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The Properties of Bianalytic Functions with Zero Arc at a Pole

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Abstract In this paper, the properties of bianalytic functions $w(z)=\bar{z}\phi_1(z)+\phi_2(z)$ with zero arc at the pole $z=0$ are discussed. Some conditions under which there exists an arc γ, an end of which is $z=0$, such that $w(z)=0$ for $\forall z\in\gamma\backslash\{0\}$ are given. Secondly, that the limit set of $w(z)$ is a circle or line as $z\to 0$ is proved in this case. Finally, two numerical examples are given to illustrate our results.

Keywords bianalytic functions with zero arc; pole; convergence to a circle or line; sufficient condition.

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1. Introduction

The bianalytic functions have been introduced to study physical fields with divergence or rotation at the present time, and their theories and applications have been discussed by many researchers. In [1–5], Wang and others studied some properties and boundary value problems of bianalytic functions. Mazalov presented an example of a nonconstant bianalytic function vanishing everywhere on a nowhere analytic boundary in [6] and proved that an arbitrary function continuous on a compact set X⊂C and holomorphic in the interior of X can be approximated by functions bianalytic in neighborhoods of X with arbitrary accuracy in [7]. In [8], Zheng and Zheng applied bianalytic functions to the problems of plane elastic mechanics. In this paper, the limit properties of bianalytic functions with zero arc at the pole $z=0$ are discussed and some conditions under which there exists an arc γ, an end of which is $z=0$, such that $w(z)=0$ for $\forall z\in\gamma\backslash\{0\}$ are given. Finally, that the limit set of $w(z)$ is a circle or line as $z\to 0$ is proved in this case.

2. Definition and main results

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Definition *Let $w(z) = \bar{z}\phi_1(z) + \phi_2(z)$ be a bianalytic function defined in a domain D and suppose that each analytic function $\phi_i(z)$ has a pole of order k_i $(i = 1, 2)$ at $z = a$ (We permit $k_i = 0$. $z = a$ is said to be a pole of order 0 of $\phi_i(z)$ if $z = a$ is a removable singular point of $\phi_i(z)$). If $0 \le k_i < \infty$ $(i = 1, 2)$ and $k_1^2 + k_2^2 \neq 0$, then $z = a$ is called a pole of order (k_1, k_2) of $w(z)$.*

Without loss of generality, we need only discuss the behavior of $w(z)$ near $z = 0$, in which case it follows $a = 0$. If $a \neq 0$, under the transformation $z(\zeta) = \zeta + a$, we can discuss the behavior of $w(z(\zeta)) = \bar{\zeta}\phi_1(z(\zeta)) + \bar{a}\phi_1(z(\zeta)) + \phi_2(z(\zeta))$ near $\zeta = 0$ similarly.

Obviously, if each analytic function $\phi_i(z)$ has a pole of order k_i $(i = 1, 2)$ at $z = 0$, then $\phi_i(z)$ has the Laurent series respresentation:

$$\phi_1(z) = \frac{a_{-k_1}}{z^{k_1}} + \frac{a_{-k_1+1}}{z^{k_1-1}} + \cdots + a_0 + a_1 z + \cdots = \frac{1}{z^{k_1}}\psi_1(z); \tag{1}$$

$$\phi_2(z) = \frac{b_{-k_2}}{z^{k_2}} + \frac{b_{-k_2+1}}{z^{k_2-1}} + \cdots + b_0 + b_1 z + \cdots = \frac{1}{z^{k_2}}\psi_2(z). \tag{2}$$

If $k_1 \ge 1$, then $a_{-k_1} \neq 0$ in (1). It is the same to k_2. Some notations in the equations (1) and (2) are used in this paper.

Lemma 1 *Suppose that the bianalytic function $w(z) = \bar{z}\phi_1(z) + \phi_2(z)$ has a pole of order (k_1, k_2) at $z = 0$. If there exists an arc γ, an end of which is $z = 0$, such that $w(z) = 0$ for $\forall z \in \gamma\backslash\{0\}$, then $k_1 = k_2 + 1$ and $|a_{-k_1}| = |b_{-k_2}|$.*

Proof The proof is by contradiction.

If $k_1 \le k_2$, it follows from the equations (1) and (2) that

$$w(z) = \frac{\bar{z}z^{k_2-k_1}\psi_1(z) + \psi_2(z)}{z^{k_2}}.$$

Since $w(z) = 0$ for all $z \in \gamma\backslash\{0\}$, we obtain

$$\bar{z}z^{k_2-k_1}\psi_1(z) + \psi_2(z) = 0 \quad \text{for all} \quad z \in \gamma\backslash\{0\}.$$

But

$$\lim_{z\to 0}(\bar{z}z^{k_2-k_1}\psi_1(z) + \psi_2(z)) = b_{-k_2} \neq 0,$$

we have a contradiction. □

If $k_1 > k_2 + 1$, another contradiction is provided in a similar way as shown before.

So it follows $k_1 = k_2 + 1$, in which case we have

$$w(z) = \frac{\bar{z}\psi_1(z) + z\psi_2(z)}{z^{k_1}}.$$

Thus

$$\bar{z}\psi_1(z) + z\psi_2(z) = 0 \quad \text{for all} \quad z \in \gamma\backslash\{0\}. \tag{3}$$

Therefore

$$\lim_{\substack{z\to 0\\ z\in\gamma}}\left|\frac{\bar{z}}{z}\right| = \lim_{\substack{z\to 0\\ z\in\gamma}}\left| -\frac{\psi_2(z)}{\psi_1(z)}\right| = \left| -\frac{b_{-k_2}}{a_{-k_1}}\right| = 1.$$

Lemma 2 *Suppose that the bianalytic function $w(z) = \bar{z}\phi_1(z) + \phi_2(z)$ has a pole of order (k_1, k_2) at $z = 0$. Then there exists a line segment γ, an end of which is $z = 0$, such that $w(z) = 0$ for $\forall z \in \gamma\backslash\{0\}$ if and only if $\frac{\phi_2(z)}{\phi_1(z)} = e^{i\theta_0}z$, where $\theta_0 = \arg\frac{b_{-k_2}}{a_{-k_1}}$.*

Proof "If" part is obvious so that we need only prove "only if" part.

By the conditions of "only if" part and the proof of Lemma 1, we know that $k_1 = k_2 + 1$ and the equation (3) are valid. Denote the inclination of the line segment γ by α. Thus there exists a deleted neighborhood $U^0(0)$ of $z = 0$ such that

$$\frac{\psi_2(z)}{\psi_1(z)} = -e^{-2i\alpha} \quad \text{for all} \;\; z \in \gamma \cap U^0(0).$$

Note that $\frac{\psi_2(z)}{\psi_1(z)}$ is analytic at $z = 0$, we know from the unique theorem of analytic function and the equation (3) that

$$\frac{\phi_2(z)}{\phi_1(z)} = \frac{\psi_2(z)}{\psi_1(z)}z = e^{i\theta_0}z.$$

Theorem *Suppose that*

1) The bianalytic function $w(z) = \bar{z}\phi_1(z) + \phi_2(z)$ has a pole of order (k_1, k_2) at $z = 0$;

2) There exists an arc γ, an end of which is $z = 0$, such that $w(z) = 0$ for $\forall z \in \gamma\backslash\{0\}$. Then:

(i) If $k_1 = 1$, the limit set of $w(z)$ is the circle $|w - b_0| = |a_{-1}|$ which goes through the origin as $z \to 0$;

(ii) If $k_1 > 1$, the limit set of $w(z)$ is the line $\frac{w+\bar{w}}{2}\sin\alpha - \frac{w-\bar{w}}{2i}\cos\alpha = 0$ as $z \to 0$, where $\alpha = \frac{1+k_1}{2}\theta_0 - \frac{k_1}{2}\pi + \arg a_{-k_1}$ and $\theta_0 = \arg\frac{b_{-k_2}}{a_{-k_1}}$.

Proof To prove part (i), since $k_1 = 1$, we know from Lemma 1 that $k_2 = 0$. So we obtain from the equations (1) and (2) that

$$w(z) = \bar{z}\phi_1(z) + \phi_2(z) = \frac{\bar{z}}{z}\psi_1(z) + \psi_2(z), \quad \lim_{z\to 0}\psi_1(z) = a_{-1} \;\; \text{and} \;\; \lim_{z\to 0}\psi_2(z) = b_0.$$

Set $z = re^{i\theta}$. We have

$$w(z) \to a_{-1}e^{-2i\theta} + b_0 \;\; \text{as} \;\; r \to 0. \tag{4}$$

By (4) and Lemma 1, part (i) holds.

To show part (ii), by Lemma 1 and the equations (1) and (2), we gain $k_1 = k_2 + 1$ and

$$\frac{\phi_2(z)}{\phi_1(z)} = e^{i\theta_0}z + c_1z^2 + c_2z^3 + c_3z^4 + \cdots,$$

where c_i is a constant for $i = 1, 2, \ldots$. Thus

$$w(z) = \frac{(\bar{z} + e^{i\theta_0}z + c_1z^2 + c_2z^3 + \cdots)}{z^{k_1}}\psi_1(z). \tag{5}$$

Therefore, we need only consider the limit behavior of $\frac{\bar{z}+e^{i\theta_0}z+c_1z^2+c_2z^3+\cdots}{z^{k_1}}$ to obtain that of $w(z)$ as $z \to 0$.

Set

$$f(z) = c_1z + c_2z^2 + c_3z^3 + \cdots \; (\text{where } z = re^{i\theta} = x + iy \text{ and } \theta \in [-\pi - \frac{\theta_0}{2}, \pi - \frac{\theta_0}{2}]),$$

$$U(r,\theta)=U(x,y)=\mathrm{Re}f(z), \quad \text{and} \quad V(r,\theta)=V(x,y)=\mathrm{Im}f(z).$$

We have

$$\frac{\bar{z}+e^{i\theta_0}z+c_1z^2+c_2z^3+\cdots}{z^{k_1}}=\frac{\dfrac{\bar{z}}{z}+e^{i\theta_0}+f(z)}{z^{k_1-1}}=\frac{e^{-2i\theta}+e^{i\theta_0}+U(r,\theta)+iV(r,\theta)}{r^{k_1-1}e^{i(k_1-1)\theta}}$$
$$=\frac{e^{i(\frac{\theta_0}{2}-k_1\theta)}\Big[2\cos(\theta+\frac{\theta_0}{2})+e^{i(\theta-\frac{\theta_0}{2})}(U(r,\theta)+iV(r,\theta))\Big]}{r^{k_1-1}}. \tag{6}$$

On the other hand, we know from the equation (3) that

$$\lim_{\substack{z\to 0\\ z\in\gamma}} e^{-2i\theta}=\lim_{\substack{z\to 0\\ z\in\gamma}}\frac{\bar{z}}{z}=\lim_{\substack{z\to 0\\ z\in\gamma}}\Big(-\frac{\psi_2(z)}{\psi_1(z)}\Big)=-\frac{b_{-k_2}}{a_{-k_1}}=-e^{i\theta_0}.$$

Hence there are two cases to be discussed:

Case 1 If the point z approaches the origin along the arc γ, it follows $\theta\to\frac{\pi}{2}-\frac{\theta_0}{2}$.

In this case, since the Condition 2) and the equations (5) and (6) are valid, it follows that, for each positive number r_n small enough, there exists an angle $\theta_n\in[-\frac{\theta_0}{2},\pi-\frac{\theta_0}{2}]$ such that

$$2\cos(\theta_n+\frac{\theta_0}{2})+e^{i(\theta_n-\frac{\theta_0}{2})}(U(r_n,\theta_n)+iV(r_n,\theta_n))=0. \tag{7}$$

Thus

$$\lim_{r_n\to 0}\theta_n=\frac{\pi}{2}-\frac{\theta_0}{2}. \tag{8}$$

Now set

$$g(r,\theta)=2\cos(\theta+\frac{\theta_0}{2})+e^{i(\theta-\frac{\theta_0}{2})}(U(r,\theta)+iV(r,\theta)), \tag{9}$$

$$p(r,\theta)=\mathrm{Re}g(r,\theta)=2\cos(\theta+\frac{\theta_0}{2})+U(r,\theta)\cos(\theta-\frac{\theta_0}{2})-V(r,\theta)\sin(\theta-\frac{\theta_0}{2}), \tag{10}$$

$$q(r,\theta)=\mathrm{Im}\,g(r,\theta)=U(r,\theta)\sin(\theta-\frac{\theta_0}{2})+V(r,\theta)\cos(\theta-\frac{\theta_0}{2}). \tag{11}$$

In view of the equations (10) and (11), we have

$$\begin{aligned}p_\theta(r,\theta)=&-2\sin(\theta+\frac{\theta_0}{2})-U(r,\theta)\sin(\theta-\frac{\theta_0}{2})-V(r,\theta)\cos(\theta-\frac{\theta_0}{2})+U_\theta(r,\theta)\cos(\theta-\frac{\theta_0}{2})-\\ &V_\theta(r,\theta)\sin(\theta-\frac{\theta_0}{2}),\\ q_\theta(r,\theta)=&U(r,\theta)\cos(\theta-\frac{\theta_0}{2})-V(r,\theta)\sin(\theta-\frac{\theta_0}{2})+U_\theta(r,\theta)\sin(\theta-\frac{\theta_0}{2})+V_\theta(r,\theta)\cos(\theta-\frac{\theta_0}{2}),\end{aligned}$$

where

$$U_\theta(r,\theta)=-U_x(x,y)r\sin\theta+U_y(x,y)r\cos\theta, \quad V_\theta(r,\theta)=-V_x(x,y)r\sin\theta+V_y(x,y)r\cos\theta.$$

Note that $w(z)=0$ for all $z\in\gamma\backslash\{0\}$, we obtain that, for each positive number r_n small enough, the image of the point set $\{r_ne^{i\theta}|\,\theta\in[-\frac{\theta_0}{2},\pi-\frac{\theta_0}{2}]\}$ under the transformation (9) is the continuous curve c_n which goes through the origin and whose real part approaches enough the closed interval $[-2,2]$ in the *poq*-plane. Therefore, given any $a\in(-\infty,0)\bigcup(0,+\infty)$, it follows that, for the above r_n, there are two angles $\theta'_n\in[-\frac{\theta_0}{2},\pi-\frac{\theta_0}{2}]$ and ϕ_n such that

$$2\cos(\theta'_n+\frac{\theta_0}{2})+e^{i(\theta'_n-\frac{\theta_0}{2})}(U(r_n,\theta'_n)+iV(r_n,\theta'_n))=ar_n^{k_1-1}e^{i\phi_n}. \tag{12}$$

Since $k_1 > 1$ and $\lim_{r\to 0} f(z) = 0$, we have

$$\lim_{r_n\to 0} \theta'_n = \frac{\pi}{2} - \frac{\theta_0}{2}. \tag{13}$$

According to the equations (7), (10) and (11), we have

$$p(r_n, \theta_n) = 0, \quad \text{and} \quad q(r_n, \theta_n) = 0.$$

And so we obtain from the mean value theorem for derivatives and the equations (10), (11) and (12) that

$$\tan\phi_n = \frac{q(r_n, \theta'_n)}{p(r_n, \theta'_n)} = \frac{q(r_n, \theta'_n) - q(r_n, \theta_n)}{p(r_n, \theta'_n) - p(r_n, \theta_n)} = \frac{q_\theta(r_n, \theta_1)}{p_\theta(r_n, \theta_2)},$$

where θ_i $(i = 1, 2)$ is between θ'_n and θ_n. It follows from the structures of the functions $p_\theta(r_n, \theta)$ and $q_\theta(r_n, \theta)$ and the equations (8) and (13) that

$$\lim_{r_n\to 0} \tan\phi_n = 0.$$

Hence we have $\phi_n \to 0$ as $r_n \to 0$. Furthermore, it follows from the equations (6), (12) and (13) that

$$w(r_n e^{i\theta'_n}) \to a\, a_{-k_1} e^{i(\frac{1+k_1}{2}\theta_0 - \frac{k_1}{2}\pi)} \quad \text{as} \quad r_n \to 0.$$

Obviously, we know from the conditions of Theorem and the equation (6) that $a = 0$ and $a = \infty$ are both the accumulation points of $w(z)$ as $z \to 0$. Therefore every point on the line $\frac{w+\bar{w}}{2}\sin\alpha - \frac{w-\bar{w}}{2i}\cos\alpha = 0$ $(\alpha = \frac{1+k_1}{2}\theta_0 - \frac{k_1}{2}\pi + \arg a_{-k_1})$ is the limit point of $w(z)$ as $z \to 0$.

Case 2 If the point z approaches the origin along the arc γ, it follows $\theta \to -\frac{\pi}{2} - \frac{\theta_0}{2}$.

In this case, the above result can be proved in a similar way as shown before.

Now let $w_0 \neq 0$ and suppose that $\arg w_0 \neq \frac{1+k_1}{2}\theta_0 - \frac{k_1}{2}\pi + \arg a_{-k_1} + k\pi$, where $k \in N$. If there exists a sequence $\{r_n^{(1)} e^{i\theta_n^{(1)}}\}$ $(n = 1, 2, 3, \ldots)$ such that $w(r_n^{(1)} e^{i\theta_n^{(1)}}) \to w_0$ as $r_n^{(1)} \to 0$, then from the equations (5) and (6), we similarly know either

$$\theta_n^{(1)} \to \frac{\arg w_0 - \arg a_{-k_1} - \frac{\theta_0}{2}}{-k_1}, \quad \text{or} \quad \theta_n^{(1)} \to \frac{\arg w_0 - \arg a_{-k_1} - \frac{\theta_0}{2} - \pi}{-k_1}.$$

Thus

$$\lim_{r_n^{(1)}\to 0} \cos\left(\frac{\theta_0}{2} + \theta_n^{(1)}\right) \neq 0.$$

Hence

$$w(r_n^{(1)} e^{i\theta_n^{(1)}}) \to \infty \quad \text{as} \quad r_n^{(1)} \to 0.$$

We have a contradiction and so part (ii) is valid.

This completes the proof. □

Remark If the Condition 2) is replaced by "there exists an arc γ, an interior of which is $z = 0$, such that $w(z) = 0$ for $\forall z \in \gamma\backslash\{0\}$", then the results of the theorem are also valid.

3. Examples

Example 1 Consider the bianalytic function $w(z) = \frac{\bar{z}+c_1z+c_2z^2+\cdots+c_iz^i+\cdots}{z^n}$, where c_i is a parameter for all $i \in N^*$ and n is a fixed number of N^*. Then:

1) According to Lemmas 1 and 2, there exists a line segment γ, an end of which is $z = 0$, such that $w(z) = 0$ for $\forall z \in \gamma\backslash\{0\}$ if and only if $|c_1| = 1$ and $c_i = 0$ $(i = 2, 3\ldots)$.

2) If $n = 1$, $c_1 = e^{i\frac{\pi}{4}}$, and $c_i = 0$ $(i = 2, 3\ldots)$, then it follows by Theorem that the limit set of $w(z)$ is the circle $|w - e^{i\frac{\pi}{4}}| = 1$ as $z \to 0$.

3) If $n = 2$, $c_1 = e^{i\frac{\pi}{4}}$, and $c_i = 0$ $(i = 2, 3\ldots)$, then it follows by Theorem that the limit set of $w(z)$ is the line $\frac{w+\bar{w}}{2}\sin(-\frac{5\pi}{8}) - \frac{w-\bar{w}}{2i}\cos(-\frac{5\pi}{8}) = 0$ as $z \to 0$.

Example 2 Given the complex parameter equation $\bar{z} = -1+z+\sqrt{1-4z}$ of the parabola $x = y^2$. We obtain from Theorem that the limit set of the bianalytic function $w(z) = \frac{\bar{z}+1-z-\sqrt{1-4z}}{z^2}$ is the line $w - \bar{w} = 0$ as $z \to 0$, where the branch $\sqrt{1-4z}$ takes value 1 at $z = 0$.

On the other hand, we denote the path $x = y^2 + cy^k$ by R, where c is a real constant and the parameter k is positive. In view of the given branch, we have

$$\sqrt{1-4z} = 1 - 2z - 2z^2 - 4z^3 + \cdots\ .$$

Thus

$$w(z) = \frac{\bar{z} + z + 2z^2 + 4z^3 + \cdots}{z^2},$$

and so we need only consider the limit behavior of $\frac{\bar{z}+z+2z^2}{z^2}$ as $z \to 0$. Obviously, it follows that

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in R}} \frac{\bar{z}+z+2z^2}{z^2} = \lim_{y\to 0} \frac{2cy^k + 4cy^{k+2} + 2c^2y^{2k} + 2y^4 + 4y^3i + 4cy^{k+1}i}{2cy^{k+2} + c^2y^{2k} - y^2 + y^4 + 2y^3i + 2cy^{k+1}i}.$$

Thus, by letting z approach the origin along the path R, we obtain from the L' Hospital rule some results: If $0 < k < 2$ and $c \neq 0$, then $w(z) \to \infty$; If $k = 2$, then $w(z) \to -2c$; If $k > 2$, then $w(z) \to 0$. Hence it illustrates our results.

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