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Two-Queue Polling Model with a Timer and a Randomly-Timed Gated Mechanism

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Abstract In this paper, we consider two-queue polling model with a Timer and a Randomly-Timed Gated (RTG) mechanism. At queue Q_1, we employ a Timer $T^{(1)}$: whenever the server polls queue Q_1 and finds it empty, it activates a Timer. If a customer arrives before the Timer expires, a busy period starts in accordance with exhaustive service discipline. However, if the Timer is shorter than the interarrival time to queue Q_1, the server does not wait any more and switches back to queue Q_2. At queue Q_2, we operate a RTG mechanism $T^{(2)}$, that is, whenever the server reenters queue Q_2, an exponential time $T^{(2)}$ is activated. If the server empties the queue before $T^{(2)}$, it immediately leaves for queue Q_1. Otherwise, the server completes all the work accumulated up to time $T^{(2)}$ and leaves. Under the assumption of Poisson arrivals, general service and switchover time distributions, we obtain probability generating function (PGF) of the queue lengths at polling instant and mean cycle length and Laplace Stieltjes transform (LST) of the workload.

Keywords polling; exhaustive; Timer; Randomly-Timed Gated.

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1. Introduction

We consider that a single server attends two queues, denoted Q_1 and Q_2, by alternating its service between them. Q_1 exercises an extra priority over Q_2 through a Timer and a RTG mechanism, operating as follows. Whenever the server polls Q_1 and finds it empty, it activates a Timer $T^{(1)}$. If a customer arrives before the Timer expires, a busy period starts in accordance with exhaustive service discipline. However, if the Timer is shorter than the interarrival time to Q_1, the server does not wait any more and switches back to Q_2. For RTG policy, whenever the server reenters Q_2, an exponential time $T^{(2)}$ is activated. If the server empties the queue before $T^{(2)}$, it immediately leaves for Q_1. Otherwise, the server completes all the work accumulated up to time $T^{(2)}$ and leaves.

In queueing models, a single server is shared by multiple users, performs several tasks, or attends several channels that are widely used to describe computer networks, manufacturing and production lines, and communication systems. In these models, service periods are usually

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controlled by some mechanisms. The most studied constraining mechanisms are Exhaustive, Gated and 1-limited. Under the Exhaustive regime, the server completes all the work in the system. Under the Gated regime, the server performs only the work present in the system at moment of its arrival. Under the 1-limited regime, at most one job is served. See literatures as in [1], [6], [7] and [8]. However, in various real-world system, the controlled mechanism is governed by Timers that limit the server's sojourn at a queue[2]. Eliazar and Yachiali[5] studied Randomly-Timed Gated queueing system and derived the joint transforms of two key characteristics: the length of a busy period starting with r jobs, and the number of jobs left behind at the end of such busy period. Boxma[3] considered two-queue polling models with a patient server. The service discipline in each queue is either exhaustive or 1-limited. In this paper, we extend the analysis of [5] to two-queue polling models with a patient server. This will extend the configuration of [3].

This paper is organized as follows: Section 2 presents the model description and preliminary results. In Section 3, we study queue lengths and derive multi-dimensional PGFs of the system's state at polling instant. In Section 4, we calculate mean cycle length. Applying level-crossing argument,we obtain the steady-state distribution of the workload in Section 5.

2. Model description and preliminary results

We consider a polling system consisting of two queues Q_1 and Q_2 with infinite buffer capacity each, attended by a single server that alternates between the queues. The model and the parameters of the system are as follows: For $i = 1, 2$

Arrival process $\{A^{(i)}(t), t \geq 0\}$: Poisson process with rate λ_i.

Service times $B^{(i)}$: the distribution $B^{(i)}(\cdot)$, and LST $\widetilde{B}^{(i)}(\cdot)$. Successive i.i.d service times are denoted by $B_n^{(i)}(\cdot), n = 1, 2, \ldots,$ $\rho_i = \lambda_i EB^{(i)}$ and $\rho = \rho_1 + \rho_2$. We assume henceforth that the stability condition $\rho < 1$.

Timer $T^{(1)}$: the distribution $T^{(1)}(\cdot)$, LST $\widetilde{T}^{(1)}(\cdot)$.

RTG mechanism $T^{(2)}$: exponential distribution random variable with mean μ^{-1}.

Switchover times $D^{(i)}$: switching from Q_i to the other queue , the distribution $D^{(i)}(\cdot)$, mean d_i, LST $\widetilde{D}^{(i)}(\cdot)$, $d = d_1 + d_2$.

Independence assumption: the arrival process $\{A^{(i)}(t), t \geq 0\}$, the service times $B^{(i)}$, Switchover times $D^{(i)}$, Timer $T^{(1)}$ and $T^{(2)}$ are mutually independent.

$\theta_r^{(i)}$: the length of a busy period generated by r awaiting jobs in standard $M/G/1$ queue, LST $\widetilde{\theta}_r^{(i)}(\cdot)$ and denote $\widetilde{\theta}^{(i)}(\cdot) = \widetilde{\theta}_1^{(i)}(\cdot)$.

$\Delta_r^{(i)}$: the length of a busy period initiated by r awaiting jobs in the model with RTG mechanism.

Y_r: queue size at the end of the busy period initiated by r awaiting jobs at Q_2.

Next, we give a proposition which considers the model with just one queue with RTG mechanism[5]. The multi-dimensional PGFs of the system's state at polling instant are computed. We will use it to analyze the state of Q_2. The notations have the same meaning as defined above.

Proposition 2.1 *Under RTG regime, $T \sim \mathrm{Exp}(\mu)$, arrival process $\{A(t), t \geq 0\}$ is a Poisson process with rate λ, service time $\{B_n\}$, LST $\widetilde{B}(\cdot)$. Δ_r is the length of a busy period initiated by r awaiting jobs and Y_r is queue size at the end of Δ_r. Define the joint transform*

$$\Phi_r(\omega, z) = E[e^{-\omega\Delta_r} z^{Y_r}].$$

Then the joint distribution of (Δ_r, Y_r) satisfies

$$\begin{aligned}(\Delta_r, Y_r) \stackrel{d}{=} &1_{(T>B_1)}(B_1 + \Delta_{r+A(B_1)-1}, Y_{r+A(B_1)-1}) + \\ &1_{(T\leq B_1)}(\sum_{i=1}^{r+A(T)} B_i, A(\sum_{i=1}^{r+A(T)} B_i) - A(T)),\end{aligned}$$

and $\Phi_r(\omega, z)$ satisfies

$$\begin{cases} \Phi_0(\omega, z) = 1, \\ \Phi_r(\omega, z) = \sum_{j=0}^{\infty} a_j(\omega)\Phi_{r+j-1}(\omega, z) + c(\omega, z)\xi(\omega, z)^r, \quad r \geq 1 \end{cases} \tag{1}$$

and the equation has a unique solution given by

$$\Phi_r(\omega, z) = \psi(\omega, z)[\xi(\omega, z)]^r + (1 - \psi(\omega, z))[\widetilde{\theta}(\omega + \mu)]^r, \tag{2}$$

where

$$\begin{aligned} a_j(\omega) &= E[\frac{(\lambda B)^j}{j!} e^{-(\omega+\mu+\lambda)B}], \\ \xi(\omega, z) &= \widetilde{B}(\omega + \lambda(1 - z)), \\ \psi(\omega, z) &= \frac{\mu}{\lambda(z - \widetilde{B}(\omega + \lambda(1 - z))) + \mu}, \\ c(\omega, z) &= \frac{\mu}{\widetilde{B}(\omega + \lambda(1 - z))} \frac{\widetilde{B}(\omega + \lambda(1 - z)) - \widetilde{B}(\omega + \lambda(1 - \widetilde{B}(\omega + \lambda(1 - z))) + \mu)}{\lambda(z - \widetilde{B}(\omega + \lambda(1 - z))) + \mu}. \end{aligned}$$

Let $\widetilde{\Delta}_r(\omega) = \Phi_r(\omega, 1)$ and $\widetilde{Y}_r(z) = \Phi_r(0, z)$. We also obtain

1) $\forall \omega \geq 0$, $\forall r = 0, 1, 2, \ldots$

$$\widetilde{\Delta}_r(\omega) = \psi(\omega)\widetilde{B}(\omega)^r + (1 - \psi(\omega))[\widetilde{\theta}(\omega + \mu)]^r, \tag{3}$$

where $\psi(\omega) = \psi(\omega, 1) = \frac{\mu}{\mu+\lambda(1-\widetilde{B}(\omega))}$.

2) $\forall z \geq 0$, $\forall r = 0, 1, 2, \ldots$

$$\widetilde{Y}_r(z) = \alpha(z)\widetilde{B}(\lambda(1 - z))^r + (1 - \alpha(z))\widetilde{\theta}(\mu)^r, \tag{4}$$

where $\alpha(z) = \psi(0, z) = \frac{\mu}{\mu+\lambda(z-\widetilde{B}(\lambda(1-z)))}$.

We can calculate state-dependent performance measures

$$E\Delta_r = -\frac{\mathrm{d}}{\mathrm{d}\omega}\widetilde{\Delta}_r(\omega)|_{\omega=0} = EB[r + \frac{\lambda}{\mu}(1 - \widetilde{\theta}(\mu)^r)], \tag{5}$$

$$EY_r = \frac{\mathrm{d}}{\mathrm{d}z}\widetilde{Y}_r(z)|_{z=1} = \lambda[rEB - (1 - \lambda EB)\frac{1 - \widetilde{\theta}(\mu)^r}{\mu}]. \tag{6}$$

3. Queue length

In this section, we construct the evolution equations for the queue lengths at polling instant (a moment when the server enters the system, following an intermission interval). We restrict ourselves to the stationary situation.

Let X_i^j $(i,j=1,2)$ be the number of jobs at Q_j when Q_i is polled, with joint PGF $F_i(z_1,z_2)=E[z_1^{X_i^1}z_2^{X_i^2}]$ $(i=1,2)$. Let IA be the interarrival time at Q_1 and $M=\min\{IA,T^{(1)}\}$. Then

$$\begin{aligned}
&P(IA\le T^{(1)})=1-\widetilde{T}^{(1)}(\lambda_1),\\
&EM=E\min\{IA,T^{(1)}\}=E[IA1(IA\le T^{(1)})]+E[T^{(1)}1(IA>T^{(1)})]\\
&\quad=E\int_0^{T^{(1)}}t\lambda_1e^{-\lambda_1t}\mathrm{d}t+E[T^{(1)}e^{-\lambda_1T^{(1)}}]=\frac{1}{\lambda_1}(1-\widetilde{T}^{(1)}(\lambda_1)),
\end{aligned}\tag{7}$$

where $1(A)$ denotes the indicator function of the event A.

Proposition 3.1 *At polling instant, the system's law of motion is given by*

$$X_1^1\stackrel{d}{=}X_2^1+A^{(1)}(\Delta_{X_2^2})+A^{(1)}(D^{(2)}),$$

$$X_1^2\stackrel{d}{=}A^{(2)}(D^{(2)})+Y_{X_2^2},\quad X_2^1\stackrel{d}{=}A^{(1)}(D^{(1)}),$$

$$X_2^2\stackrel{d}{=}\begin{cases}X_1^2+A^{(2)}(\theta_{X_1^1}^{(1)})+A^{(2)}(D^{(1)}), & \text{if } X_1^1>0,\\ X_1^2+A^{(2)}(M)+A^{(2)}(\theta_1^{(1)})1(IA\le T^{(1)})+A^{(2)}(D^{(1)}), & \text{if } X_1^1=0.\end{cases}\tag{8}$$

Remark X_1^1, the number of jobs at queue Q_1 when queue Q_1 is polled, is equal to the sum of X_2^1 and the new arrivals to queue Q_1 during the time $\Delta_{X_2^2}^{(2)}+D^{(2)}$ when the server is at queue Q_2.

X_1^2, the number of jobs at queue Q_2 when queue Q_1 is polled, equals the sum of $Y_{X_2^2}$ (remaining jobs at the end of $\Delta_{X_2^2}^{(2)}$) and the number of arrivals to queue Q_2 during the time $D^{(2)}$.

X_2^1, the number of jobs at queue Q_1 when queue Q_2 is polled, equals the number of arrivals to queue Q_1 during the time $D^{(1)}$ because queue Q_1 is exhaustively served.

Finally, since a Timer is employed at Q_1, we need consider whether the number of jobs at queue Q_1 are more than zero or not. X_2^2, the number of jobs at queue Q_2 when queue Q_2 is polled, if $X_1^1>0$, equals the sum of X_1^2 and the number of arrivals to queue Q_2 during the time $\theta_{X_1^1}^{(1)}+D^{(1)}$, and if $X_1^1=0$, equals the sum of X_1^2 and the number of arrivals to queue Q_2 during the time $T^{(1)}1(IA>T^{(1)})+(IA+\theta_1^{(1)})1(IA\le T^{(1)})+D^{(1)}$, which equals to $M+\theta_1^{(1)}1(IA\le T^{(1)})+D^{(1)}$.

Theorem 3.1 *For* $\forall z_1,z_2\ge0$, *the joint PGF* $F_i(z_1,z_2)$ $(i=1,2)$ *satisfies the following equations.*

$$F_1(z_1,z_2)=\widetilde{D}^{(2)}(\omega_1+\omega_2)[\psi(\omega_1,z_2)F_2(z_1,\xi(\omega_1,z_2))+(1-\psi(\omega_1,z_2))F_2(z_1,\widetilde{\theta}^{(2)}(\omega_1+\mu))],\tag{9}$$

$$F_2(z_1,z_2)=\widetilde{D}^{(1)}(\omega_1+\omega_2)[F_1(\widetilde{\theta}^{(1)}(\omega_2),z_2)+F_1(0,z_2)(h(z_2)-1)],\tag{10}$$

where $\omega_i=\lambda_i(1-z_i)$, $h(z_2)=E[z_2^{A^{(2)}(M)+A^{(2)}(\theta_1^{(1)})1(IA\le T^{(1)})}]$.

Proof For $i=1,2$, write $\omega_i=\lambda_i(1-z_i)$

$$E[z_1^{A^{(1)}(D^{(i)})}z_2^{A^{(2)}(D^{(i)})}]=\widetilde{D}^{(i)}(\lambda_1(1-z_1)+\lambda_2(1-z_2))=\widetilde{D}^{(i)}(\omega_1+\omega_2).\tag{11}$$

Applying Proposition 2.1 and independence assumption, we have

$$\begin{aligned} E[z_1^{A^{(1)}(\Delta_r^{(2)})} z_2^{Y_r}] &= \int\int E(z_1^{A^{(1)}(t)}) z_2^x \Phi_r(t,x)\mathrm{d}t\mathrm{d}x \\ &= \int\int \sum_{k=0}^{\infty} \frac{(\lambda_1 t)^k}{k!} e^{-\lambda_1 t} z_1^k z_2^x \Phi_r(t,x)\mathrm{d}t\mathrm{d}x \\ &= \int\int e^{-\lambda_1 t(1-z_1)} z_2^x \Phi_r(t,x)\mathrm{d}t\mathrm{d}x \\ &= \Phi_r(\omega_1, z_2), \end{aligned} \tag{12}$$

where $\Phi_r(t,x)$ is the joint distribution of (Δ_r, Y_r). Thus, we obtain

$$E[z_1^{A^{(1)}(\Delta_{X_2^2}^{(2)})} z_2^{Y_{X_2^2}} | X_2^2] = \Phi_{X_2^2}(\omega_1, z_2). \tag{13}$$

Applying relationship (2), we have

$$\begin{aligned} & E[z_1^{X_2^1 + A^{(1)}(\Delta_{X_2^2})} z_2^{Y_{X_2^2}}] \\ &= E[z_1^{X_2^1 + A^{(1)}(\Delta_{X_2^2})} 1(X_2^2 > 0) z_2^{Y_{X_2^2}}] + E[z_1^{X_2^1} 1(X_2^2 = 0)] \\ &= E[z_1^{X_2^1} 1(X_2^2 > 0) E[z_1^{A^{(1)}(\Delta_{X_2^2}^{(2)})} z_2^{Y_{X_2^2}} | X_2^2]] + E[z_1^{X_2^1} 1(X_2^2 = 0)] \\ &= E[z_1^{X_2^1} 1(X_2^2 > 0) \Phi_{X_2^2}(\omega_1, z_2)] + E[z_1^{X_2^1} 1(X_2^2 = 0)] \\ &= E[z_1^{X_2^1} (\psi(\omega_1, z_2) \xi(\omega_1, z_2)^{X_2^2} + (1 - \psi(\omega_1, z_2)) \widetilde{\theta}^{(2)}(\omega_1 + \mu)^{X_2^2})] \\ &= \psi(\omega_1, z_2) E[z_1^{X_2^1} \xi(\omega_1, z_2)^{X_2^2}] + (1 - \psi(\omega_1, z_2)) E[z_1^{X_2^1} \widetilde{\theta}^{(2)}(\omega_1 + \mu)^{X_2^2}] \\ &= \psi(\omega_1, z_2) F_2(z_1, \xi(\omega_1, z_2)) + (1 - \psi(\omega_1, z_2)) F_2(z_1, \widetilde{\theta}^{(2)}(\omega_1 + \mu)). \end{aligned} \tag{14}$$

Thus

$$\begin{aligned} F_1(z_1, z_2) &= E[z_1^{X_1^1} z_2^{X_1^2}] \\ &= E[z_1^{X_2^1 + A^{(1)}(\Delta_{X_2^2}) + A^{(1)}(D^{(2)})} z_2^{A^{(2)}(D^{(2)}) + Y_{X_2^2}}] \\ &= E[z_1^{A^{(1)}(D^{(2)})} z_2^{A^{(2)}(D^{(2)})}] E[z_1^{X_2^1 + A^{(1)}(\Delta_{X_2^2})} z_2^{Y_{X_2^2}}] \\ &= \widetilde{D}^{(2)}(\omega_1 + \omega_2)[\psi(\omega_1, z_2) F_2(z_1, \xi(\omega_1, z_2)) + \\ &\quad (1 - \psi(\omega_1, z_2)) F_2(z_1, \widetilde{\theta}^{(2)}(\omega_1 + \mu))]. \end{aligned} \tag{15}$$

On the other hand,

$$\begin{aligned} E[z_2^{A^{(2)}(\theta_r^{(1)})}] &= \int \sum_{k=0}^{\infty} \frac{(\lambda_2 t)^k}{k!} e^{-\lambda_2 t} z_2^k \mathrm{d}P(\theta_r^{(1)} \le t) \\ &= \int e^{-\lambda_2 t(1-z_2)} \mathrm{d}P(\theta_r^{(1)} \le t) \\ &= \widetilde{\theta}_r^{(1)}(\omega_2) = (\widetilde{\theta}^{(1)}(\omega_2))^r, \end{aligned} \tag{16}$$

$$E[z_2^{A^{(2)}(\theta_{X_1^1}^{(1)})} | X_1^1] = (\widetilde{\theta}^{(1)}(\omega_2))^{X_1^1}, \tag{17}$$

$$E[z_2^{X_1^2} z_2^{A^{(2)}(\theta^{(1)}_{X_1^1})}] = \sum_{i,j=0}^{\infty} E z_2^{A^{(2)}(\theta_i^{(1)})} z_2^j f(i,j)$$
$$= \sum_{i,j=0}^{\infty} (\widetilde{\theta}^{(1)}(\omega_2))^i z_2^j f(i,j) = F_1(\widetilde{\theta}^{(1)}(\omega_2), z_2), \tag{18}$$

where $f(i,j)$ is the joint density function of (X_1^1, X_1^2). We can obtain

$$\begin{aligned}
F_2(z_1,z_2) &= E[z_1^{X_2^1} z_2^{X_2^2}] \\
&= E\Big[z_1^{A^{(1)}(D^{(1)})} z_2^{X_1^2 + A^{(2)}(\theta^{(1)}_{X_1^1})1(X_1^1>0) + (A^{(2)}(M) + A^{(2)}(\theta_1^{(1)})1(IA\le T^{(1)}))1(X_1^1=0) + A^{(2)}(D^{(1)})}\Big] \\
&= E[z_1^{A^{(1)}(D^{(1)})} z_2^{A^{(2)}(D^{(1)})}](E[z_2^{X_1^2} z_2^{A^{(2)}(\theta^{(1)}_{X_1^1})} 1(X_1^1>0)] + \\
&\quad E[z_2^{X_1^2} z_2^{A^{(2)}(M) + A^{(2)}(\theta_1^{(1)})1(IA\le T^{(1)})} 1(X_1^1=0)]) \\
&= \widetilde{D}^{(1)}(\omega_1+\omega_2)(E[z_2^{X_1^2} z_2^{A^{(2)}(\theta^{(1)}_{X_1^1})} 1(X_1^1>0)] + \\
&\quad E[z_2^{X_1^2} 1(X_1^1=0)] + E[z_2^{X_1^2} 1(X_1^1=0)](h(z_2)-1)) \\
&= \widetilde{D}^{(1)}(\omega_1+\omega_2)(E[z_2^{X_1^2} z_2^{A^{(2)}(\theta^{(1)}_{X_1^1})}] + E[z_2^{X_1^2} 1(X_1^1=0)](h(z_2)-1)) \\
&= \widetilde{D}^{(1)}(\omega_1+\omega_2)[F_1(\widetilde{\theta}^{(1)}(\omega_2), z_2) + F_1(0,z_2)(h(z_2)-1)],
\end{aligned} \tag{19}$$

where $h(z_2) = E[z_2^{A^{(2)}(M) + A^{(2)}(\theta_1^1)1(IA\le T^{(1)})}]$. □

4. Cycle lengths

Clearly, at the polling instant, the PGFs of the number of jobs at queue Q_1 and Q_2 are given by

$$E[z^{X_1^1}] = F_1(z,1) = \widehat{X}_1^1(z), E[z^{X_2^2}] = F_2(1,z) = \widehat{X}_2^2(z).$$

Moreover, from Proposition 3.1, one observes that

$$X_1^1 = A^{(1)}(D^{(1)}) + A^{(1)}(\Delta^{(2)}_{X_2^2}) + A^{(1)}(D^{(2)}),$$

$$X_2^2 = \begin{cases} A^{(2)}(D^{(2)}) + Y_{X_2^2} + A^{(2)}(\theta^{(1)}_{X_1^1}) + A^{(2)}(D^{(1)}), & \text{if } X_1^1 > 0, \\ A^{(2)}(D^{(2)}) + Y_{X_2^2} + A^{(2)}(M), & \\ +A^{(2)}(\theta_1^{(1)})1(IA \le T^{(1)}) + A^{(2)}(D^{(1)}), & \text{if } X_1^1 = 0. \end{cases} \tag{20}$$

Define the cycle time length C as the time length between two successive polling instants by the server of queue Q_1. Then

$$C = \Delta^{(1)} + M1(X_1^1 = 0) + D^{(1)} + \Delta^{(2)} + D^{(2)}. \tag{21}$$

Where

$$\Delta^{(1)} = \begin{cases} \theta^{(1)}_{X_1^1}, & \text{if } X_1^1 > 0, \\ \theta_1^{(1)} 1(IA \le T^{(1)}), & \text{if } X_1^1 = 0, \end{cases} \tag{22}$$

and $\Delta^{(2)}$ denotes the busy period at Q_2 during a cycle. In the following, we derive the mean of the cycle time length C. Firstly,

$$E\Delta^{(1)} = E[\theta^{(1)}_{X_1^1} 1(X_1^1 > 0)] + E[\theta^{(1)}_1 1(IA \le T^{(1)}) 1(X_1^1 = 0)]$$
$$= E\theta^{(1)}_{X_1^1} + E[\theta^{(1)}_1 1(IA \le T^{(1)})] P(X_1^1 = 0). \quad (23)$$

Since $\theta^{(1)}_{X_1^1} = 0$ when $X_1^1 = 0$, and since queue Q_1 is exhaustive, we have

$$E\theta^{(1)}_{X_1^1} = \frac{EX_1^1 EB^{(1)}}{1-\rho_1}, \quad (24)$$

and

$$EX_1^1 = EA^{(1)}(D^{(1)} + D^{(2)} + \Delta^{(2)}) = \lambda_1(d + E\Delta^{(2)}), \quad (25)$$

$$E\theta^{(1)}_{X_1^1} = \frac{\rho_1}{1-\rho_1}(d + E\Delta^{(2)}), \quad (26)$$

$$E[\theta^{(1)}_1 1(IA \le T^{(1)})] = \frac{EB^{(1)}}{1-\rho_1}(1 - \widetilde{T}^{(1)}(\lambda_1)) = \frac{\rho_1}{1-\rho_1} EM. \quad (27)$$

Therefore, it follows from (21), (23), (26) and (27) that

$$E\Delta^{(1)} = \frac{\rho_1}{1-\rho_1}(d + E\Delta^{(2)}) + \frac{\rho_1}{1-\rho_1} EMP(X_1^1 = 0)$$
$$= \frac{\rho_1}{1-\rho_1}(EC - E\Delta^{(1)}),$$
$$E\Delta^{(1)} = \rho_1 EC. \quad (28)$$

From Proposition 2.1 (since queue Q_2 is RTG)

$$E\Delta^{(2)} = E[E[\Delta^{(2)}|X_2^2]] = EX_2^2 EB^{(2)} + \rho_2\gamma, \quad (29)$$

where

$$\gamma = \frac{1 - \widehat{X}_2^2(\widetilde{\theta}^{(2)}(\mu))}{\mu}.$$

By the Proposition 3.1, we have

$$EX_2^2 = E[A^{(2)}(D^{(1)} + D^{(2)}) + Y_{X_2^2} + A^{(2)}(\theta^{(1)}_{X_1^1}) 1(X_1^1 > 0) +$$
$$(A^{(2)}(M) + A^{(2)}(\theta^{(1)}_1) 1(IA \le T^{(1)})) 1(X_1^1 = 0)]$$
$$= \lambda_2[d + EX_2^2 EB^{(2)} - (1-\rho_2)\gamma + E\theta^{(1)}_{X_1^1} P(X_1^1 > 0) +$$
$$EMP(X_1^1 = 0) + \frac{\rho_1}{1-\rho_1} EMP(X_1^1 = 0)],$$

where

$$E[A^{(2)}(\theta^{(1)}_{X_1^1}) 1(X_1^1 > 0)] = E[E[A^{(2)}(\theta^{(1)}_{X_1^1}) 1(X_1^1 > 0)|X_1^1]]$$
$$= \lambda_2 E\theta^{(1)}_{X_1^1} P(X_1^1 > 0).$$

Thus

$$EX_2^2 = \frac{\lambda_2}{1-\rho_2}(d + E\Delta^{(1)} + EMP(X_1^1 = 0)) - \lambda_2\gamma. \quad (30)$$

From (29) and (30), we have

$$E\Delta^{(2)} = \frac{\rho_2}{1-\rho_2}(d + E\Delta^{(1)} + EMP(X_1^1 = 0))$$

$$= \frac{\rho_2}{1-\rho_2}(EC - E\Delta^{(2)}),$$

$$E\Delta^{(2)} = \rho_2 EC. \tag{31}$$

It follows from (21), (28) and (31) that

$$\begin{aligned} EC &= ED^{(1)} + ED^{(2)} + E\Delta^{(1)} + E\Delta^{(2)} + EMP(X_1^1 = 0) \\ &= d + \rho_1 EC + \rho_2 EC + EMP(X_1^1 = 0) \\ &= d + \rho EC + EMP(X_1^1 = 0). \end{aligned}$$

Hence

$$EC = \frac{d + EMP(X_1^1 = 0)}{(1-\rho)}.$$

From (28) and (31), we can see that ρ_1 and ρ_2 denote the server's busy fraction at Q_1 and Q_2 during a cycle, respectively. Therefore, we have the following theorem.

Theorem 4.1 *1) The mean cycle length* $EC = (d + EMP(X_1^1 = 0))/(1-\rho)$, *2) The server's busy fraction* $P_{\text{busy}} = (E\Delta^{(1)} + E\Delta^{(2)})/EC = \rho$.

By the law of motion (8) and the expression for $E\Delta^{(1)}, E\Delta^{(2)}$, we can easily get the following results.

Theorem 4.2 *The expectations of* $EX_i^j (i,j = 1,2)$ *are*

$$\begin{aligned} EX_1^1 &= \lambda_1 (d + \frac{\rho_2}{1-\rho}(d + EMP(X_1^1 = 0))), \\ EX_1^2 &= \lambda_2 d_2 + \frac{\lambda_2 \rho_2}{1-\rho}(d + EMP(X_1^1 = 0)) - \lambda_2 \gamma, \\ EX_2^1 &= \lambda_1 d_1, \\ EX_2^2 &= \frac{\lambda_2}{1-\rho}(d + EMP(X_1^1 = 0)) - \lambda_2 \gamma. \end{aligned}$$

5. Workload

In this section, we only consider zero switchover time (non-zero switchover time is similar). This model can also be viewed as $M/G/1$ with server intermissions, whose arrival intensity is $\lambda = \lambda_1 + \lambda_2$, service time $B = \frac{\lambda_1}{\lambda_1+\lambda_2}B^{(1)} + \frac{\lambda_2}{\lambda_1+\lambda_2}B^{(2)}$, transfic intensity $\rho = \rho_1 + \rho_2$, and intermission time M. Then let V be workload process with the distribution function $V(\cdot)$, the density function $v(\cdot)$ and LST $\widetilde{V}(\cdot)$. Applying Level-crossing theory[4], we know that the rate of upcrossing the level x is equal to the rate of downcrossing the level x. Denote $v_0(x) = \frac{\mathrm{d}}{\mathrm{d}x}P\ (V \le x, \text{Timer on})$. Since a downcrossing is only possible when the Timer is off, the rate of downcrossing the level x is $v(x) - v_0(x)$. Therefore, we have

$$v(x) - v_0(x) = \lambda \int_{0-}^{x} \overline{B}(x-y)v(y)\mathrm{d}y. \tag{32}$$

Then there is the stochastic decomposition

$$V(x) = V_{M/G/1} * V_0(x),$$

where $\overline{B}(x) = 1 - B(x)$, and $V_{M/G/1}$is the workload of the standard $M/G/1$ model.

Taking LST in the both sides of (32), we have

$$\widetilde{V}(s) - E[e^{-sV}1(\text{Timer on})] = \rho\beta(s)\widetilde{V}(s),$$

where $\beta(s) = \frac{1}{EB}\int_0^\infty e^{-sx}\overline{B}(x)\mathrm{d}x$. From Theorem 4.1, $P(\text{Timer on}) = 1-\rho$, thus

$$\widetilde{V}(s) = \frac{1-\rho}{1-\rho\beta(s)}E[e^{-sV}|\text{Timer on}]. \tag{33}$$

Since $\{V|_{\text{Timer on}}(t), t \geq 0\}$ is a regeneration process, we obtain

$$\begin{aligned}
E[e^{-sV|_{\text{Timer on}}}] &= \frac{1}{EM}E\int_0^M e^{-sV|_{\text{Timer on}}(t)}\mathrm{d}t \\
&= \frac{1}{EM}E\int_0^\infty 1(M>t)e^{-s(\lambda_2^{-1}Y_{X_2^2}+\sum_{i=0}^{A^{(2)}(t)}B_i^{(2)})}\mathrm{d}t \\
&= \frac{Ee^{-s\lambda_2^{-1}Y_{X_2^2}}}{EM}E\int_0^\infty 1(M>t)e^{-s\sum_{i=0}^{A^{(2)}(t)}B_i^{(2)}}\mathrm{d}t \\
&= \frac{Ee^{-s\lambda_2^{-1}Y_{X_2^2}}}{EM}\int_0^\infty P(M>t)e^{-\lambda_2(1-\widetilde{B}^{(2)}(s))t}\mathrm{d}t \\
&= \frac{Ee^{-s\lambda_2^{-1}Y_{X_2^2}}}{EM}\frac{1-\widetilde{M}(\lambda_2(1-\widetilde{B}^{(2)}(s))}{\lambda_2(1-\widetilde{B}^{(2)}(s))}.
\end{aligned}$$

Therefore, we have

$$E[V|_{\text{Timer on}}] = \lambda_2^{-1}EY_{X_2^2} + \frac{\rho_2 EM^2}{2EM}.$$

From (33), we can obtain the mean workload

$$\begin{aligned}
EV &= EV_{M/G/1} + E[V|_{\text{Timer on}}] \\
&= \frac{\sum_{i=1}^{2}\lambda_i EB^{(i)2}}{2(1-\rho)} + \frac{\rho_2}{1-\rho}EMP(X_1^1=0) - \gamma + \frac{\rho_2 EM^2}{2EM}.
\end{aligned}$$

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