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Sequences of Lower Bounds for the Perron Root of a Nonnegative Irreducible Matrix

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Abstract Estimate bounds for the Perron root of a nonnegative matrix are important in theory of nonnegative matrices. It is more practical when the bounds are expressed as an easily calculated function in elements of matrices. For the Perron root of nonnegative irreducible matrices, three sequences of lower bounds are presented by means of constructing shifted matrices, whose convergence is studied. The comparisons of the sequences with known ones are supplemented with a numerical example.

Keywords nonnegative irreducible matrix; shifted matrix; Perron root; lower bound.

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1. Introduction

Let $A = (a_{ij})$ be a nonnegative $n \times n$ matrix, denoted by $A \geq 0$. By the Perron-Frobenius theorem, A possesses a nonnegative eigenvalue $\rho(A)$ (the spectral radius of A), called the Perron root of A.

Now we introduce some general definitions which will be referred to in the next section.

Definition 1.1[1] *For $n \geq 2$, an $n \times n$ complex matrix A is reducible if there exists an $n \times n$ permutation matrix P such that*

$$PAP^{\mathrm{T}} = \begin{pmatrix} B & C \\ O & D \end{pmatrix},$$

where B is an $r \times r$ submatrix and D is an $(n-r) \times (n-r)$ submatrix, where $1 \leq r < n$. If no such permutation matrix exists, then A is irreducible. If A is a 1×1 complex matrix, then A is irreducible if its single entry is nonzero, and reducible otherwise.

Definition 1.2[2] *Let $A \geq 0$ be an irreducible $n \times n$ matrix, and let k be the number of eigenvalues of A of modulus $\rho(A)$. If $k = 1$, then A is primitive. If $k > 1$, then A is cyclic of index k.*

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Let $A \geq 0$ be an irreducible $n \times n$ matrix and $a = \min_i a_{ii}$. The matrix $B := A - aI$ is called the shifted matrix of A and is again nonnegative irreducible.

Bounds for $\rho(A)$ have been obtained by several authors. Szulc[3] showed that

$$\rho(A) \geq a + \begin{cases} K & n = 2, 4, 6, \ldots \\ \max\left\{K, \sqrt{\frac{2}{n-1} \sum\limits_{1 \leq i < j \leq n} a_{ij} a_{ji}}\right\} & n = 3, 5, 7, \ldots \end{cases}, \tag{1}$$

where $a = \min_i a_{ii}$, $K^2 = \frac{1}{n} \mathrm{tr}\left((A - aI)^2\right)$. Later, the results in (1) were improved by Szulc[4]. Let $A = (a_{ij})_{n \times n} \geq 0$ with $n \geq 2$. Assume that there exists a pair (k, s) such that

$$(a_{ss} - a_{kk})\, a_{ks} > 0,$$

and define

$$M = \min\left\{\min_{i \in \{j : a_{kj} > 0\}} \left\{\frac{a_{si}}{a_{ki}}\right\}, \frac{a_{ss} - a_{kk}}{2a_{ks}}\right\},$$

$$\overline{a} = \min\left\{\min_{i \in \{1, \ldots, n\} \setminus \{k, s\}} \{a_{ii}\}, a_{kk} + M a_{ks}\right\},$$

$$\overline{K}^2 = \frac{1}{n} \mathrm{tr}\left((A - \overline{a} I)^2\right).$$

Then

$$\rho(A) \geq \overline{a} + \begin{cases} \overline{K} & n = 2, 4, 6, \ldots \\ \max\left\{\overline{K}, \sqrt{\frac{2}{n-1}\left(\sum\limits_{1 \leq i < j \leq n} a_{ij} a_{ji} + M a_{ks}(a_{ss} - a_{kk}) - (M a_{ks})^2\right)}\right\} & \\ & n = 3, 5, 7, \ldots \end{cases}. \tag{2}$$

Bounds for the Perron root of a nonnegative irreducible partitioned matrix were studied by Deutsch[8]. Let $A = (a_{ij})_{n \times n} \geq 0$ be irreducible and in the form

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{pmatrix},$$

where A_{ii} is a square matrix of order n_i, $\sum_{i=1}^{k} n_i = n$, $1 \leq i \leq k$. Let p_{ij} and q_{ij} be the minimum sum and maximal sum of row of A_{ij}, respectively, $1 \leq i, j \leq k$ and denote

$$P(A) = (p_{ij})_{k \times k}, \quad Q(A) = (q_{ij})_{k \times k}.$$

Then

$$\rho(P(A)) \leq \rho(A) \leq \rho(Q(A)), \tag{3}$$

and either

$$\rho(P(A)) = \rho(Q(A)),$$

or

$$\rho(P(A)) < \rho(A) < \rho(Q(A)).$$

Recently, Kolotilina[6] showed that

$$\rho(A) \geq \max_i \left\{ \frac{a_{ii}+a}{2} + \left(\frac{(a_{ii}-a)^2}{4} + \sum_{j\neq i} a_{ij}a_{ji} \right)^{\frac{1}{2}} \right\}, \tag{4}$$

where $a = \min_i a_{ii}$.

In this paper, three sequences of lower bounds for the Perron root of nonnegative irreducible matrices were presented by means of constructing shifted matrices, whose convergence is studied.

2. Main results

We shall make use of the following lemmas.

Lemma 2.1[2] *Let $A = (a_{ij})_{n\times n} \geq 0$ be an irreducible cyclic matrix of index k $(k > 1)$. Then there exists an $n \times n$ permutation matrix P such that*

$$PAP^{\mathrm{T}} = \begin{pmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & & \vdots \\ \vdots & & & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & A_{k-1k} \\ A_{k1} & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{5}$$

where the null diagonal submatrices are square of order n_i, $1 \leq i \leq k$, $\sum_{i=1}^{k} n_i = n$.

We say that (5) is the norm form of an irreducible $n \times n$ matrix $A \geq 0$, which is cyclic of index k $(k > 1)$.

Lemma 2.2[2] *Let $A = (a_{ij})_{n\times n} \geq 0$ be an irreducible cyclic matrix of index k $(k > 1)$. Then there exists an $n \times n$ permutation matrix P such that*

$$PA^{jk}P^{\mathrm{T}} = \begin{pmatrix} C_1^j & & & 0 \\ & C_2^j & & \\ & & \ddots & \\ 0 & & & C_k^j \end{pmatrix}, \tag{6}$$

where each diagonal submatrix C_i is square and primitive with

$$\rho(C_1) = \rho(C_2) = \cdots = \rho(C_k) = \rho^k(A). \tag{7}$$

We present now our main results.

Theorem 2.3 *Let $A = (a_{ij})_{n\times n} \geq 0$ be irreducible, $a = \min_i a_{ii}$, $B := A - aI$. Then*

$$\varepsilon_0 \leq \varepsilon_1 \leq \cdots \varepsilon_t \leq \rho(A), \tag{8}$$

$$\lim_{t\to\infty} \varepsilon_t = \rho(A),$$

where, $\varepsilon_t = a + \left(\frac{\mathrm{tr}B^{2^t}}{n}\right)^{2^{-t}}$.

Proof Note that

$$(\rho(A) - a)^{2^t} = \rho(B)^{2^t} = \rho(B^{2^t}) \geq \frac{\mathrm{tr}B^{2^t}}{n},$$

we have $\varepsilon_t \leq \rho(A)$ immediately. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of B. Since, for any positive integer t,

$$\left(\lambda_1^{2^t} - \frac{\mathrm{tr}B^{2^t}}{n}\right)^2 + \left(\lambda_2^{2^t} - \frac{\mathrm{tr}B^{2^t}}{n}\right)^2 + \cdots + \left(\lambda_n^{2^t} - \frac{\mathrm{tr}B^{2^t}}{n}\right)^2$$

$$\mathrm{tr}\left(B^{2^t} - \frac{\mathrm{tr}B^{2^t}}{n} I\right)^2 = \mathrm{tr}B^{2^{t+1}} - \frac{\left(\mathrm{tr}B^{2^t}\right)^2}{n} \geq 0,$$

we have

$$\left(\frac{\mathrm{tr}B^{2^t}}{n}\right)^{2^{-t}} \leq \left(\frac{\mathrm{tr}B^{2^{t+1}}}{n}\right)^{2^{-(t+1)}},$$

this implies that

$$\varepsilon_t \leq \varepsilon_{t+1}.$$

(8) is thus proved. □

To prove $\lim_{t\to\infty} \varepsilon_t = \rho(A)$, we consider the following two cases.

Case 1 If A is a primitive matrix, then $B := A - aI$ is also a primitive matrix. Since, for any primitive matrix B[2],

$$\lim_{m\to\infty} (\mathrm{tr}B^m)^{1/m} = \rho(B),$$

we have

$$\lim_{t\to\infty} \varepsilon_t = \lim_{t\to\infty} \left[a + \left(\frac{\mathrm{tr}B^{2^t}}{n}\right)^{2^{-t}}\right] = a + \rho\,(B) = \rho\,(A)\,.$$

Case 2 If A is a cyclic matrix of index k $(k > 1)$, from the norm form of nonnegative irreducible $n \times n$ cyclic matrix of index k $(k > 1)$, we have $a = 0$. Then $B = A$ is also cyclic of index k $(k > 1)$. And from Lemma 2.2, we have

$$\lim_{m\to\infty} \left(\frac{\mathrm{tr}A^{mk}}{n}\right)^{1/mk} = \lim_{m\to\infty} \left(\sum_{i=1}^{k} \frac{\mathrm{tr}C_i^m}{n}\right)^{1/mk} = \rho\,(A)\,.$$

Now let

$$mk = 2^t,$$

we obtain

$$\lim_{t\to\infty} \varepsilon_t = \lim_{t\to\infty} \left(\frac{\mathrm{tr}A^{2^t}}{n}\right)^{2^{-t}} = \rho(A),$$

since $t \to \infty$ as $m \to \infty$. The conclusion now follows.

Remark 2.4 If we set $t=2$ in (8), then $\rho(A) \geq \min_i a_{ii} + \left(\frac{\mathrm{tr}B^2}{n}\right)^{1/2}$, which yields the result of Szulc[3] and can be further improved by (8).

Remark 2.5 Let $t=2$ in (8), then the inequality $\min_i a_{ii} + \left(\frac{\mathrm{tr}B^2}{n}\right)^{1/2} \geq \left(\frac{\mathrm{tr}A^2}{n}\right)^{1/2}$. This shows that lower bounds for the Perron root were improved by constructing shifted matrices.

Theorem 2.6 *Let $A=(a_{ij})_{n\times n} \geq 0$ be an irreducible matrix, $a=\min_i a_{ii}$, $B := A - aI$. Then*

$$\sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_t \leq \rho(A), \tag{9}$$

$$\lim_{t\to\infty} \sigma_t = \rho\left(A\right),$$

where $\sigma_t = a + \max_i \left[\left(B^{2^t}\right)_{ii}\right]^{2^{-t}}$.

Proof Note that

$$\left(\rho\left(A\right)-a\right)^{2^t} = \rho\left(B\right)^{2^t} = \rho\left(B^{2^t}\right) \geq \max_i \left[\left(B^{2^t}\right)_{ii}\right],$$

inequality $\sigma_t \leq \rho\left(A\right)$ is immediate. Since, for any positive integer t,

$$\begin{aligned}\max_i \left[(B^{2^t})_{ii}\right]^2 &= \max_i \left[b_{ii}^{(2^t)}\right]^2 \leq \max_i \left(\sum_{k=1}^{n} b_{ik}^{(2^t)} b_{ki}^{(2^t)}\right)\\ &= \max_i \left\{[(B^{2^t})^2]_{ii}\right\} = \max_i \left[(B^{2^{t+1}})_{ii}\right],\end{aligned}$$

so that

$$\max_i \left[\left(B^{2^t}\right)_{ii}\right]^{2^{-t}} \leq \max_i \left[\left(B^{2^{t+1}}\right)_{ii}\right]^{2^{-(t+1)}},$$

that is

$$\sigma_t \leq \sigma_{t+1}.$$

(9) is thus proved. From Theorem 2.3 and note that

$$a + \left(\frac{\mathrm{tr}B^{2^t}}{n}\right)^{2^{-t}} \leq a + \max_i \left[\left(B^{2^t}\right)_{ii}\right]^{2^{-t}} \leq \rho(A), \tag{10}$$

we get $\lim_{t\to\infty} \sigma_t = \rho\left(A\right)$, so that the proof is completed. □

Remark 2.7 From (10), one can get $\varepsilon_t \leq \sigma_t \leq \rho(A)$.

Remark 2.8 Since

$$\max_i \left[\left(A^2\right)_{ii}\right]^{1/2} \leq \min_i a_{ii} + \max_i \left[\left(B^2\right)_{ii}\right]^{1/2} \leq \rho(A),$$

it follows that improvements on lower bounds for the Perron root can be achieved by constructing shifted matrices.

Let $A \geq 0$, Schwenk[5] and Kolotilina[6] showed that

$$\rho(S(A)) \leq \rho(A),$$

where $S(A)$ is the matrix whose entry in row i, column j is given by $\sqrt{a_{ij}a_{ji}}$ and is called geometric symmetrization of A. Further, Szyld[7] proved the following theorem.

Therorem 2.9 *Let $A = (a_{ij})_{n\times n} \geq 0$, and let $\beta_t = \rho(S(A^{2^t}))^{2^{-t}}$. Then*

$$\beta_0 \leq \beta_1 \leq \cdots \leq \beta_t \leq \rho(A).$$

Next we give the last sequence of lower bounds for the Perron root.

Therorem 2.10 *Let $A = (a_{ij})_{n\times n} \geq 0$ be an irreducible matrix, $a = \min_i a_{ii}$, $B := A - aI$. Then*

$$\rho_0 \leq \rho_1 \leq \cdots \leq \rho_t \leq \rho(A), \tag{11}$$

$$\lim_{t\to\infty} \rho_t = \rho(A),$$

where $\rho_t = a + \rho(S(B^{2^t}))^{2^{-t}}$.

Proof Since

$$(\rho(A) - a)^{2^t} = \rho(B)^{2^t} = \rho(B^{2^t}) \geq \rho(S(B^{2^t})),$$

we have $\rho_t \leq \rho(A)$. To prove

$$\rho_0 \leq \rho_1 \leq \cdots \leq \rho_t \leq \rho(A),$$

we need to prove that

$$a + \rho\left(S(B)\right) \leq a + \rho\left(S(B^2)\right)^{1/2} \leq \cdots \leq a + \rho\left(S(B^{2^t})\right)^{2^{-t}} \leq \rho(A),$$

this is equivalent to

$$\rho\left(S(B)\right) \leq \rho\left(S(B^2)\right)^{1/2} \leq \cdots \leq \rho\left(S(B^{2^t})\right)^{2^{-t}} \leq \rho(A) - a = \rho\left(B\right),$$

which is clearly true from Theorem 2.9 and since

$$\rho(A) \geq a + \rho(S(B^{2^t}))^{2^{-t}} \geq a + \max_i \left[\left(S\left(B^{2^t}\right)\right)_{ii}\right]^{2^{-t}} = a + \max_i \left[\left(B^{2^t}\right)_{ii}\right]^{2^{-t}}, \tag{12}$$

we get

$$\lim_{t\to\infty} = [a + \rho(S(B^{2^t}))^{2^{-t}}] = \rho(A).$$

Hence, the theorem is proved. □

Remark 2.11 It is convenient to know that $\sigma_t \leq \rho_t \leq \rho(A)$ from (12).

3. Numerical example

Example 3.1 We use the example in [6]. Consider the following matrix:

$$A = \begin{pmatrix} 1 & 1 & \vdots & 2 \\ 2 & 1 & \vdots & 3 \\ \cdots & \cdots & \cdots & \cdots \\ 2 & 3 & \vdots & 5 \end{pmatrix}, \quad \rho(A) = 7.5311.$$

We have the comparison results in Table 1.

$\rho(A) \geq 1 + \left(\frac{46}{3}\right)^{1/2} \approx 4.915$	(1) (see [3])
$\rho(A) \geq 1 + 19^{1/2} \approx 5.3588$	(2)$k = 1, s = 3$ (see [4])
$\rho(A) > \rho(P(A)) = \rho\left(\begin{bmatrix} 2 & 2 \\ 5 & 5 \end{bmatrix}\right) = 7$	(3) (see [8])
$\rho(A) > 1 + [17.5 + (132.25 + 378)^{1/2}]^{1/2} \approx 7.331$	(4) applying to $(A - I)^2$ (see [6])
$\rho(A) \geq \varepsilon_5 = 7.3107 \geq \varepsilon_3 = 6.6931 \geq \varepsilon_1 = 4.9158$	(8)
$\rho(A) \geq \sigma_5 = 7.4498 \geq \sigma_3 = 7.2117 \geq \sigma_1 = 6.3852$	(9)
$\rho(A) = \rho_2 = 7.5311 \geq \rho_1 = 7.5293$	(11)

Table 1 Lower bounds for the Perron root

References

[1] HORN R A, JOHNSON C R. *Matrix Analysis* [M]. Cambridge University Press, Cambridge, 1985.

[2] VARGA R S. *Matrix Iterative Analysis* [M]. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1962.

[3] SZULC T. *A lower bound for the Perron root of a nonnegative matrix* [J]. Linear Algebra Appl., 1988, **101**: 181–186.

[4] SZULC T. *A lower bound for the Perron root of a nonnegative matrix. II* [J]. Linear Algebra Appl., 1989, **112**: 19–27.

[5] SCHWENK A J. *Tight bounds on the spectral radius of asymmetric nonnegative matrices* [J]. Linear Algebra Appl., 1986, **75**: 257–265.

[6] KOLOTILINA L YU. *Lower bounds for the Perron root of a nonnegative matrix* [J]. Linear Algebra Appl., 1993, **180**: 133–151.

[7] SZYLD D B. *A sequence of lower bounds for the spectral radius of nonnegative matrices* [J]. Linear Algebra Appl., 1992, **174**: 239–242.

[8] DEUTSCH E. *Bounds for the Perron root of a nonnegative irreducible partitioned matrix* [J]. Pacific J. Math., 1981, **92**(1): 49–56.