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An H^1-Galerkin Nonconforming Mixed Finite Element Method for Integro-Differential Equation of Parabolic Type

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Abstract H^1-Galerkin nonconforming mixed finite element methods are analyzed for integro-differential equation of parabolic type. By use of the typical characteristic of the elements, we obtain that the Galerkin mixed approximations have the same rates of convergence as in the classical mixed method, but without LBB stability condition.

Keywords H^1-Galerkin mixed method; integro-differential equation of parabolic type; nonconforming; semi-discrete scheme; full discrete scheme; error estimates.

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1. Introduction

In this paper, we discuss a new mixed finite element method for the following integro-differential equation of parabolic problem

$$\begin{cases} \text{(a) } u_t - \nabla\cdot(a(X)\nabla u + \int_0^t b(t,\tau)\nabla u(\tau)\mathrm{d}\tau) = f(X,t), & X\in\Omega, t\in(0,T],\\ \text{(b) } u(X,t)=0, & X\in\partial\Omega, t\in[0,T],\\ \text{(c) } u(X,0)=u_0(X), & X\in\Omega, \end{cases} \tag{1}$$

where $\nabla\cdot$ and ∇ denote the gradient and the divergence of functions, respectively, $X=(x,y), u_t=\frac{\partial u}{\partial t}$. Assume that the coefficients a, b, u_0 and the forcing function f are sufficiently smooth with $|b|\le a_1, 0<a_0\le a\le a_1$, for some positive constants a_0 and a_1.

In recent years, integro-differential equations of parabolic type have received a great deal of attention. This type of equations often occur in applications such as heat conduction in material with memory, compression of viscoelastic media, nuclear reactor dynamics, etc.. A lot of investigations have been devoted to above problems, such as Chen et al[1], Xu[2], Thomee[3], Mclean[4], etc. When our primary concern is to obtain u and $p=a\nabla u$, we first split (a) of (1) into a system of two equations and then use classical mixed methods[5]. However, this procedure has to satisfy the LBB stability condition on the approximating spaces, which restricts the choice

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of finite element spaces. In order to avoid the condition, Pani[6,7] introduced an alternate mixed finite element procedure. The method is a non-symmetric version of least square method. It takes advantage of the least-square method and yields a better rate of convergence for the stress than the conventional method.

Recently, H^1-Galerkin mixed finite element method has been developed deeply. For example, Guo[8] studied the one dimensional regularized long wave equations and gave the error estimates of semi-discrete and fully-discrete scheme. Wang[9] considered the integro-differential equations of hyperbolic type and gave the semi-discrete error estimates. But all the results obtained previously are just for conforming finite elements. Stynes and Tobisk[10] have pointed out that nonconforming finite element approximations are appropriate, with the striking advantage that the unknowns are associated with the element faces, each degree of freedom belongs to at most elements. This results in cheap local communication and can be parallelized in a highly efficient manner on MIMD-machines.

In the present article, we will focus on the nonconforming mixed finite element approximation to (1). We make the best use of characteristics of the elements, such as $(\nabla(u-I_h^1u),\nabla v_h)=0$, $\forall u\in H^1(\Omega)$, $v_h\in V_h$, $(\nabla\cdot(p-I_h^2p),\nabla\cdot\psi)=0$, $\forall p\in H(\mathrm{div};\Omega)$, $\psi\in W_h$ and the novel techniques of the boundary estimation (see Lemmas 1-3 for detail). It is proved that the same estimates as in the traditional mixed finite element methods can be obtained under semi-discrete and full discrete scheme without LBB stability condition.

A brief outline of this paper is as follows. In next section, we construct nonconforming rectangular elements. In Section 3, the error analysis of the semi-discrete H^1-Galerkin nonconforming mixed finite method is given. At last, we give the convergence results of the fully-discrete scheme.

Throughout this paper, C denotes a general positive constant which is independent of h, where $h=\max\limits_K h_K$, h_K is the diameter of the finite element K.

2. Construction of the element

Assume that $\hat{K}=[0,1]\times[0,1]$ is $\hat{x}-\hat{y}$ plane. Let $\hat{A}_1=(-1,-1)$, $\hat{A}_2=(1,-1)$, $\hat{A}_3=(1,1)$ and $\hat{A}_4=(-1,1)$ be the four vertices, $\hat{l}_1=\overline{\hat{A}_1\hat{A}_2}$, $\hat{l}_2=\overline{\hat{A}_2\hat{A}_3}$, $\hat{l}_3=\overline{\hat{A}_3\hat{A}_4}$ and $\hat{l}_4=\overline{\hat{A}_4\hat{A}_1}$ be the four edges.

We define the finite element $(\hat{K},\hat{P}^i,\hat{\Sigma}^i)$ $(i=1,2,3)$

$$\hat{\Sigma}^1=\{\hat{v}_1,\hat{v}_2,\hat{v}_3,\hat{v}_4,\hat{v}_5\},\quad \hat{P}^1=\mathrm{span}\{1,\hat{x},\hat{y},\varphi(\hat{x}),\varphi(\hat{y})\},$$

$$\hat{\Sigma}^2=\{\hat{q}_1,\hat{q}_2,\hat{q}_3,\hat{q}_4\},\quad \hat{P}^2=\mathrm{span}\{1,\hat{x},\hat{y},\hat{y}^2\},$$

$$\hat{\Sigma}^3=\{\hat{p}_1,\hat{p}_2,\hat{p}_3,\hat{p}_4\},\quad \hat{P}^3=\mathrm{span}\{1,\hat{x},\hat{y},\hat{x}^2\},$$

where

$$\hat{v}_5=\frac{1}{|\hat{K}|}\int_{\hat{K}}\hat{v}\mathrm{d}\hat{x}\mathrm{d}\hat{y},\quad \varphi(t)=\frac{1}{2}(3t^2-1),$$

$$\hat{v}_i=\frac{1}{|\hat{l}_i|}\int_{\hat{l}_i}\hat{v}\mathrm{d}\hat{s},\ \ \hat{q}_i=\frac{1}{|\hat{l}_i|}\int_{\hat{l}_i}\hat{q}\mathrm{d}\hat{s},\ \ \hat{p}_i=\frac{1}{|\hat{l}_i|}\int_{\hat{l}_i}\hat{p}\mathrm{d}\hat{s},\quad i=1,2,3,4.$$

Remark 1 When $i = 1$, we refer to [11,12].

The interpolations defined above are properly posed and the interpolation functions can be expressed as follows:

$$\hat{\Pi}^1\hat{v} = \hat{v}_5 + \frac{1}{2}(\hat{v}_2 - \hat{v}_4)\hat{x} + \frac{1}{2}(\hat{v}_3 - \hat{v}_1)\hat{y} + \frac{1}{2}(\hat{v}_2 + \hat{v}_4 - 2\hat{v}_5)\varphi(\hat{x}) + \frac{1}{2}(\hat{v}_3 + \hat{v}_1 - 2\hat{v}_5)\varphi(\hat{y}),$$

$$\hat{\Pi}^2\hat{q} = \frac{1}{4}(-\hat{q}_1 - \hat{q}_3 + 3\hat{q}_2 + 3\hat{q}_4) + \frac{1}{2}(\hat{q}_2 - \hat{q}_4)\hat{x} + \frac{1}{2}(\hat{q}_3 - \hat{q}_1)\hat{y} + \frac{3}{4}(\hat{q}_1 + \hat{q}_3 - \hat{q}_2 - \hat{q}_4)\hat{y}^2$$

and

$$\hat{\Pi}^3\hat{p} = \frac{1}{4}(3\hat{p}_1 + 3\hat{p}_3 - \hat{p}_2 - \hat{p}_4) + \frac{1}{2}(\hat{p}_2 - \hat{p}_4)\hat{x} + \frac{1}{2}(\hat{p}_3 - \hat{p}_1)\hat{y} + \frac{3}{4}(-\hat{p}_1 - \hat{p}_3 + \hat{p}_2 + \hat{p}_4)\hat{x}^2,$$

respectively.

Let $\Omega \subset R^2$ be a polygon domain with boundaries $\partial\Omega$ parallel to the coordinate axes. $\mathcal{J}_h$ be a family of decomposition of Ω with $\bar{\Omega} = \bigcup_{K\in\mathcal{J}_h} K$. Let (x_K, y_K) be the barycenter of element K, and $2h_x$, $2h_y$ are the two sides, respectively. $A_1(x_K - h_x, y_K - h_y)$, $A_2(x_K + h_x, y_K - h_y)$, $A_3(x_K + h_x, y_K + h_y)$ and $A_4(x_K - h_x, y_K + h_y)$ are the four vertices and $l_1 = \overline{A_1A_2}$, $l_2 = \overline{A_2A_3}$, $l_3 = \overline{A_3A_4}$ and $l_4 = \overline{A_4A_1}$ are the edges. Thus there exists an affine mapping $F_K : \hat{K} \to K$:

$$\begin{cases} x = x_K + h_x\hat{x}, \\ y = y_K + h_y\hat{y}. \end{cases} \tag{2}$$

The associated finite element spaces V_h and W_h are defined as

$$V_h = \{v; v\,|_K = \hat{v} \circ F_K^{-1}, \hat{v} \in \hat{P}^1, \int_F [v] = 0, F \subset \partial K\},$$

$$W_h = \{w = (w_1, w_2); w\,|_K = (\hat{w}_1 \circ F_K^{-1}, \hat{w}_2 \circ F_K^{-1}), \hat{w} \in \hat{P}^2 \times \hat{P}^3, \int_F [w] = 0, F \subset \partial K\},$$

where $[\varphi]$ denotes the jump of φ across the boundary F, $[\varphi] = \varphi$ if $F \subset \partial\Omega$.

Then for all $v \in H^2(\Omega), w = (w_1, w_2) \in (H^1(\Omega))^2$, we define the interpolation operators I_h^1 and I_h^2 as

$$I_h^1 : H^2(\Omega) \to V_h, I_h^1|_K = I_K^1, I_K^1 v = (\hat{\Pi}^1\hat{v}) \circ F_K^{-1},$$
$$I_h^2 : (H^1(\Omega))^2 \to W_h, I_h^2|_K = I_K^2, I_K^2 w = (\hat{\Pi}^2\hat{w}_1 \circ F_K^{-1}, \hat{\Pi}^3\hat{w}_2 \circ F_K^{-1}).$$

3. H^1-Galerkin methods in the semi-discrete scheme

In this section, we mainly study the semi-discrete scheme for (1).

For H^1-Galerkin mixed finite element procedure, we first split (a) of (1) into the following system of two equations

$$\nabla u = \alpha p, \quad u_t - \nabla \cdot p - \int_0^t \nabla \cdot (\beta p)(\tau)\mathrm{d}\tau = f, \tag{3}$$

where $\alpha = 1/a$ and $\beta = b\alpha$.

Let $L^2(\Omega)$ be the set of square integrable functions on Ω and $(L^2(\Omega))^2$ the space of two dimensional vectors which have all components in $L^2(\Omega)$ with its norm $\|\cdot\|$. Let $H(\Omega; \mathrm{div})$ be the space of vectors in $(L^2(\Omega))^2$ which has divergence in $L^2(\Omega)$ with norm $\|\cdot\|^2_{H(\Omega;\mathrm{div})} = \|\cdot\|^2+\|\nabla\cdot\|^2$. $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product. For our subsequent use, we also use the standard Sobolev space $W^{m,p}(\Omega)$ with a norm $\|\cdot\|_{m,p}$. Especially for $p=2$, we denote $H^m(\Omega)=W^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$.

The weak formulation is now defined to be a pair $\{u,p\}$: $[0,T]\mapsto H^1_0(\Omega)\times H(\mathrm{div};\Omega)$ satisfying

$$\begin{cases} (\nabla u,\nabla v) = (\alpha p,\nabla v), \forall v\in H^1_0(\Omega), \\ (\alpha p_t, w) + (\nabla\cdot p,\nabla\cdot w) + \int_0^t(\nabla\cdot(\beta p)(\tau),\nabla\cdot w)\mathrm{d}\tau = -(f,\nabla\cdot w), \forall w\in H(\mathrm{div};\Omega), \\ u(X,0)=u_0(X), X\in\Omega. \end{cases} \tag{4}$$

The semi-discrete H^1-Galerkin finite element procedure for the system is determined as a pair $\{u_h,p_h\}$: $[0,T]\to V_h\times W_h$ satisfying

$$\begin{cases} (\nabla u_h,\nabla v_h) = (\alpha p_h,\nabla v_h), & \forall v_h\in V_h, \\ (\alpha p_{ht}, w_h) + (\nabla\cdot p_h,\nabla\cdot w_h) + \int_0^t(\nabla\cdot(\beta p_h)(\tau),\nabla\cdot w_h)\mathrm{d}\tau = -(f,\nabla\cdot w_h), & \forall w_h\in W_h, \\ u_h(X,0)=I^1_h u(X,0), & X\in\Omega. \end{cases} \tag{5}$$

For all $w_h\in W_h, v_h\in V_h$, we define

$$\|w_h\|_h = \Big(\sum_K \|w_h\|^2 + \|\nabla\cdot w_h\|^2\Big)^{\frac12}, \quad |v_h|_h = \Big(\sum_K |v_h|^2_{1,K}\Big)^{\frac12}.$$

It is easy to see that $\|\cdot\|_h$ and $|\cdot|_h$ are norms of W_h and V_h.

Theorem 1 *Problem (5) has a unique solution.*

Proof Let $\{\phi_i\}_{i=1}^{r_1}$ and $\{\psi_j\}_{j=1}^{r_2}$ be the bases of V_h and W_h. Let

$$u_h = \sum_{i=1}^{r_1} h_i(t)\phi_i,\ p_h = \sum_{j=1}^{r_2} g_j(t)\psi_j,\ v_h=\phi_j,\ w_h=\psi_i,$$

so that (5) can be rewritten as

$$\begin{cases} \text{(a) } AH(t) = BG(t), \\ \text{(b) } D\frac{\mathrm{d}G(t)}{\mathrm{d}t} + EG(t) + \int_0^t FG(\tau)\mathrm{d}\tau = N, \end{cases} \tag{6}$$

where

$$H(t) = (h_1(t),\dots,h_{r_1}(t))',\quad G(t)=(g_1(t),\dots,g_{r_2}(t))',$$

$$A = ((\nabla\phi_i,\nabla\phi_j))_{r_1\times r_1},\quad B=((\alpha\psi_i,\nabla\phi_j))_{r_1\times r_2},\quad D=((\alpha\psi_i,\psi_j))_{r_2\times r_2},$$

$$E = ((\nabla\cdot\psi_i,\nabla\cdot\psi_j))_{r_2\times r_2},\quad F=((\nabla\cdot(\beta\psi_i),\nabla\cdot\psi_j))_{r_2\times r_2},\quad N=-((f,\nabla\cdot\psi_j))_{1\times r_2},$$

((6)b) is an ordinary differential equation about the vector $G(t)$. It has a unique solution with initial value $H(0)$ (see [13] for details). Applying the theory of differential equations, we can obtain a unique solution of (6). Therefore, (5) has a unique solution.

Now we give the following lemmas which will play an important role in our error analysis.

Lemma 1 *For $u\in H^1(\Omega)$, there holds that $(\nabla(u-I^1_h u),\nabla v)=0$, $\forall v\in V_h$.*

Proof Since for all $v \in V_h, v|_K \in \text{span}\{1, x, y, x^2, y^2\}$, we have that $\triangle v|_K$ and $\frac{\partial v}{\partial n}|_K$ are constants. By Green's formula and the interpolation definition, we obtain

$$\begin{aligned}(\nabla u - \nabla I_h^1 u, \nabla v) &= \sum_K \int_K \nabla(u - I_K^1 u)\nabla v \mathrm{d}x\mathrm{d}y \\ &= -\sum_K \int_K (u - I_K^1 u)\Delta v \mathrm{d}x\mathrm{d}y + \sum_K \int_{\partial K} (u - I_K^1 u)\frac{\partial v}{\partial n}\mathrm{d}s \\ &= 0.\end{aligned}$$

The proof is completed. □

Lemma 2 *For $p \in H(\text{div}; \Omega)$, there holds that $(\nabla \cdot (p - I_h^2 p), \nabla \cdot \psi) = 0$, for all $\psi \in W_h$.*

Proof Since $\psi \in W_h$, $\psi|_K \in \text{span}\{1, x, y, y^2\} \times \text{span}\{1, x, y, x^2\}$, we have $\nabla(\nabla \cdot \psi)|_K = 0$ and $\nabla \cdot \psi|_{\partial K}$ is a constant. Applying Green's formula and the interpolation definition yields

$$\begin{aligned}(\nabla \cdot (p - I_h^2 p), \nabla \cdot \psi) &= \sum_K \int_K \nabla \cdot (p - I_K^2 p)\nabla \cdot \psi \mathrm{d}x\mathrm{d}y \\ &= -\sum_K \int_K (p - I_K^2 p)\nabla(\nabla \cdot \psi)\mathrm{d}x\mathrm{d}y + \sum_K \int_{\partial K} (p - I_K^2 p) \cdot n(\nabla \cdot \psi)\mathrm{d}s \\ &= 0.\end{aligned}$$

The proof is completed. □

Lemma 3 *Suppose $u_t \in H^2(\Omega)$. There holds*

$$\Big|\sum_K \int_{\partial K} u_t(w \cdot n)\mathrm{d}s\Big| \le Ch|u_t|_2\|w\|, \quad \forall w \in W_h. \tag{7}$$

Proof Let $w = (w_1, w_2) \in W_h$. Then

$$\sum_K \int_{\partial K} u_t(w \cdot n)\mathrm{d}s = \sum_K \int_{\partial K} (u_t w_1 n_1 + u_t w_2 n_2)\mathrm{d}s = \sum_K (\sum_{i=1}^4 I_i), \tag{8}$$

where

$$\begin{aligned}I_1 &= \int_{l_1} -\Big(u_t - \frac{1}{|l_1|}\int_{l_1} u_t \mathrm{d}x\Big)\Big(w_2 - \frac{1}{|l_1|}\int_{l_1} w_2 \mathrm{d}x\Big)\mathrm{d}x, \\ I_2 &= \int_{l_2} \Big(u_t - \frac{1}{|l_2|}\int_{l_2} u_t \mathrm{d}y\Big)\Big(w_1 - \frac{1}{|l_2|}\int_{l_2} w_1 \mathrm{d}y\Big)\mathrm{d}y, \\ I_3 &= \int_{l_3} \Big(u_t - \frac{1}{|l_3|}\int_{l_3} u_t \mathrm{d}x\Big)\Big(w_2 - \frac{1}{|l_3|}\int_{l_3} w_2 \mathrm{d}x\Big)\mathrm{d}x,\end{aligned}$$

and

$$I_4 = \int_{l_4} -\Big(u_t - \frac{1}{|l_4|}\int_{l_4} u_t \mathrm{d}y\Big)\Big(w_1 - \frac{1}{|l_4|}\int_{l_4} w_1 \mathrm{d}y\Big)\mathrm{d}y.$$

On the one hand, we note that

$$\begin{aligned}&u_t(x, y_K + h_y) - u_t(x, y_K - h_y) - \frac{1}{2h_x}\int_{x_K - h_x}^{x_K + h_x} (u_t(x, y_K + h_y) - u_t(x, y_K - h_y))\mathrm{d}x \\ &= \frac{1}{2h_x}\int_{x_K - h_x}^{x_K + h_x} \Big(\int_t^x \frac{\partial u_t}{\partial z}(z, y_K + h_y)\mathrm{d}z\Big)\mathrm{d}t - \frac{1}{2h_x}\int_{x_K - h_x}^{x_K + h_x} \Big(\int_t^x \frac{\partial u_t}{\partial z}(z, y_K - h_y)\mathrm{d}z\Big)\mathrm{d}t\end{aligned}$$

$$= \frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\int_t^x\int_{y_K-h_y}^{y_K+h_y}\frac{\partial^2 u_t}{\partial z\partial y}(z,y)\mathrm{d}y\mathrm{d}z\mathrm{d}t, \tag{9}$$

and

$$\begin{aligned} &w_2(x,y_K\pm h_y)-\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}w_2(x,y_K\pm h_y)\mathrm{d}x\\ &=\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\Big(\int_t^x\frac{\partial w_2}{\partial z}(z,y_K\pm h_y)\mathrm{d}z\Big)\mathrm{d}t. \end{aligned} \tag{10}$$

On the other hand, since $w=(w_1,w_2)\in\mathrm{span}\{1,x,y,y^2\}\times\mathrm{span}\{1,x,y,x^2\}$, there holds

$$\frac{\partial w_2}{\partial x}(x,y_K-h_y)=\frac{\partial w_2}{\partial x}(x,y_K+h_y). \tag{11}$$

Further, using Cauchy-Schwartz's inequality, we get

$$\begin{aligned} |I_1+I_3| =&\Big|\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\Big[\int_{x_K-h_x}^{x_K+h_x}\mathrm{d}t\int_t^x\int_{y_K-h_y}^{y_K+h_y}\frac{\partial^2 u}{\partial y\partial z}(z,y)\mathrm{d}y\Big]\\ &\Big[\frac{1}{2h_x}\int_{x_K-h_x}^{x_K+h_x}\mathrm{d}t\int_t^x\frac{\partial w_2}{\partial z}(z,y_K+h_y)\mathrm{d}z)\mathrm{d}t\Big]\mathrm{d}x\Big|\\ \le&\frac{4h_x^2}{3}\|\frac{\partial^2 u_t}{\partial x\partial y}\|_{0,K}\|\frac{\partial w_2}{\partial x}\|_{0,K}. \end{aligned} \tag{12}$$

Similarly, we have

$$|I_2+I_4|\le\frac{4h_y^2}{3}\|\frac{\partial^2 u_t}{\partial x\partial y}\|_{0,K}\|\frac{\partial w_1}{\partial y}\|_{0,K}.$$

Note that

$$\|\frac{\partial w_2}{\partial x}\|_{0,K}\le Ch_x^{-1}\|w_2\|_{0,K},\|\frac{\partial w_1}{\partial y}\|_{0,K}\le Ch_y^{-1}\|w_1\|_{0,K}. \tag{13}$$

So

$$\Big|\sum_K\int_{\partial K}u_t(w\cdot n)\mathrm{d}s\Big|\le Ch|u_t|_2\|w\|.$$

The proof is completed. □

Lemma 4[14] *Suppose that $u\in H^2(\Omega)$, $p\in(H^2(\Omega))^2$ and $p_t\in(H^1(\Omega))^2$. Then there hold*

$$|u-I_h^1u|_h\le Ch|u|_2,\quad \|p-I_h^2p\|\le Ch|p|_1,$$

$$\|p_t-I_h^2p_t\|\le Ch|p_t|_1,\quad \|\nabla\cdot(p-I_h^2p)\|\le Ch|p|_2.$$

Theorem 2 *Let u, $u_t\in H^2(\Omega)$, $p\in(H^2(\Omega))^2$ and $p_t\in(H^1(\Omega))^2$. Then there hold*

$$|u-u_h|_h\le Ch\Big[|u|_2+|p|_1+\Big(\int_0^t(|p_t|_1^2+|u_t|_2^2)\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big], \tag{14}$$

and

$$\|p-p_h\|_h\le Ch\Big[|p|_1+|p|_2+\Big(\int_0^t(|p_t|_1^2+|u_t|_2^2)\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big]. \tag{15}$$

Proof Let $u - u_h = (u - I_h^1 u) + (I_h^1 u - u_h) = \eta + \xi$, $p - p_h = (p - I_h^2 p) + (I_h^2 p - p_h) = \sigma + \theta$. It is easy to see that for any $v_h \in V_h, w_h \in W_h$ there hold

$$\begin{cases} \text{(a) } (\nabla\xi, \nabla v_h) = (\alpha\theta, \nabla v_h) + (\alpha\sigma, \nabla v_h), \\ \text{(b) } (\alpha\theta_t, w_h) + (\nabla\cdot\theta, \nabla\cdot w_h) + (\theta, w_h) = (\theta, w_h) - (\alpha\sigma_t, w_h) - \\ \qquad \int_0^t (\nabla\cdot(\beta\theta), \nabla\cdot w_h)\mathrm{d}\tau + \sum_K \int_{\partial K} u_t w_h \cdot n ds. \end{cases} \tag{16}$$

Setting $v_h = \xi$ in (16(a)) and using the Cauchy-Schwartz's inequality gives

$$\|\nabla\xi\| \le C(\|\sigma\| + \|\theta\|). \tag{17}$$

Further, choosing $w_h = \theta_t$ in (16(b)) leads to

$$\begin{aligned} \|\alpha^{\frac{1}{2}}\theta_t\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_h^2 =& (\theta, \theta_t) - (\sigma_t, \theta_t) - \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^t (\nabla\cdot(\beta\theta), \nabla\cdot\theta)\mathrm{d}\tau\Big) + \\ & \int_0^t (\nabla\cdot(\beta_t\theta), \nabla\cdot\theta)\mathrm{d}\tau + (\nabla\cdot(\beta\theta), \nabla\cdot\theta) + \sum_K \int_{\partial K} u_t\theta_t \cdot n\mathrm{d}s. \end{aligned} \tag{18}$$

Applying ε-Young's inequality and Lemma 3, we get

$$\begin{aligned} \|\alpha^{\frac{1}{2}}\theta_t\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_h^2 \le& C(\varepsilon)\Big(\|\sigma_t\|^2 + \|\theta\|_h^2 + h^2|u_t|_2^2 + \int_0^t \|\theta(\tau)\|_h^2\mathrm{d}\tau\Big) + \\ & \varepsilon\|\theta_t\|^2 - \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^t (\nabla\cdot(\beta\theta), \nabla\cdot\theta)\mathrm{d}\tau\Big). \end{aligned} \tag{19}$$

With sufficiently small ε, we have

$$\begin{aligned} \|\theta_t\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_h^2 \le& C\Big(\|\sigma_t\|^2 + \|\theta\|_h^2 + h^2|u_t|_2^2 + \int_0^t \|\theta(\tau)\|_h^2\mathrm{d}\tau\Big) - \\ & \frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_0^t (\nabla\cdot(\beta\theta), \nabla\cdot\theta)\mathrm{d}\tau\Big). \end{aligned} \tag{20}$$

Integrating the both sides of (20) with respect to time from 0 to t, and noting $\theta(0) = 0$, we obtain

$$\int_0^t \|\theta_t(\tau)\|^2\mathrm{d}\tau + \|\theta\|_h^2 \le C\int_0^t (\|\theta(\tau)\|_h^2 + \|\sigma_t(\tau)\|^2 + h^2|u_t(\tau)|_2^2)\mathrm{d}\tau - \int_0^t (\nabla\cdot(\beta\theta), \nabla\cdot\theta)\mathrm{d}\tau. \tag{21}$$

Hence, using ε-Young's inequality gives

$$\int_0^t \|\theta_t(\tau)\|^2\mathrm{d}\tau + \|\theta\|_h^2 \le C\int_0^t (\|\theta(\tau)\|_h^2 + \|\sigma_t(\tau)\|^2 + h^2|u_t(\tau)|_2^2)\mathrm{d}\tau + \varepsilon\|\theta\|_h^2. \tag{22}$$

For sufficiently small ε, by Gronwall's lemma and Lemma 4, we have

$$\int_0^t \|\theta_t(\tau)\|^2\mathrm{d}\tau + \|\theta\|_h^2 \le Ch^2\int_0^t (|p_t(\tau)|_1^2 + |u_t(\tau)|_2^2)\mathrm{d}\tau. \tag{23}$$

Combining Lemma 4, (17) and (23) yields

$$|u - u_h|_h \le \|\nabla\eta\| + \|\nabla\xi\| \le Ch\Big[|u|_2 + |p|_1 + \Big(\int_0^t (|p_t(\tau)|_1^2 + |u_t(\tau)|_2^2)\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big], \tag{24}$$

and

$$\|p - p_h\|_h \le \|\sigma\|_h + \|\theta\|_h \le Ch\Big[|p|_1 + |p|_2 + \Big(\int_0^t (|p_t|_1^2 + |u_t|_2^2)\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big]. \tag{25}$$

The proof is completed. □

4. Fully-discrete scheme and error estimates

In this section, we briefly describe a fully scheme for approximating a pair of solution $\{u, p\}$ of (4) and discuss a priori error bounds.

Let $0 = t_0 < t_1 < \cdots < t_M = T$ be a given partition of the time interval $[0, T]$ with step length $\triangle t = T/M$ for some positive integer M. Let $u_h(t_n) \in V_h, p_h(t_n) \in W_h$ be the approximations of u and p at time $t = t_n = n\triangle t$. The time discretization will be based on the backward difference quotient $\bar{\partial}_t p_h(t_n) = (p_h(t_n) - p_h(t_{n-1}))/\triangle t$. The integral term then has to be evaluated by numerical quadrature from the values of the $p_h(t_n)$. We shall approximate ϕ in $\int_0^{t_n} \beta(t_n, \tau)\phi(\tau)\mathrm{d}\tau$ by the piecewise constant function taking the value $\phi(t_j)$, and thus

$$\int_0^{t_n} \beta(t_n, \tau)\phi(\tau)\mathrm{d}\tau \approx \sum_{j=0}^{n-1} \triangle t \beta(t_n, t_j)\phi(t_j).$$

We now determine a pair of $\{u_h(t_n), p_h(t_n)\} \in V_h \times W_h$, satisfying

$$\begin{cases} \text{(a) } (\nabla u_h(t_n), \nabla v_h) = (\alpha p_h(t_n), \nabla v_h), \ \forall v_h \in V_h, \\ \text{(b) } (\alpha \bar{\partial}_t p_h(t_n), w_h) + (\nabla \cdot p_h(t_n), \nabla \cdot w_h) + \sum\limits_{j=0}^{n-1} \triangle t \beta(t_n, t_j)(\nabla \cdot p_h(t_j), \nabla \cdot w_h) \\ \qquad = -(f(t_n), \nabla \cdot w_h), \ \forall w_h \in W_h, \end{cases} \tag{26}$$

Theorem 3 *Problem (26) has a unique solution.*

Proof Set $w_h = p_h(t_n)$ in (26(b)). It follows that

$$\begin{aligned} &(\alpha \bar{\partial}_t p_h(t_n), p_h(t_n)) + (\nabla \cdot p_h(t_n), \nabla \cdot p_h(t_n)) \\ &\quad = -\sum_{j=0}^{n-1} \triangle t \beta(t_n, t_j)(\nabla \cdot p_h(t_j), \nabla \cdot p_h(t_n)) - (f(t_n), \nabla \cdot p_h(t_n)). \end{aligned} \tag{27}$$

Using ε-Young inequality gives

$$\begin{aligned} &(2\triangle t)^{-1}(\|\alpha^{\frac{1}{2}} p_h(t_n)\|^2 - \|\alpha^{\frac{1}{2}} p_h(t_{n-1})\|^2) + \|\nabla \cdot p_h(t_n)\|^2 \\ &\quad \le \sum_{j=0}^{n-1} \triangle t \|\nabla \cdot p_h(t_j)\|^2 + \|f(t_n)\|^2 + \varepsilon \|\nabla \cdot p_h(t_n)\|^2. \end{aligned} \tag{28}$$

Summing from 1 to n with sufficiently small ε yields

$$\begin{aligned} &(2\triangle t)^{-1}(\|\alpha^{\frac{1}{2}} p_h(t_n)\|^2 - \|\alpha^{\frac{1}{2}} p_h(0)\|^2) + \sum_{i=1}^{n} \|\nabla \cdot p_h(t_i)\|^2 \\ &\quad \le \sum_{i=1}^{n} (\sum_{j=0}^{i-1} \triangle t \|\nabla \cdot p_h(t_j)\|^2) + \sum_{i=1}^{n} \|f(t_i)\|^2. \end{aligned} \tag{29}$$

Applying discrete Gronwall's lemma, we get

$$\sum_{i=1}^{n} \|\nabla \cdot p_h(t_i)\|^2 \le C \sum_{i=1}^{n} \|f(t_i)\|^2. \tag{30}$$

Therefore, there holds

$$\|p_h(t_n)\| \le C(\triangle t)^{\frac{1}{2}}(\|p_h(0)\| + \sum_{i=1}^{n} \|f(t_i)\|). \tag{31}$$

The problem (26(b)) has a unique solution according to the theory of differential equations[13]. Then (26) has a unique solution.

Theorem 4 *For u, u_t, $u_{tt} \in H^2(\Omega)$, p, $p_t \in (H^1(\Omega))^2$, $p_{tt} \in (L^2(\Omega))^2$, there hold*

$$\begin{aligned}|u(t_n) - u_h(t_n)|_h \le & Ch\Big[|u|_2 + |p|_1 + \Big(\int_0^{t_n} (\triangle t|p_t(\tau)|_1^2 + \triangle t|u_{tt}|_2^2 + |u_t(\tau)|_2^2)\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big] + \\ & C(\triangle t)\Big(\int_0^{t_n} \|p_{tt}(\tau)\|^2 \mathrm{d}\tau\Big)^{\frac{1}{2}} + C(\triangle t)^{\frac{3}{2}}\Big(\int_0^{t_n} (|p_t(\tau)|_1^2 + |p(\tau)|_1^2)\mathrm{d}\tau\Big)^{\frac{1}{2}}, \end{aligned} \tag{32}$$

and

$$\begin{aligned}\|p(t_n) - p_h(t_n)\|_h \le & Ch\Big[|p|_2 + |p|_1 + \Big(\int_0^{t_n} (\triangle t|p_t(\tau)|_1^2 + \triangle t|u_{tt}|_2^2 + |u_t(\tau)|_2^2)\mathrm{d}\tau\Big)^{\frac{1}{2}}\Big] + \\ & C(\triangle t)\Big(\int_0^{t_n} \|p_{tt}(\tau)\|^2 \mathrm{d}\tau\Big)^{\frac{1}{2}} + C(\triangle t)^{\frac{3}{2}}\Big(\int_0^{t_n} (|p_t(\tau)|_1^2 + |p(\tau)|_1^2)\mathrm{d}\tau\Big)^{\frac{1}{2}}. \end{aligned} \tag{33}$$

Proof Let $u(t_n) - u_h(t_n) = u(t_n) - I_h^1 u(t_n) + I_h^1 u(t_n) - u_h(t_n) = \eta^n + \xi^n$, $p(t_n) - p_h(t_n) = p(t_n) - I_h^2 p(t_n) + I_h^2 p(t_n) - p_h(t_n) = \sigma^n + \theta^n$.

For all $v_h \in V_h, w_h \in W_h$, we have by our definitions:

$$\begin{cases} \text{(a) } (\nabla \xi^n, \nabla v_h) = (\alpha(\sigma^n + \theta^n), \nabla v_h), \\ \text{(b) } (\alpha \bar{\partial}_t \theta^n, w_h) + (\nabla \cdot \theta^n, \nabla \cdot w_h) + \sum_{j=0}^{n-1} \triangle t(\beta(t_n, t_j)\nabla \cdot \theta^j, \nabla \cdot w_h) \\ \qquad = (\varepsilon_n^1, v_h) + (\varepsilon_n^2, \nabla \cdot w_h) + \sum_K \int_{\partial K} u_t(t_n) \cdot n w_h \mathrm{d}s, \end{cases} \tag{34}$$

where

$$\begin{aligned}\varepsilon_n^1 =& \bar{\partial}_t I_h^2 p(t_n) - p_t(t_n) = \bar{\partial}_t \sigma^n + \bar{\partial}_t p(t_n) - p_t(t_n) \\ =& \frac{1}{\triangle t}\int_{t_{n-1}}^{t_n} \sigma_t(\tau)\mathrm{d}\tau + \frac{1}{\triangle t}\int_{t_{n-1}}^{t_n} (t_n - \tau)p_{tt}(\tau)\mathrm{d}\tau, \\ \varepsilon_n^2 =& \sum_{j=0}^{n-1} \triangle t \beta(t_n, t_j)\nabla \cdot p(t_j) - \int_0^{t_n} \beta(t_n, \tau)\nabla \cdot p(\tau)\mathrm{d}\tau \\ =& -\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{\tau}^{t_j} (\beta_t(t_n, s)\nabla \cdot p(s) + \beta(t_n, s)\nabla \cdot p_t(s))\mathrm{d}s\mathrm{d}\tau. \end{aligned}$$

For ε_n^1 and ε_n^2, we have the following estimates

$$\|\varepsilon_n^1\|^2 \le \int_{t_{n-1}}^{t_n} \|\sigma_t(\tau)\|^2 \mathrm{d}\tau + \triangle t \int_{t_{n-1}}^{t_n} \|p_{tt}(\tau)\|^2 \mathrm{d}\tau,$$

$$\|\varepsilon_n^2\|^2 \le C(\triangle t)^2 \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} (|p_t(\tau)|_1^2 + |p(\tau)|_1^2)\mathrm{d}\tau.$$

Setting $v_h = \xi^n$ in (34(a)) and using Cauchy-Schwartz inequality, with the boundary of α, yields

$$\|\nabla \xi^n\| \le C(\|\sigma^n\| + \|\theta^n\|). \tag{35}$$

Similarly, by choosing $w_h = \bar{\partial}_t\theta^n$ in (34(b)) and using Cauchy-Schwartz's inequality and ε-Young's inequality, we obtain

$$\begin{aligned}
&\|\alpha^{\frac{1}{2}}\bar{\partial}_t\theta^n\|^2 + \frac{1}{2\triangle t}(\|\theta^n\|_h^2 - \|\theta^{n-1}\|_h^2 + \|\theta^n - \theta^{n-1}\|_h^2) \\
&= (\varepsilon_n^1, \bar{\partial}_t\theta^n) + (\varepsilon_n^2, \bar{\partial}_t\nabla\cdot\theta^n) - \sum_{j=0}^{n-1}\triangle t(\beta(t_n,t_j)\nabla\cdot\theta^j, \bar{\partial}_t\nabla\cdot\theta^n) + \sum_K\int_{\partial K} u_t(t_n)n\cdot w_h\mathrm{d}s \\
&\le \|\varepsilon_n^1\|^2 + \|\varepsilon_n^2\|^2 + \varepsilon\|\bar{\partial}_t\theta^n\|^2 + \varepsilon\triangle t^{-1}\|\theta^n - \theta^{n-1}\|_h^2 + \sum_{j=0}^{n-1}(\triangle t\|\theta^j\|_h^2) + Ch^2|u_t(t_n)|_2^2. \qquad (36)
\end{aligned}$$

With sufficiently small ε, and the boundary of α, we get

$$\|\theta^n\|_h^2 - \|\theta^{n-1}\|_h^2 \le C(\triangle t)\Big[\|\varepsilon_n^1\|^2 + \|\varepsilon_n^2\|^2 + \sum_{j=0}^{n-1}(\triangle t\|\theta^j\|_h^2) + h^2|u_t(t_n)|_2^2\Big]. \qquad (37)$$

Summing from 1 to n with respect to time and noting $\theta(0) = 0$, we obtain

$$\|\theta^n\|_h^2 \le C\triangle t)\Big[\sum_{i=1}^{n}(\|\varepsilon_i^1\|^2 + \|\varepsilon_i^2\|^2 + h^2|u_t(t_i)|_2^2) + \sum_{i=1}^{n}\triangle t\|\theta^j\|_h^2\Big]. \qquad (38)$$

Note that $\sum_{i=0}^{n-1}|u_t(t_i)|_2^2 \le \int_0^{t_n}|u_{tt}|_2^2\mathrm{d}\tau + \frac{1}{\triangle t}\int_0^{t_n}|u_t|_2^2\mathrm{d}\tau$. Applying discrete Gronwall's lemma and Lemma 2 gives

$$\begin{aligned}
\|\theta^n\|_h^2 \le & Ch^2\Big[\int_0^{t_n}(\triangle t|p_t(\tau)|_1^2 + \triangle t|u_{tt}|_2^2 + |u_t(\tau)|_2^2)\mathrm{d}\tau\Big] + C(\triangle t)^2\int_0^{t_n}\|p_{tt}(\tau)\|^2\mathrm{d}\tau + \\
& C(\triangle t)^3\int_0^{t_n}(|p_t(\tau)|_1^2 + |p(\tau)|_1^2)\mathrm{d}\tau. \qquad (39)
\end{aligned}$$

Substituting these estimates into (35), we have

$$\begin{aligned}
\|\nabla\xi^n\| \le & Ch\Big[(\int_0^{t_n}(\triangle t|p_t(\tau)|_1^2 + \triangle t|u_{tt}|_2^2 + |u_t(\tau)|_2^2)\mathrm{d}\tau)^{\frac{1}{2}}\Big] + C(\triangle t)\Big(\int_0^{t_n}\|p_{tt}(\tau)\|^2\mathrm{d}\tau\Big)^{\frac{1}{2}} + \\
& C(\triangle t)^{\frac{3}{2}}\Big(\int_0^{t_n}(|p_t(\tau)|_1^2 + |p(\tau)|_1^2)\mathrm{d}\tau\Big)^{\frac{1}{2}}. \qquad (40)
\end{aligned}$$

With the help of triangle inequality, Lemma 2, (39) and (40), we complete the proof. □

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