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Weighted Norm Inequalities for Potential Type Operators

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Abstract Let Φ be a non-negative locally integrable function on $\mathbb{R}^n$ and satisfy some weak growth conditions, define the potential type operator T_Φ by

$$T_\Phi f(x)=\int_{\mathbb{R}^n}\Phi(x-y)f(y)\mathrm{d}y.$$

The aim of this paper is to give several strong type and weak type weighted norm inequalities for the potential type operator T_Φ.

Keywords potential type operators; weight; maximal function.

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1. Introduction

For a non-negative, locally integrable function Φ on $\mathbb{R}^n$, define the potential type operator T_Φ by

$$T_\Phi f(x)=\int_{\mathbb{R}^n}\Phi(x-y)f(y)\mathrm{d}y. \tag{1.1}$$

Although the basic example is provided by the Riesz potentials or fractional integrals I_α, defined by the kernel $\Phi(x)=|x|^{\alpha-n}$, $0<\alpha<n$, there are other important examples such as the bessel potentials. They are denoted by $J_{\beta,\lambda}$, $\beta,\lambda>0$, and the kernel $\Phi(x)=K_{\beta,\lambda}(x)$ is best defined by means of its Fourier transform $\hat{K}_{\beta,\lambda}(\xi)=(\lambda^2+|\xi|^2)^{-\beta/2}$.

Now assume that the kernel Φ satisfies the following weak growth condition: there are constants δ, c, $0\le\varepsilon<1$, with the property that for all $k\in\mathbb{Z}$,

$$\sup_{2^k<|x|\le 2^{k+1}}\Phi(x)\le\frac{c}{2^{kn}}\int_{\delta(1-\varepsilon)2^k<|x|\le 2\delta(1+\varepsilon)2^k}\Phi(x)\mathrm{d}x. \tag{1.2}$$

For any kernel Φ, we define the corresponding positive function $\widetilde{\Phi}$ as follows

$$\widetilde{\Phi}(t)=\int_{|x|\le t}\Phi(x)\mathrm{d}x, t\ge 0.$$

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Pérez[1] studied the two-weight strong type (p,q) inequalities for T_Φ, $1<p\le q<\infty$. In this paper, using the techniques developed in [1,2], we give several weighted norm inequalities for the potential type operators.

2. Preliminaries and main results

To state our result, we recall the mean Luxemburg norm on cube (see [3] for further information). A function $B:[0,\infty)\to[0,\infty)$ is a Young function if it is continuous, convex, increasing function with $B(0)=0$ and such that $B(t)\to\infty$ as $t\to\infty$. Given a locally integrable function f and a Young function B, define the mean Luxemburg norm of f on a cube Q by

$$\|f\|_{B,Q}=\inf\Big\{\lambda>0:\frac{1}{|Q|}\int_Q B\Big(\frac{|f(x)|}{\lambda}\Big)\mathrm{d}x\le 1\Big\}.$$

Specially, for the Young function $B(t)=t\log(1+t)^\delta$, $\delta>0$, its Luxemburg norm is also denoted by $\|\cdot\|_{L(\log L)^\delta,Q}$.

A Young function B is doubling if it satisfies $B(2t)\le CB(t)$ for any $t>0$. For a Young function B, there exists a complementary Young function $\bar{B}$ such that $t\le B^{-1}(t)\bar{B}^{-1}(t)\le 2t$ for all $t>0$. For any Young function B, the following Hölder's inequality is true:

$$\frac{1}{|Q|}\int_Q|f(x)g(x)|\mathrm{d}x\le 2\|f\|_{B,Q}\|g\|_{\bar{B},Q}. \tag{2.1}$$

For a non-decreasing function $\phi:[0,\infty)\to[0,\infty)$ and the Young function $B(t)=t\log(1+t)^\delta$, $\delta\ge 0$, we define the maximal operator associated to ϕ and B by

$$M_{\phi,L(\log L)^\delta}f(x)=\sup_{Q\ni x}\phi(l(Q))\|f\|_{L(\log L)^\delta,Q},$$

where $l(Q)$ denotes the side-length of cube Q. When $\delta=0$, we write $M_{\phi,L(\log L)^\delta}$ as M_ϕ.

Our main results are the following theorems.

Theorem 2.1 *Let T_Φ be the potential type operator defined by (1.1) with Φ satisfying condition (1.2).*

(1) If $0<p\le 1$, then there is a constant $C>0$ such that for any weight w and all f,

$$\int_{\mathbb{R}^n}|T_\Phi f(x)|^p w(x)\mathrm{d}x\le C\int_{\mathbb{R}^n}(M_{\widetilde{\Phi}}f(x))^p Mw(x)\mathrm{d}x. \tag{2.2}$$

(2) If $1<p<\infty$, then there is a constant $C>0$ such that for any weight w and all f,

$$\int_{\mathbb{R}^n}|T_\Phi f(x)|^p w(x)\mathrm{d}x\le C\int_{\mathbb{R}^n}(M_{\widetilde{\Phi}}f(x))^p M^{[p]+1}w(x)\mathrm{d}x. \tag{2.3}$$

Theorem 2.2 *Let T_Φ be the potential type operator defined by (1.1) with Φ satisfying condition (1.2). If $1<p<\infty$, then there is a constant $C>0$ such that for any weight w and all f,*

$$\int_{\mathbb{R}^n}|T_\Phi f(x)|^p w(x)\mathrm{d}x\le C\int_{\mathbb{R}^n}|f(x)|^p M(M_{\widetilde{\Phi}^p,L(\log L)^{[p]}})w(x)\mathrm{d}x. \tag{2.4}$$

Theorem 2.3 *Let T_Φ be the potential type operator defined by (1.1) with Φ satisfying condition (1.2)* *(a) If $0 < p \le 1$, then for any $\delta > 0$ there is a constant $C > 0$ such that for any weight w and all f,*

$$\|T_\Phi f\|_{L^{p,\infty}(w)} \le C\|M_{\widetilde{\Phi}} f\|_{L^{p,\infty}(M_{L(\log L)^\delta}(w))}. \tag{2.5}$$

(b) If $1 < p < \infty$, then for any $\delta > 0$ there is a constant $C > 0$ such that for any weight w and all f,

$$\|T_\Phi f\|_{L^{p,\infty}(w)} \le C\|M_{\widetilde{\Phi}} f\|_{L^{p,\infty}(M_{L(\log L)^{p-1+\delta}}(w))}. \tag{2.6}$$

Remark 2.4 For $0 < \alpha < n$, the case $\Phi(x) = |x|^{\alpha-n}$ corresponds to the Riesz potential I_α of order α. In this case $\widetilde{\Phi}(t) \cong t^\alpha$ and the maximal operator $M_{\widetilde{\Phi}}$ is the classical fractional maximal operator M_α. For the fractional integral I_α, Pérez[4] obtained the result for $1 \le p < \infty$ in Theorem 2.1 and a similar result as in Theorem 2.2, Carro, et al[2] got the result for $0 < p \le 1$ in Theorem 2.1 and the result for $p = 1$ in Theorem 2.3.

Let $1 < p < \infty$. We say that a doubling Young function B satisfies the B_p-condition if there is a positive constant c such that

$$\int_c^\infty \frac{B(t)}{t^p}\frac{\mathrm{d}t}{t} \approx \int_c^\infty \Big(\frac{t^{p'}}{\bar{B}(t)}\Big)^{p-1}\frac{\mathrm{d}t}{t} < \infty.$$

Lemma 2.5[5] *Let B be a doubling Young function and p satisfy $1 < p < \infty$. Then $B \in B_p$ if and only if $M_B : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded.*

3. Proof of the theorems

We need the following Lemmas.

Lemma 3.1[1] *Let T_Φ be the potential type operator defined by (1.1) with Φ satisfying condition (1.2), f and g be nonnegative bounded functions with compact support, and let μ be a non-negative measure finite on compact sets. Let $a > 2^n$. Then there exist a family of cubes $\{Q_{k,j}\}$, and a family of pairwise disjoint subsets $\{E_{k,j}\}$, $E_{k,j} \subset Q_{k,j}$, with $|Q_{k,j}| < (1 - 2^n/a)^{-1}|E_{k,j}|$ for all k, j, such that*

$$\int_{\mathbb{R}^n} T_\Phi f(x)g(x)\mathrm{d}\mu(x) \le C\sum_{k,j}\frac{\tilde{\Phi}(l(\gamma Q_{k,j}))}{|\gamma Q_{k,j}|}\int_{\gamma Q_{k,j}} f(y)\mathrm{d}y\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} g(y)\mathrm{d}\mu(y)|E_{k,j}|, \tag{3.1}$$

where $\gamma = \max\{3, \delta(1+\varepsilon)\}$, δ, ε are the numbers provided by condition (1.2).

A weight v satisfies RH_∞ condition, if there is a constant $C > 0$ such that for each cube Q,

$$\operatorname*{ess\,sup}_{x\in Q} v(x) \le \frac{C}{|Q|}\int_Q v.$$

It is very easy to check that $RH_\infty \subset A_\infty$.

Lemma 3.2 *Let T_Φ be the potential type operator defined by (1.1) with Φ satisfying condition (1.2) and v be a weight satisfying the RH_∞ condition. Then there is a constant C such that for any weight w and all positive f,*

$$\int_{\mathbb{R}^n} T_\Phi f(x)w(x)v(x)\mathrm{d}x \le C\int_{\mathbb{R}^n} M_{\widehat{\Phi}} f(x)Mw(x)v(x)\mathrm{d}x. \tag{3.2}$$

Proof We start with inequality (3.1) with g replaced by w and $d\mu$ replaced by $v(x)\mathrm{d}x$:

$$\begin{aligned}&\int_{\mathbb{R}^n} T_\Phi f(x)w(x)v(x)\mathrm{d}x\\&\leq C\sum_{k,j}\frac{\tilde{\Phi}(l(\gamma Q_{k,j}))}{|\gamma Q_{k,j}|}\int_{\gamma Q_{k,j}} f(y)\mathrm{d}y\int_{Q_{k,j}} w(y)v(y)\mathrm{d}y\\&\leq C\sum_{k,j}\frac{\tilde{\Phi}(l(\gamma Q_{k,j}))}{|\gamma Q_{k,j}|}\int_{\gamma Q_{k,j}} f(y)\mathrm{d}y\int_{Q_{k,j}} w(y)\mathrm{d}y \ \mathrm{ess}\sup_{x\in Q_{k,j}} v\\&\leq C\sum_{k,j}\frac{\tilde{\Phi}(l(\gamma Q_{k,j}))}{|\gamma Q_{k,j}|}\int_{\gamma Q_{k,j}} f(y)\mathrm{d}y\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} w(y)\mathrm{d}yv(Q_{k,j}).\end{aligned}$$

Since $v\in RH_\infty$, by the properties of the sets $\{E_{k,j}\}$, we have $v(Q_{k,j})\leq Cv(E_{k,j})$. Combining this with the fact that the family $\{E_{k,j}\}$ is formed by pairwise disjoint subsets, with $E_{k,j}\subset Q_{k,j}$, we continue with

$$\begin{aligned}&\int_{\mathbb{R}^n} T_\Phi f(x)w(x)v(x)\mathrm{d}x\\&\leq C\sum_{k,j}\frac{\tilde{\Phi}(l(\gamma Q_{k,j}))}{|\gamma Q_{k,j}|}\int_{\gamma Q_{k,j}} f(y)\mathrm{d}y\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}} w(y)\mathrm{d}yv(E_{k,j})\\&\leq C\sum_{k,j}\int_{E_{k,j}} M_{\tilde{\Phi}}f(x)Mw(x)v(x)\mathrm{d}x\\&\leq C\int_{\mathbb{R}^n} M_{\tilde{\Phi}}f(x)Mw(x)v(x)\mathrm{d}x.\end{aligned}$$

This completes the proof of Lemma 2.2. □

Lemma 3.3[6] *Let g be any function such that Mg is finite a.e. Then $(Mg)^{-\alpha}\in RH_\infty, \alpha>0$.*

Proof of Theorem 2.1 Using (3.2) with $v(x)=1$ and the extrapolation Theorem 1.1 in [7], we obtain (2.3) immediately. The case $\Phi(x)=|x|^{\alpha-n}$ of (2.2) was proved in [4] and the same proof works for (2.2) with the obvious changes, and we omit the details. This concludes the proof of Theorem 2.1. □

Proof of Theorem 2.2 In fact we will prove something sharper than (2.4): for $1<p<\infty$ and $\delta>0$, there is a constant C such that for any weight w and all f,

$$\int_{\mathbb{R}^n}|T_\Phi f(y)|^p w(y)\mathrm{d}y\leq C\int_{\mathbb{R}^n}|f(y)|^p M(M_{\widetilde{\Phi}^p,L(\log L)^{p-1+\delta}})w(y)\mathrm{d}y, \tag{3.3}$$

where $M_{\widetilde{\Phi}^p,L(\log L)^{p-1+\delta}}$ denotes the maximal operator associated to $\widetilde{\Phi}^p$ and

$$B(t)=t\log(1+t)^{p-1+\delta},\quad t>0.$$

Selecting $\delta>0$ such that $p-1+\delta=[p]$, we get (2.4).

By Lemma 3.1, we have

$$\int_{\mathbb{R}^n}|T_\Phi f(x)|w(x)\mathrm{d}x$$

$$\leq C\sum_{k,j}\frac{1}{|Q_{k,j}|}\int_{Q_{k,j}}|f(x)|\mathrm{d}x\frac{\widetilde{\Phi}(\gamma l(Q_{k,j}))}{|\gamma Q_{k,j}|}\int_{\gamma Q_{k,j}}w(x)\mathrm{d}x|E_{k,j}|$$
$$\leq C\sum_{k,j}\int_{E_{k,j}}Mf(y)M_{\widetilde{\Phi}}w(y)\mathrm{d}y\leq C\int_{\mathbb{R}^n}Mf(y)M_{\widetilde{\Phi}}w(y)\mathrm{d}y. \tag{3.4}$$

Our argument will be based on duality. By (3.4), there is a constant C such that for all $g\geq 0$,

$$\int_{\mathbb{R}^n}|T_\Phi f(y)|g(y)w(y)^{1/p}\mathrm{d}y\leq C\int_{\mathbb{R}^n}Mf(y)M_{\widetilde{\Phi}}(gw^{1/p})(y)\mathrm{d}y.$$

We choose $A(t)\approx t^{p'}(\log t)^{-1-(p'-1)\delta}$ for large t, and then $\bar{A}(t)\approx t^p(\log t)^{p-1+\delta}$. It is easy to check that $A\in B_{p'}$. By the Hölder inequality (2.1) and Lemma 2.5, we can continue with

$$\begin{aligned}
&\leq C\int_{\mathbb{R}^n}Mf(y)M_{\widetilde{\Phi},\bar{A}}(w^{1/p})(y)M_Ag(y)\mathrm{d}y\\
&\leq C\Big(\int_{\mathbb{R}^n}Mf(y)^p(M_{\widetilde{\Phi},\bar{A}}(w^{1/p})(y))^p\mathrm{d}y\Big)^{1/p}\Big(\int_{\mathbb{R}^n}M_Ag(y)^{p'}\mathrm{d}y\Big)^{1/p'}\\
&\leq C\Big(\int_{\mathbb{R}^n}Mf(y)^pM_{\widetilde{\Phi}^p,L(\log L)^{p-1+\delta}}(w)(y)\mathrm{d}y\Big)^{1/p}\Big(\int_{\mathbb{R}^n}g(y)^{p'}\mathrm{d}y\Big)^{1/p'}\\
&\leq C\Big(\int_{\mathbb{R}^n}|f(y)|^pM(M_{\widetilde{\Phi}^p,L(\log L)^{p-1+\delta}})w(y)\mathrm{d}y\Big)^{1/p}\Big(\int_{\mathbb{R}^n}g(y)^{p'}\mathrm{d}y\Big)^{1/p'}.
\end{aligned}$$

Thus (3.3) is true. This concludes the proof of Theorem 2.2. □

Proof of Theorem 2.3 By standard density arguments, we may assume that both f and the weight w are non-negative bounded function with compact support. Raising the quantity $\|T_\Phi f\|_{L^{p,\infty}(w)}$ to the power $1/q$, with $pq>1$ ($q>1$ will be choosen at the end of the proof), gives

$$\begin{aligned}
\|T_\Phi f\|_{L^{p,\infty}(w)}^{1/q}&=\|(T_\Phi f)^{1/q}\|_{L^{pq,\infty}(w)}\\
&=\sup_{g\in L^{(pq)',1}(w),\|g\|_{L^{(pq)',1}(w)}=1}\int_{\mathbb{R}^n}(T_\Phi f(x))^{1/q}g(x)w(x)\mathrm{d}x.
\end{aligned}$$

The last equality follows since $L^{p',1}(w)$ and $L^{p,\infty}(w)$ are associate spaces. Fixing one of these g's, and using (2.2) and Hölder's inequality for Lorentz spaces, for any $\varepsilon>0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}(T_\Phi f(x))^{1/q}g(x)w(x)\mathrm{d}x\\
&\leq C\int_{\mathbb{R}^n}(M_{\widetilde{\Phi}}f(x))^{1/q}M(gw)(x)\mathrm{d}x\\
&=C\int_{\mathbb{R}^n}(M_{\widetilde{\Phi}}f(x))^{1/q}\frac{M(gw)(x)}{M_{L(\log L)^{pq-1+2\varepsilon}}w(x)}M_{L(\log L)^{pq-1+2\varepsilon}}w(x)\mathrm{d}x\\
&\leq C\|(M_{\widetilde{\Phi}}f)^{1/q}\|_{L^{pq,\infty}(M_{L(\log L)^{pq-1+2\varepsilon}}w)}\|\frac{M(gw)}{M_{L(\log L)^{pq-1+2\varepsilon}}w}\|_{L^{(pq)',1}(M_{L(\log L)^{pq-1+2\varepsilon}}w)}\\
&=C\|M_{\widetilde{\Phi}}f\|_{L^{p,\infty}(M_{L(\log L)^{pq-1+2\varepsilon}}w)}^{1/q}\Big\|\frac{M(gw)}{M_{L(\log L)^{pq-1+2\varepsilon}}w}\Big\|_{L^{(pq)',1}(M_{L(\log L)^{pq-1+2\varepsilon}}w)}.
\end{aligned}$$

To conclude the proof, we just need to show that

$$\Big\|\frac{M(gw)}{M_{L(\log L)^{pq-1+2\varepsilon}}w}\Big\|_{L^{(pq)',1}(M_{L(\log L)^{pq-1+2\varepsilon}}w)} \le C\|g\|_{L^{(pq)',1}(w)},$$

or equivalently

$$S:\ L^{(pq)',1}(w) \to L^{(pq)',1}(M_{L(\log L)^{pq-1+2\varepsilon}}w),$$

where

$$Sf = \frac{M(fw)}{M_{L(\log L)^{pq-1+2\varepsilon}}w}.$$

Notice that $Mw \le M_{L(\log L)^{pq-1+2\varepsilon}}w$ for each w. With this we trivially have

$$S:\ L^{\infty}(w) \to L^{\infty}(M_{L(\log L)^{pq-1+\varepsilon}}w).$$

Therefore by Marcinkiewicz's interpolation theorem for Lorentz spaces due to [3], it will be enough to show that: for $\varepsilon > 0$,

$$S:\ L^{(pq+\varepsilon)'}(w) \to L^{(pq+\varepsilon)'}(M_{L(\log L)^{pq-1+2\varepsilon}}w).$$

Which amounts to proving

$$\int_{\mathbb{R}^n} M(wf)(y)^{(pq+\varepsilon)'}M_{L(\log L)^{pq-1+2\varepsilon}}w(y)^{1-(pq+\varepsilon)'}\mathrm{d}y \le C\int_{\mathbb{R}^n} f(y)^{(pq+\varepsilon)'}w(y)\mathrm{d}y.$$

But this result follows from [5]: indeed it is shown there that, for $r > 1$, $\eta > 0$,

$$\int_{\mathbb{R}^n} M(f)(y)^{r'}M_{L(\log L)^{r-1+\eta}}(w)(y)^{1-r'}\mathrm{d}y \le C\int_{\mathbb{R}^n} f(y)^{r'}w(y)^{1-r'}\mathrm{d}y.$$

We finally choose the appropriate parameters. Let $r = pq+\varepsilon$, $\eta = \varepsilon$. This shows that for any $pq > 1$ and $\varepsilon > 0$,

$$\|T_\Phi f\|_{L^{p,\infty}(w)} \le C\|M_\Phi f\|_{L^{p,\infty}(M_{L(\log L)^{pq-1+2\varepsilon}}(w))}.$$

We conclude the proof of (2.5) by choosing $q = \frac{1}{p}(1+\delta-2\varepsilon)$ and (2.6) by choosing $q = 1+\frac{\delta-2\varepsilon}{p}$ for $1 < p < \infty$, where ε satisfies $0 < 2\varepsilon < \delta$. This concludes the proof of Theorem 2.3. □

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