega,\rho)$, then u is a harmonic function on Ω. For each $x\in\mathcal{K}$,

$$u(x)=\frac{1}{V(B(x,r))}\int_{B(x,r)}u(y)\mathrm{d}V(y).$$

According to Hölder inequality, we have

$$\begin{aligned}|u(x)|&\le\frac{1}{V(B(x,r))}\int_{B(x,r)}|u(y)|\mathrm{d}V(y)\\&=\frac{1}{V(B(x,r))}\int_{B(x,r)}|u(y)|\rho(y)^{1/p}\frac{1}{\rho(y)^{1/p}}\mathrm{d}V(y)\\&\le\frac{1}{V(B(x,r))}\Big(\int_{B(x,r)}|u|^p\rho\mathrm{d}V(y)\Big)^{1/p}\Big(\int_{B(x,r)}\frac{1}{\rho^{q/p}}\mathrm{d}V(y)\Big)^{1/q}\\&\le\frac{1}{r^nV(B)}\parallel u\parallel_p\Big(\int_{\mathcal{K}^*}\frac{1}{\rho^{q/p}}\mathrm{d}V\Big)^{1/q}.\end{aligned}\tag{2}$$

$$|u_j(x)-u_m(x)|\le\frac{1}{r^nV(B)}\parallel u_j-u_m\parallel_p\Big(\int_{\mathcal{K}^*}\frac{1}{\rho^{q/p}}\mathrm{d}V\Big)^{1/q}$$

for all $x\in\mathcal{K}$ and j,m. The inequality above implies that$\{u_j\}$ converges uniformly on $\mathcal{K}$. It follows from Theorem 1.2 that $\{u_j\}$ converges uniformly on compact subset of Ω to a harmonic function u on Ω and $\Delta_1u=0$, $\Delta_2u=0$. Because $\{u_j\rho^{1/p}\}$ has a subsequence converging to f poinwise almost everywhere on Ω, f equals to $u\rho^{1/p}$ almost everywhere on Ω, and thus $u\in D^p(\Omega,\rho)$. $\lim_{j\to\infty}(\int_\Omega\rho|u_j-u|^p\mathrm{d}V)^{1/p}=0$. □

Take $p=2$. Then the above theorem shows that $D^2(\Omega,\rho)$ is a Hilbert space with inner product

$$\langle u,v\rangle=\int_\Omega\rho u\overline{v}\mathrm{d}V.$$

For each $x\in\Omega$, the map $u\to u(x)$ is a bounded linear functional on the Hilbert space $D^2(\Omega,\rho)$ (by formula (2)). Thus there exists a unique function $T_\Omega(x,)\in D^2(\Omega,\rho)$ such that

$$u(x)=\int_\Omega\rho(y)u(y)\overline{T_\Omega(x,y)}\mathrm{d}V(y)\tag{3}$$

for every $u\in D^2(\Omega,\rho)$. The function T_Ω, which can be viewed as a function on $\Omega\times\Omega$, is called the reproducing kernel of Ω.

Lemma 1.5 *$T_\Omega(x,y)$ has the following properties:*

(a) T_Ω is real valued;

(b) If $\{u_m\}$ is the orthonormal basis of $D^2(\Omega,\rho)$, then

$$T_\Omega(x,y)=\sum_{m=1}^{\infty}\overline{u_m(x)}u_m(y)$$

for all $x,y\in\Omega$;

(c) $T_\Omega(x,y)=T_\Omega(y,x)$ for all $x,y\in\Omega$;

(d) $\| T_\Omega(x,)\|_2=\sqrt{T_\Omega(x,x)}$ for all $x\in\Omega$.

Theorem 1.6 *Let Ω_1,Ω_2 be the bounded domain of R^n, $\Omega=\Omega_1\times\Omega_2$ and $\rho(x)\equiv 1$. Then $T_\Omega(x,y)=R_{\Omega_1}(x^{(1)},y^{(1)})\times R_{\Omega_2}(x^{(2)},y^{(2)})$, where $R_{\Omega_1},R_{\Omega_2}$ are the harmonic Bergman kernels of Ω_1,Ω_2, respectively.*

Proof Suppose that $t=(t^{(1)},t^{(2)})\in\Omega$. We have

$$\begin{aligned}
R_{\Omega_1}(x^{(1)},t^{(1)})R_{\Omega_2}(x^{(2)},t^{(2)}) &= \int_{\Omega_1\times\Omega_2} R_{\Omega_1}(y^{(1)},t^{(1)})R_{\Omega_2}(y^{(2)},t^{(2)})T_\Omega(x,y)\mathrm{d}V(y)\\
&= \int_{\Omega_1} R_{\Omega_1}(y^{(1)},t^{(1)})\mathrm{d}V(y^{(1)})\int_{\Omega_2} R_{\Omega_2}(y^{(2)},t^{(2)})T_\Omega((x^{(1)},x^{(2)}),(y^{(1)},y^{(2)}))\mathrm{d}V(y^{(2)})\\
&= \int_{\Omega_1} R_{\Omega_1}(y^{(1)},t^{(1)})T_\Omega((x^{(1)},x^{(2)}),(y^{(1)},t^{(2)}))\mathrm{d}V(y^{(1)})\\
&= T_\Omega((x^{(1)},x^{(2)}),(t^{(1)},t^{(2)})).
\end{aligned}$$

Lemma 1.7 *Let B denote the unit ball in R^n, $\rho\equiv 1$. Then $T_{B\times B}(x,y)=R_B(x^{(1)},y^{(1)})\times R_B(x^{(2)},y^{(2)})$, where*

$$R_B(x^{(1)},y^{(1)})=\frac{(n-4)|x^{(1)}|^4|y^{(1)}|^4+(8x^{(1)}\cdot y^{(1)}-2n-4)|x^{(1)}|^2|y^{(1)}|^2+n}{nV(B)(1-2x^{(1)}\cdot y^{(1)}+|x^{(1)}|^2|y^{(1)}|^2)^{1+n/2}}.$$

2. The reproducing kernel for $B\times B$

Theorem 2.1 *Suppose that $x=(x^{(1)},x^{(2)})\in R^{2n}$, and $p(x)$ is a polynomial on R^{2n} about x satisfying $\Delta_1 p(x)=0,\Delta_2 p(x)=0$. Then $p(x)$ can be written in the following form: $p(x)=\sum_{j=1}^{N} f_j(x^{(1)})g_j(x^{(2)})$, where f_j, g_j are homogeneous harmonic polynomials on R^n about $x^{(1)}$, $x^{(2)}$, respectively.*

Proof $p(x)$ can be written in the following form: $p(x)=\sum_\alpha (x^{(1)})^\alpha g_\alpha(x^{(2)})$, where $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)$, $g_\alpha(x^{(2)})$ is a polynomial about $x^{(2)}$. Because $\Delta_2 p(x)=\sum_\alpha (x^{(1)})^\alpha\Delta_2 g_\alpha(x^{(2)})=0$, we have $\Delta_2 g_\alpha(x^{(2)})=0$. Thus $g_\alpha(x^{(2)})$ is a harmonic polynomial on R^n. There exists nonnegative integer N_α such that $g_\alpha=\sum_{j=0}^{N_\alpha} g_\alpha^{(j)}(x^{(2)})$, where $g_\alpha^{(j)}(x^{(2)})$ is a homogeneous harmonic polynomial on R^n about $x^{(2)}$ of degree j. Let $m=\max\{N_\alpha|\alpha\}$ and $\dim H_K(R^n)=L_k, k=0,1,\dots,m$. Suppose that the basis for the $H_K(R^n)$ are $p_{k1},p_{k2},\dots,p_{kL_k}$. $p(x)$ can be written in the following form: $p(x)=\sum_{k=0}^{m}\sum_{j=1}^{L_k} p_{kj}(x^{(2)})f_{kj}(x^{(1)})$, where $f_{kj}(x^{(1)})$ is a polynomial about $x^{(1)}$. Because $\Delta_1 p(x)=0$, we have $\sum_{k=0}^{m}\sum_{j=1}^{L_k} p_{kj}(x^{(2)})\Delta_1 f_{kj}(x^{(1)})=0$. For each $t\in R$, $\sum_{k=0}^{m}\sum_{j=1}^{L_k} p_{kj}(tx^{(2)})\Delta_1 f_{kj}(x^{(1)})=\sum_{k=0}^{m} t^k\sum_{j=1}^{L_k} p_{kj}(x^{(2)})\Delta_1 f_{kj}(x^{(1)})=0$. Thus $\sum_{j=1}^{L_k} p_{kj}(x^{(2)})\Delta_1 f_{kj}(x^{(1)})=0$. Because $p_{k1},p_{k2},\dots,p_{kL_k}$ are the basis of $H_K(R^n)$, we have

$\Delta_1 f_{kj}(x^{(1)})=0$. $f_{kj}(x^{(1)})$ is the sum of some homogeneous harmonic polynomials on R^n. The desired result is proved. $\square$

Theorem 2.2 *Let f be a function defined on $B_n\times B_n$ satisfying $\Delta_1 f=0$, $\Delta_2 f=0$. Then f can be written in the following form:*

$$f(x^{(1)},x^{(2)})=\sum_{m=0}^{\infty}\sum_{j+k=m} f_j^{(m)}(x^{(1)})g_k^{(m)}(x^{(2)}), \tag{4}$$

where $f_j^{(m)}(x^{(1)})$ is a homogeneous harmonic polynomial on R^n about $x^{(1)}$ of degree j and $g_k^{(m)}(x^{(2)})$ is a homogeneous harmonic polynomial on R^n about $x^{(2)}$ of degree k. For each fixed point $x^{(2)}\in B_n$, $\sum_{m=0}^{\infty}\sum_{j+k=m} f_j^{(m)}(x^{(1)})g_k^{(m)}(x^{(2)})$ as a series about $x^{(1)}$ converges uniformly on any compact subset of B_n.

Proof It follows from $B_{2n}\subset B_n\times B_n$ that f is harmonic on B_{2n}. According to Lemma 1.1, we can write f in the following form: $f(x^{(1)},x^{(2)})=\sum_{m=0}^{\infty}p_m(x^{(1)},x^{(2)}), x\in B_{2n}$, where $p_m(x^{(1)},x^{(2)})$ is a homogeneous harmonic polynomial on R^{2n} about x of degree m. Because $\Delta_1 f(x^{(1)},x^{(2)})=\sum_{m=0}^{\infty}\Delta_1 p_m(x^{(1)},x^{(2)})=0, \Delta_2 f(x^{(1)},x^{(2)})=\sum_{m=0}^{\infty}\Delta_2 p_m(x^{(1)},x^{(2)})=0$, $\Delta_1 p_m(x^{(1)},x^{(2)})=0, \Delta_2 p_m(x^{(1)},x^{(2)})=0$. Theorem 2.1 shows that we can write f in the following form:

$$f(x^{(1)},x^{(2)})=\sum_{m=0}^{\infty}\sum_{j+k=m} f_j^{(m)}(x^{(1)})g_k^{(m)}(x^{(2)}),\quad x\in B_{2n},$$

where $f_j^{(m)}(x^{(1)})$ is a homogeneous harmonic polynomial on R^n about $x^{(1)}$ of degree j and $g_k^{(m)}(x^{(2)})$ is a homogeneous harmonic polynomial on R^n about $x^{(2)}$ of degree k. For each fixed point $x^{(2)}\in B_n$, $f(x^{(1)},x^{(2)})$ as a function about $x^{(1)}$ is harmonic. For each fixed point $x^{(1)}\in B_n$, $f(x^{(1)},x^{(2)})$ as a function about $x^{(2)}$ is harmonic. Lemma 1.1 shows that formula (4) holds for any $x\in B_n\times B_n$. $\square$

Theorem 2.3 *Suppose that f is a function defined on the open set containing $\overline{B_n\times B_n}$ satisfying $\Delta_1 f=0,\Delta_2 f=0$. Then we can write f in the following form:*

$$f(x^{(1)},x^{(2)})=\sum_{m=0}^{\infty}\sum_{j+k=m} f_j^{(m)}(x^{(1)})g_k^{(m)}(x^{(2)}),$$

where $f_j^{(m)}(x^{(1)})$ and $g_k^{(m)}(x^{(2)})$ are the same as in Theorem 2.2. The series converges uniformly on any compact subset of $B_n\times B_n$.

Proof Suppose that $p(t,\xi)$ is the Poisson kernel for B_n. Then $p(t,\xi)=\frac{1-|t|^2}{|t-\xi|^n}$, and we can also write $p(t,\xi)$ in the following form: $p(t,\xi)=\sum_{m=0}^{\infty}Z_m(t,\xi)$, $t\in B_n$, $\xi\in S$, where $Z_m(t,\xi)$ is the zonal harmonic of degree m. The series converges absolutely and uniformly on $K\times S$ for every compact set $K\subset B_n$.

$$f(x^{(1)},x^{(2)})=\int_S f(\xi^{(1)},x^{(2)})p(x^{(1)},\xi^{(1)})\mathrm{d}\sigma(\xi^{(1)}))$$

$$= \int_S p(x^{(1)}, \xi^{(1)}) \mathrm{d}\sigma(\xi^{(1)}) \int_S f(\xi^{(1)}, \xi^{(2)}) p(x^{(2)}, \xi^{(2)}) \mathrm{d}\sigma(\xi^{(2)})$$
$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \int_{S\times S} f(\xi^{(1)}, \xi^{(2)}) Z_j(x^{(1)}, \xi^{(1)}) Z_k(x^{(2)}, \xi^{(2)}) \mathrm{d}\sigma(\xi^{(1)}) \mathrm{d}\sigma(\xi^{(2)})$$
$$= \sum_{m=0}^{\infty}\sum_{j+k=m} \int_{S\times S} f(\xi^{(1)}, \xi^{(2)}) Z_j(x^{(1)}, \xi^{(1)}) Z_k(x^{(2)}, \xi^{(2)}) \mathrm{d}\sigma(\xi^{(1)}) \mathrm{d}\sigma(\xi^{(2)}).$$

Theorem 2.2 shows that we can write f in the following form:

$$f(x^{(1)}, x^{(2)}) = \sum_{m=0}^{\infty}\sum_{j+k=m} f_j^{(m)}(x^{(1)}) g_k^{(m)}(x^{(2)}),$$

where $f_j^{(m)}(x^{(1)})$ is a homogeneous harmonic polynomial on R^n about $x^{(1)}$ of degree j and $g_k^{(m)}(x^{(2)})$ is a homogeneous harmonic polynomial on R^n about $x^{(2)}$ of degree k. Hence we have

$$\sum_{j+k=m} \int_{S\times S} f(\xi^{(1)}, \xi^{(2)}) Z_j(x^{(1)}, \xi^{(1)}) Z_k(x^{(2)}, \xi^{(2)}) \mathrm{d}\sigma(\xi^{(1)}) \mathrm{d}\sigma(\xi^{(2)})$$
$$= \sum_{j+k=m} f_j^{(m)}(x^{(1)}) g_k^{(m)}(x^{(2)}).$$

By the property of zonal harmonic, there exists constant $C > 0$ such that

$$|Z_j(x^{(1)}, \xi^{(1)}) Z_k(x^{(2)}, \xi^{(2)})| \le Cj^{n-2}|x^{(1)}|^j Ck^{n-2}|x^{(2)}|^k.$$

Let $M = \sup\{f(\xi^{(1)}, \xi^{(2)}) | (\xi^{(1)}, \xi^{(2)}) \in S \times S\}$.

$$\int_{S\times S} f(\xi^{(1)}, \xi^{(2)}) Z_j(x^{(1)}, \xi^{(1)}) Z_k(x^{(2)}, \xi^{(2)}) \mathrm{d}\sigma(\xi^{(1)}) \mathrm{d}\sigma(\xi^{(2)}) \le MCj^{n-2}|x^{(1)}|^j Ck^{n-2}|x^{(2)}|^k.$$

Thus the desired result is proved. □

Theorem 2.4 *Suppose that $\rho(x)$ is continuous on $\overline{B_n \times B_n}$, $\rho(x) > 0$, for each $x \in B_n \times B_n$. For each $f \in D^2(B_n \times B_n, \rho)$ and $\varepsilon > 0$, there exists a polynomial $u(x) = \sum_i p_i(x^{(1)}) q_i(x^{(2)})$, where $p_i(x^{(1)})$ is a homogeneous harmonic polynomial on R^n about $x^{(1)}$ and $q_i(x^{(2)})$ is a homogeneous harmonic polynomial on R^n about $x^{(2)}$, such that $(\int_{B_n\times B_n} \rho(y)|f(y) - u(y)|^2 \mathrm{d}V(y))^{1/2} < \varepsilon$.*

Proof There exists $M > 0$ such that $\rho(x) < M$, $x \in B_n \times B_n$. For each $f \in D^2(B_n \times B_n, \rho)$ and $\varepsilon > 0$, there exists a continuous function g on R^{2n} with compact support such that $(\int_{B_n\times B_n} |f(y) - g(y)|^2 \mathrm{d}V(y))^{1/2} < \varepsilon$. For each real number r satisfying $0 < r < 1$, we have

$$\Big[\int_{B_n\times B_n} \rho(y)|f(ry) - f(y)|^2 \mathrm{d}V(y)\Big]^{1/2}$$
$$\le \Big[\int_{B_n\times B_n} \rho(y)|f(ry) - g(y)|^2 \mathrm{d}V(y)\Big]^{1/2} + \Big[\int_{B_n\times B_n} \rho(y)|g(y) - f(y)|^2 \mathrm{d}V(y)\Big]^{1/2}$$
$$\le \Big[\int_{B_n\times B_n} \rho(y)|f(ry) - g(ry)|^2 \mathrm{d}V(y)\Big]^{1/2} + \Big[\int_{B_n\times B_n} \rho(y)|g(ry) - g(y)|^2 \mathrm{d}V(y)\Big]^{1/2} + \sqrt{M}\varepsilon$$
$$\le \frac{\sqrt{M}}{r^{2n}}\Big[\int_{B_n\times B_n} \rho(y)|f(y) - g(y)|^2 \mathrm{d}V(y)\Big]^{1/2} + \Big[\int_{B_n\times B_n} \rho(y)|g(ry) - g(y)|^2 \mathrm{d}V(y)\Big]^{1/2} + \sqrt{M}\varepsilon$$

$$\le \Big(\frac{\sqrt{M}}{r^{2n}}+\sqrt{M}\Big)\varepsilon+\Big[\int_{B_n\times B_n}\rho(y)|g(ry)-g(y)|^2\mathrm{d}V(y)\Big]^{1/2}.$$

There exists $\delta>0$, such that when $0<1-r<\delta$, $(\frac{\sqrt{M}}{r^n}+\sqrt{M})<3\sqrt{M}$, $\int_{B_n\times B_n}\rho(y)|g(ry)-g(y)|^2\mathrm{d}V(y)<\varepsilon$. Hence we have $(\int_{B_n\times B_n}\rho(y)|f(ry)-f(y)|^2\mathrm{d}V(y))^{1/2}<(1+3\sqrt{M})\varepsilon$, when $0<1-r<\delta$. Theorem 2.3 shows that $f(ry)$ can be approximated uniformly on $\overline{B_n\times B_n}$ by $u(x)$. The desired result is proved. □

Theorem 2.5 *Let* $\varphi(t_1,t_2)=\sum_{i=0}^{l}\sum_{j=0}^{m}a_{ij}t_1^it_2^j$, t_1, $t_2\in R$, $a_{ij}>0$ $(i=0,1,\dots,l;\ j=0,1,\dots,m)$ *and* $\rho(x)=\varphi(|x^{(1)}|,|x^{(2)}|)$. *Then*

$$T_{B_n\times B_n}(x,y)=\frac{1}{[nV(B_n)]^2}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}Z_{k_1}(x^{(1)},y^{(1)})Z_{k_2}(x^{(2)},y^{(2)})\Big[\sum_{i=0}^{l}\sum_{j=0}^{m}a_{ij}\frac{1}{(2k_1+n+i)(2k_2+n+j)}\Big]^{-1}. \tag{5}$$

Proof Denote by $F(x,y)$ the series on the right hand of formula (5). Let $p(x^{(1)})$ be a homogeneous harmonic polynomial on R^n about $x^{(1)}$ of degree k_1 and $q(x^{(2)})$ be a homogeneous harmonic polynomial on R^n about $x^{(2)}$ of degree k_2.

$$\begin{aligned}&\int_{B_n\times B_n}\rho(y)p(y^{(1)})q(y^{(2)})Z_{k_1}(x^{(1)},y^{(1)})Z_{k_2}(x^{(2)},y^{(2)})\mathrm{d}V(y)\\&=\int_{B_n}p(y^{(1)})Z_{k_1}(x^{(1)},y^{(1)})\mathrm{d}V(y^{(1)})\int_{B_n}\varphi(|y^{(1)}|,|y^{(2)}|)q(y^{(2)})Z_{k_2}(x^{(2)},y^{(2)})\mathrm{d}V(y^{(2)})\end{aligned}$$

$$\begin{aligned}&\int_{B_n}\varphi(|y^{(1)}|,|y^{(2)}|)q(y^{(2)})Z_{k_2}(x^{(2)},y^{(2)})\mathrm{d}V(y^{(2)})\\&=nV(B)\int_0^1r^{n-1}\int_S\varphi(|y^{(1)}|,|r\xi^{(2)}|)q(r\xi^{(2)})Z_{k_2}(x^{(2)},r\xi^{(2)})\mathrm{d}\sigma(\xi^{(2)})\mathrm{d}r\\&=nV(B)\int_0^1\varphi(|y^{(1)}|,r)r^{2k_2+n-1}\mathrm{d}r\int_Sq(\xi^{(2)})Z_{k_2}(x^{(2)},\xi^{(2)})\mathrm{d}\sigma(\xi^{(2)})\\&=nV(B)q(x^{(2)})\sum_{i=0}^{l}\sum_{j=0}^{m}a_{ij}\int_0^1|y^{(1)}|^ir^{2k_2+n+j-1}\mathrm{d}r\\&=nV(B)q(x^{(2)})\sum_{i=0}^{l}\sum_{j=0}^{m}a_{ij}|y^{(1)}|^i\frac{1}{2k_2+n+j}.\end{aligned}$$

Hence

$$\begin{aligned}&\int_{B_n\times B_n}\rho(y)p(y^{(1)})q(y^{(2)})Z_{k_1}(x^{(1)},y^{(1)})Z_{k_2}(x^{(2)},y^{(2)})\mathrm{d}V(y)\\&=nV(B)q(x^{(2)})\int_{B_n}p(y^{(1)})Z_{k_1}(x^{(1)},y^{(1)})\sum_{i=0}^{l}\sum_{j=0}^{m}a_{ij}|y^{(1)}|^i\frac{1}{2k_2+n+j}\mathrm{d}V(y^{(1)})\\&=[nV(B)]^2q(x^{(2)})\sum_{i=0}^{l}\sum_{j=0}^{m}a_{ij}\frac{1}{2k_2+n+j}\int_0^1r^{n-1}\int_Sp(r\xi^{(1)})Z_{k_1}(x^{(1)},r\xi^{(1)})|r\xi^{(1)}|^i\mathrm{d}\sigma(\xi^{(1)})\mathrm{d}r\end{aligned}$$

$$= [nV(B)]^2 q(x^{(2)}) p(x^{(1)}) \Big[\sum_{i=0}^{l}\sum_{j=0}^{m} a_{ij}\frac{1}{(2k_2+n+j)(2k_1+n+i)}\Big]$$

$$p(x^{(1)})q(x^{(2)}) = \frac{1}{[nV(B_n)]^2}\int_{B_n\times B_n}\rho(y)P(y^{(1)})q(y^{(2)})\times$$

$$Z_{k_1}(x^{(1)},y^{(1)})Z_{k_2}(x^{(2)},y^{(2)})\Big[\sum_{i=0}^{l}\sum_{j=0}^{m} a_{ij}\frac{1}{(2k_1+n+i)(2k_2+n+j)}\Big]^{-1}\mathrm{d}V(y).$$

There exists $C>0$ such that

$$\Big|Z_{k_1}(x^{(1)},y^{(1)})Z_{k_2}(x^{(2)},y^{(2)})\Big[\sum_{i=0}^{l}\sum_{j=0}^{m} a_{ij}\frac{1}{(2k_1+n+i)(2k_2+n+j)}\Big]^{-1}\Big|$$
$$\le \frac{(2k_2+n)(2k_1+n)}{a_{00}}|x^{(1)}|^{k_1}|x^{(2)}|^{k_2}|y^{(1)}|^{k_1}|y^{(2)}|^{k_2}Ck_1^{n-2}Ck_2^{n-2}$$
$$\le \frac{(2k_2+n)(2k_1+n)}{a_{00}}r_1^{k_1}r_2^{k_2}Ck_1^{n-2}Ck_2^{n-2},$$

where $0<r_1,\ r_2<1,\ |x^{(1)}|<r_1,\ |x^{(2)}|<r_2,\ y^{(1)},\ y^{(2)}\in B^n$. The above inequality shows that $\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} Z_{k_1}(x^{(1)},y^{(1)})Z_{k_2}(x^{(2)},y^{(2)})[\sum_{i=0}^{l}\sum_{j=0}^{m} a_{ij}\frac{1}{(2k_1+n+i)(2k_2+n+j)}]^{-1}$ converges uniformly and absolutely on $(r_1B_n\times r_2B_n)\times(B_n\times B_n)$. Thus we have

$$p(x^{(1)})q(x^{(2)}) = \int_{B_n\times B_n}\rho(y)P(y^{(1)})q(y^{(2)})F(x,y)\mathrm{d}V(y). \tag{6}$$

According to Theorem 2.4, for each $f\in D^2(B_n\times B_n,\rho)$ there exists $\{u_j\}$ which has the same form as u in Theorem 2.4 such that $u_j\to f$ in $D^2(B_n\times B_n,\rho)$. Because u_j satisfies formula (6), we must have $f(x)=\int_{B_n\times B_n}\rho(y)f(y)F(x,y)\mathrm{d}V(y)$. The desired result is proved. □

Theorem 2.6 *Suppose that* $\rho(x)=(1-|x^{(1)}|)^L(1-|x^{(2)}|)^N$, *where* L,N *are positive integers. Then there exist positive constants* $c_0,c_1,\dots,c_{L+1}$; $b_0,b_1,\dots,b_{N+1}$ *such that*

$$T_{B_n\times B_n}(x,y) = \Big[\frac{1}{nV(B)L!}\sum_{i=0}^{L+1}c_i\frac{\mathrm{d}^iP(tx^{(1)},ty^{(1)})}{\mathrm{d}t^i}\Big|_{t=1}\Big]\times$$
$$\Big[\frac{1}{nV(B)N!}\sum_{i=0}^{N+1}b_i\frac{\mathrm{d}^iP(tx^{(2)},ty^{(2)})}{\mathrm{d}t^i}\Big|_{t=1}\Big],$$

where

$$p(t,w)=\frac{1-|t|^2|w|^2}{(1-2\langle t,w\rangle+|t|^2|w|^2)^{n/2}},\quad t,w\in B_n.$$

Proof Let $p(x^{(1)})$ be a homogeneous harmonic polynomial on R^n about $x^{(1)}$ of degree k_1 and $q(x^{(2)})$ be a homogeneous harmonic polynomial on R^n about $x^{(2)}$ of degree k_2.

$$\int_{B_n\times B_n}\rho(y)p(y^{(1)})q(y^{(2)})Z_{k_1}(x^{(1)},y^{(1)})Z_{k_2}(x^{(2)},y^{(2)})\mathrm{d}V(y)$$
$$=\int_{B_n}(1-y^{(1)})^Lp(y^{(1)})Z_{k_1}(x^{(1)},y^{(1)})\mathrm{d}V(y^{(1)})\times$$
$$\int_{B_n}(1-y^{(2)})^Nq(y^{(2)})Z_{k_2}(x^{(2)},y^{(2)})\mathrm{d}V(y^{(2)})$$

$$
\begin{aligned}
&= \{nV(B_n)\int_0^1 r^{n-1}\int_S (1-r)^L p(r\xi^{(1)})Z_{k_1}(x^{(1)},r\xi^{(1)})\mathrm{d}\sigma(\xi^{(1)})\mathrm{d}r\}\times\\
&\quad \{nV(B_n)\int_0^1 r^{n-1}\int_S (1-r)^N q(r\xi^{(2)})Z_{k_2}(x^{(2)},r\xi^{(2)})\mathrm{d}\sigma(\xi^{(2)})\mathrm{d}r\}\\
&= [nV(B_n)]^2 p(x^{(1)})q(x^{(2)})\int_0^1 r^{n-1+2k_1}(1-r)^L\mathrm{d}r\int_0^1 r^{n-1+2k_2}(1-r)^N\mathrm{d}r\\
&= [nV(B_n)]^2 p(x^{(1)})q(x^{(2)})\frac{\Gamma(n+2k_1)\Gamma(L+1)}{\Gamma(n+2k_1+L+1)}\frac{\Gamma(n+2k_2)\Gamma(N+1)}{\Gamma(n+2k_2+N+1)}
\end{aligned}
$$

$$
\begin{aligned}
p(x^{(1)})q(x^{(2)}) &= \frac{1}{[nV(B_n)]^2}\int_{B_n\times B_n}\rho(y)p(y^{(1)})q(y^{(2)})\frac{\Gamma(n+2k_1+L+1)}{\Gamma(n+2k_1)\Gamma(L+1)}\times\\
&\quad \frac{\Gamma(n+2k_2+N+1)}{\Gamma(n+2k_2)\Gamma(N+1)}Z_{k_1}(x^{(1)},y^{(1)})Z_{k_2}(x^{(2)},y^{(2)})\mathrm{d}V(y).
\end{aligned}
$$

There exists $C>0$ such that

$$
\begin{aligned}
&\Big|\frac{\Gamma(n+2k_1+L+1)}{\Gamma(n+2k_1)\Gamma(L+1)}\frac{\Gamma(n+2k_2+N+1)}{\Gamma(n+2k_2)\Gamma(N+1)}Z_{k_1}(x^{(1)},y^{(1)})Z_{k_2}(x^{(2)},y^{(2)})\Big|\\
&\quad\le \frac{\Gamma(n+2k_1+L+1)}{\Gamma(n+2k_1)\Gamma(L+1)}\frac{\Gamma(n+2k_2+N+1)}{\Gamma(n+2k_2)\Gamma(N+1)}|x^{(1)}|^{k_1}|x^{(2)}|^{k_2}|y^{(1)}|^{k_1}|y^{(2)}|^{k_2}Ck_1^{n-2}Ck_2^{n-2}\\
&\quad\le \frac{\Gamma(n+2k_1+L+1)}{\Gamma(n+2k_1)\Gamma(L+1)}\frac{\Gamma(n+2k_2+N+1)}{\Gamma(n+2k_2)\Gamma(N+1)}r_1^{k_1}r_2^{k_2}Ck_1^{n-2}Ck_2^{n-2},
\end{aligned}
$$

where $0<r_1,\ r_2<1$, $|x^{(1)}|<r_1$, $|x^{(2)}|<r_2$, $y^{(1)},\ y^{(2)}\in B^n$. The above inequality shows that

$$
T^*_{B_n\times B_n}(x,y):=\sum_{k_1=0}^{\infty}\frac{\Gamma(n+2k_1+L+1)}{\Gamma(n+2k_1)\Gamma(L+1)}Z_{k_1}(x^{(1)},y^{(1)})\sum_{k_2=0}^{\infty}\frac{\Gamma(n+2k_2+N+1)}{\Gamma(n+2k_2)\Gamma(N+1)}Z_{k_2}(x^{(2)},y^{(2)})
$$

converges uniformly and absolutely on $(r_1B_n\times r_2B_n)\times(B_n\times B_n)$. Thus we have

$$
p(x^{(1)})q(x^{(2)})=\frac{1}{[nV(B_n)]^2}\int_{B_n\times B_n}\rho(y)p(y^{(1)})q(y^{(2)})T^*_{B_n\times B_n}(x,y). \tag{7}
$$

According to Theorem 2.4, for each $f\in D^2(B_n\times B_n,\rho)$ there exists $\{u_j\}$ which has the same form as u in Theorem 2.4 such that $u_j\to f$ in $D^2(B_n\times B_n,\rho)$. Because u_j satisfies formula (7), we must have

$$
f(x)=\frac{1}{[nV(B_n)]^2}\int_{B_n\times B_n}\rho(y)f(y)T^*_{B_n\times B_n}(x,y).
$$

Hence $T_{B_n\times B_n}(x,y)=T^*_{B_n\times B_n}(x,y)$.

Next, we shall obtain the explicit formula of $T_{B_n\times B_n}(x,y)$.

Suppose that $\psi(t)=(n+t+L)(n+t+L-1)\cdots(n+t)$. It is obvious that ψ is a polynomial about t of degree $L+1$. $\Gamma(n+2k_1+L+1)/\Gamma(n+2k_1)=\psi(2k_1)$. There exist $c_0,c_1,\dots,c_{L+1}$ such that $\psi(t)=c_0+c_1t+c_2t(t-1)+\cdots c_{L+1}t(t-1)(t-2)\cdots(t-L)$.

$$
\begin{aligned}
&\sum_{k_1=0}^{\infty}\frac{\Gamma(n+2k_1+L+1)}{\Gamma(n+2k_1)\Gamma(L+1)}Z_{k_1}(x^{(1)},y^{(1)})=\frac{1}{nV(B)L!}\sum_{k_1=0}^{\infty}\psi(2k_1)Z_{k_1}(x^{(1)},y^{(1)})\\
&\quad=\frac{1}{nV(B)L!}\sum_{k_1=0}^{\infty}\sum_{i=0}^{L+1}c_i2k_1(2k_1-1)\cdots(2k_1-i+1)Z_{k_1}(x^{(1)},y^{(1)})
\end{aligned}
$$

$$= \frac{1}{nV(B)L!}\sum_{i=0}^{L+1} c_i \sum_{k_1=0}^{\infty} 2k_1(2k_1-1)\cdots(2k_1-i+1)Z_{k_1}(x^{(1)},y^{(1)})$$

$$P(x^{(1)},y^{(1)}) = \sum_{k_1=0}^{\infty} Z_{k_1}(x^{(1)},y^{(1)}) = \frac{1-|x^{(1)}|^2|y^{(1)}|^2}{(1-2\langle x^{(1)},y^{(1)}\rangle+|x^{(1)}|^2|y^{(1)}|^2)^{n/2}}, x^{(1)},y^{(1)}\in B_n.$$

Let $G(t)=\sum_{k_1=0}^{\infty} Z_{k_1}(tx^{(1)},ty^{(1)})=\sum_{k_1=0}^{\infty} t^{2k_1}Z_{k_1}(x^{(1)},y^{(1)})$

$$\left.\frac{\mathrm{d}^iG(t)}{\mathrm{d}t^i}\right|_{t=1} = \sum_{k_1=0}^{\infty} 2k_1(2k_1-1)\cdots(2k_1-i+1)Z_{k_1}(x^{(1)},y^{(1)}) = \left.\frac{\mathrm{d}^iP(tx,ty))}{\mathrm{d}t^i}\right|_{t=1}.$$

$$\frac{1}{nV(B_n)}\sum_{k_1=0}^{\infty}\frac{\Gamma(n+2k_1+L+1)}{\Gamma(n+2k_1)\Gamma(L+1)}Z_{k_1}(x^{(1)},y^{(1)}) = \frac{1}{nV(B)L!}\sum_{i=0}^{L+1} c_i \left.\frac{\mathrm{d}^iP(tx,ty))}{\mathrm{d}t^i}\right|_{t=1}.$$

We can also prove that there exist $b_0,b_1,\ldots,b_{N+1}$ such that

$$\frac{1}{nV(B_n)}\sum_{k_2=0}^{\infty}\frac{\Gamma(n+2k_2+N+1)}{\Gamma(n+2k_1)\Gamma(N+1)}Z_{k_2}(x^{(2)},y^{(2)}) = \frac{1}{nV(B)N!}\sum_{i=0}^{N+1} b_i \left.\frac{\mathrm{d}^iP(tx,ty))}{\mathrm{d}t^i}\right|_{t=1}.$$

The desired result is proved. □

3. Metric matrix induced by reproducing kernel

Let H be a Hilbert space and consider the following equivalent relation between non-zero elements $\sim$: $h_1\sim h_2 \Longleftrightarrow h_1=ch_2$, where c is a complex number. The set of all equivalence classes forms projective space $P(H)$. This is a complete metric space with respect to the distance $\varphi([h_1],[h_2])=\mathrm{dist}([h_1]\cap S,[h_2]\cap S)$, where $S\subset H$ is the unit sphere.

$$\varphi^2([h_1],[h_2]) = \inf_{t_1,t_2} \parallel \frac{e^{it_1h_1}}{\parallel h_1\parallel} - \frac{e^{it_2h_2}}{\parallel h_2\parallel}\parallel^2$$

$$= \inf_{t_1,t_2}\Big[2-2\mathrm{Re}\frac{e^{i(t_1-t_2)\langle h_1,h_2\rangle}}{\parallel h_1\parallel\parallel h_2\parallel}\Big] = 2-2\frac{|\langle h_1,h_2\rangle|}{\parallel h_1\parallel\parallel h_2\parallel}.$$

$$\varphi([h_1],[h_2]) = \sqrt{2}\Big[1-\frac{|\langle h_1,h_2\rangle|}{\parallel h_1\parallel\parallel h_2\parallel}\Big]^{1/2}. \tag{8}$$

Remark The knowledge about $P(H)$ comes from [4].

Let Ω be a bounded domain in R^{2n}. Define $\tau:\Omega\to P(D^2(\Omega,\rho))$, $\tau(x)=[T_\Omega(x,)]$, $x\in\Omega$. It is obvious that τ is an injective map. Hence

$$\phi_\Omega(p,q) := \frac{1}{\sqrt{2}}\varphi(\tau(p),\tau(q)) = \Big[1-\frac{|T_\Omega(p,q)|}{\sqrt{T_\Omega(p,p)}\sqrt{T_\Omega(q,q)}}\Big]^{1/2}, \tag{9}$$

$\forall p,q\in\Omega$, is a distance function. Let G be bounded domain in C^n and d_G be the Bergman distance of G,

$$\phi_G(z,w) = \Big[1-\frac{|K_G(z,w)|}{\sqrt{K_G(w,w)}\sqrt{K_G(z,z)}}\Big]^{1/2}, \quad z,w\in G,$$

where $K_G(z,w)$ is the Bergman kernel of G . In [5], Pflug proved that there exists $c>0$ such that $\phi_G(z,w)\le cd_G(z,w)$. In this section, we will make use of $T_\Omega(x,y)$ to construct a metric matrix and prove a similar result.

Definition 3.1 *Let Ω be a bounded domain in R^{2n}, and $\gamma:[a,b]\to\Omega$ be a curve. Define $L_{\phi_\Omega}(\gamma)=\sup\sum_{i=0}^{n-1}\phi_\Omega(\gamma(t_i),\gamma(t_{i+1}))$, where $a=t_0<t_1<t_2<\cdots<t_n=b$ is a partition of $[a,b]$.*

Definition 3.2 *Let $p,q\in\Omega$. Define $\phi^*_\Omega(p,q)=\inf\{L_{\phi_\Omega}(\gamma)|\gamma:[0,1]\to D,\ \gamma$ is a piecewise smooth curve, $\gamma(0)=p,\gamma(1)=q\}$. $\phi^*_\Omega(p,q)\geq\phi_\Omega(p,q)$.*

Theorem 3.3 *Let*

$$G_{jk}(x,y)=\frac{1}{2T^2_\Omega(x,x)}\Big\{\frac{\partial^2T_\Omega(x,x)}{\partial x_j\partial x_k}T_\Omega(x,x)-2\frac{\partial T_\Omega(x,y)}{\partial x_j}\frac{\partial T_\Omega(x,y)}{\partial x_k}-2T_\Omega(x,y)\frac{\partial^2T_\Omega(x,y)}{\partial x_j\partial x_k}\Big\}.$$

$$G_\Omega(x,x)=\{G_{jk}(x,x)\}_{1\leq j,k\leq 2n}.$$

Then $G_\Omega(x,x)$ is a real semi-definite matrix.

Proof

$$\begin{aligned}\phi^2_\Omega(p,q)&=1-\frac{|T_\Omega(p,q)|}{\sqrt{T_\Omega(p,p)}\sqrt{T_\Omega(q,q)}}=\frac{\sqrt{T_\Omega(p,p)}\sqrt{T_\Omega(q,q)}-|T_\Omega(p,q)|}{\sqrt{T_\Omega(p,p)}\sqrt{T_\Omega(q,q)}}\\&=\frac{T_\Omega(p,p)T_\Omega(q,q)-T^2_\Omega(p,q)}{\sqrt{T_\Omega(p,p)}\sqrt{T_\Omega(q,q)}(\sqrt{T_\Omega(p,p)}\sqrt{T_\Omega(q,q)}+|T_\Omega(p,q)|)}.\end{aligned}$$

Let $\gamma:[0,1]\to\Omega$, be a piecewise smooth curve . For $\forall t,s\in[0,1]$,

$$\begin{aligned}\phi^2_\Omega(\gamma(t),\gamma(s))=&\frac{1}{\sqrt{T_\Omega(\gamma(t),\gamma(t))}\sqrt{T_\Omega(\gamma(s),\gamma(s))}}\\&\frac{T_\Omega(\gamma(t),\gamma(t))T_\Omega(\gamma(s),\gamma(s))-T^2_\Omega(\gamma(t),\gamma(s))}{(\sqrt{T_\Omega(\gamma(t),\gamma(t))}\sqrt{T_\Omega(\gamma(s),\gamma(s))}+|T_\Omega(\gamma(t),\gamma(s))|)}.\end{aligned}$$

Let $\Phi(h)=T_\Omega(\gamma(s)+h(\gamma(t)-\gamma(s)),\gamma(s)+h(\gamma(t)-\gamma(s)))T_\Omega(\gamma(s),\gamma(s))-T^2_\Omega(\gamma(s)+h(\gamma(t)-\gamma(s)),\gamma(s))$, $0\leq h\leq 1$. According to Taylor theorem, there exists $\xi\in[0,1]$ such that $\Phi(1)-\Phi(0)=\Phi'(0)+\frac{\Phi^{(2)}(\xi)}{2}$. Let $\gamma(t)=(\gamma_1(t),\gamma_2(t),\dots,\gamma_{2n}(t))$.

$$\begin{aligned}\Phi'(h)=&\sum_{j=1}^{2n}[\frac{\partial T_\Omega(x,x)}{\partial x_j}(\gamma_j(t)-\gamma_j(s))T_\Omega(\gamma(s),\gamma(s))-\\&2T_\Omega(x,\gamma(s))\frac{\partial T_\Omega(x,\gamma(s))}{\partial x_j}(\gamma_j(t)-\gamma_j(s))],\end{aligned}$$

where $x=\gamma(s)+h(\gamma(t)-\gamma(s))$. $\Phi'(0)=0$.

$$\begin{aligned}\Phi^{(2)}(h)=\sum_{j=1}^{2n}\sum_{k=1}^{2n}\{&\frac{\partial^2T_\Omega(x,x)}{\partial x_j\partial x_k}(\gamma_j(t)-\gamma_j(s))(\gamma_k(t)-\gamma_k(s))T_\Omega(\gamma(s),\gamma(s))-\\&2\frac{\partial T_\Omega(x,\gamma(s))}{\partial x_k}\frac{\partial T_\Omega(x,\gamma(s))}{\partial x_j}(\gamma_j(t)-\gamma_j(s))(\gamma_k(t)-\gamma_k(s))-\\&2T_\Omega(x,\gamma(s))\frac{\partial^2T_\Omega(x,\gamma(s))}{\partial x_j\partial x_k}(\gamma_j(t)-\gamma_j(s))(\gamma_k(t)-\gamma_k(s))\}.\end{aligned}$$

Suppose that $t > s$. By the above formula, we have

$$\begin{aligned}\lim_{t\to s}\frac{\phi^2_\Omega(\gamma(t),\gamma(s))}{(t-s)^2} &= \lim_{t\to s}\frac{1}{2T^2_\Omega(\gamma(s),\gamma(s))}\frac{\Phi(1)-\Phi(0)}{(t-s)^2}\\ &= \sum_{j=1}^{2n}\sum_{k=1}^{2n}\gamma'_j(s)G_{jk}(\gamma(s),\gamma(s))\gamma'_k(s).\end{aligned}$$

Hence

$$\lim_{t\to s}\frac{\phi_\Omega(\gamma(t),\gamma(s))}{t-s} = \{\gamma'(s)G_\Omega(\gamma(s),\gamma(s))[\gamma'(s)]'\}^{1/2}, \quad t > s.$$

It is obvious that $G_\Omega(x,x)$ is semi-definite, for $\forall x\in\Omega$. □

Definition 3.4 *Define*

$$\begin{aligned}B_\Omega(p,q) = \inf\Big\{&\int_0^1[\gamma'(s)G_\Omega(\gamma(s),\gamma(s))[\gamma'(s)]']^{1/2}\mathrm{d}s|\\ &\gamma:[0,1]\to\Omega,\ \text{where } \gamma \text{ is a piecewise smooth curve}, \gamma(0)=p,\gamma(1)=q\Big\},\end{aligned}$$

for all $p,q\in\Omega$. $B_\Omega(p,q)$ *is called pseudo-distance induced by the reproducing kernel.*

Theorem 3.5 $\phi_\Omega(p,q)\le B_\Omega(p,q)$. *If* $G_\Omega(x,x)$ *is positive definite, then that* $B_\Omega(p,q)$ *is a distance and that* (Ω,ϕ_Ω) *is complete implies that* (Ω,B_Ω) *is complete.*

Proof Let

$$F(t,s) = \begin{cases}\dfrac{\phi_\Omega(\gamma(t),\gamma(s))}{t-s} - \{\gamma'(s)G_\Omega(\gamma(s),\gamma(s))[\gamma'(s)]'\}^{1/2}, & t > s,\\ 0, & t = s,\end{cases}$$

where $(t,s)\in\{(t,s)|0\le s\le t\le 1\}$. Since $F(t,s)$ is uniformly continuous on $\{(t,s)|0\le s\le t\le 1\}$, we can easily get

$$L_{\phi_\Omega}(\gamma)\le\int_0^1\{\gamma'(s)G_\Omega(\gamma(s),\gamma(s))[\gamma'(s)]'\}^{1/2}\mathrm{d}s.$$

It follows from the above definitions that

$$\phi_\Omega(p,q)\le\phi^*_\Omega(p,q)\le B_\Omega(p,q).$$

The desired result is proved. □

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