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Extended Cesàro Operators on μ-Bloch Spaces in C^n

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Abstract Let μ, $\nu \in [0,1)$ be normal functions and g be holomorphic function on the unit ball. In this paper, we prove that the generalized Cesàro operator $T_g : \beta_\mu \to \beta_\nu$ is bounded and compact.

Keywords Bloch-type space; Cesàro operator; boundedness; compactness.

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1. Introduction

Let B and D be the unit ball and the uint disc of C^n, respectively, and ∂B the boundary of B. $H(B)$ denotes the class of all holomorphic functions on B. A positive continuous function $\mu(r)$ on $[0,1)$ is called normal if there is a constant $b>0$ such that

$$\lim_{r\to 1-}\mu(r)=0,\quad \lim_{r\to 1-}\frac{\mu(r)}{(1-r)^b}=\infty.$$

These normal functions are more general than usual normal functions.

If μ is normal, we extend it to B by $\mu(z)=\mu(|z|)$. It is well known that a holomorphic function f belongs to μ-Bloch space β_μ if and only if

$$\|f\|_\mu=\sup_{z\in B}\mu(|z|)|\nabla f(z)|<\infty,$$

and f belongs to little μ-Bloch space $\beta_{\mu,0}$ if and only if

$$\lim_{|z|\to 1-}\mu(|z|)|\nabla f(z)|=0.$$

Here $\nabla f(z)=(\frac{\partial f}{\partial z_1},\ldots,\frac{\partial f}{\partial z_n})$ is the complex gradient of f. Obviously, both β_μ and $\beta_{\mu,0}$ are Banach spaces under the norm $\|f\|_{\beta_\mu}=|f(0)|+\|f\|_\mu$ and $\beta_{\mu,0}$ is a closed subspace of β_μ. When $\mu(r)=(1-r^2)^\alpha$ with $\alpha\ge 1$, μ-Bloch space is α-Bloch space, the weighted Bloch space.

Let $g\in H(B)$. The extended Cesàro operator T_g on $H(B)$ with symbol g is defined as

$$T_gf(z)=\int_0^1 f(tz)Rg(tz)\frac{\mathrm{d}t}{t},\quad f\in H(B), z\in B$$

where $Rg(z)=\langle\nabla g(z),\overline{z}\rangle$ is radial derivative of f.

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For a holomorphic $f(z)$ on D, the extended Cesàro operator acting on f was proved to be bounded or compact on the Hardy spaces, Bergman spaces, weighted Bergman spaces and μ-Bloch spaces[1−6].

In the unit ball, Hu[7−9] obtained some necessary and sufficient conditions that T_g is bounded or compact on $\beta_{(1-r^2)^\alpha}$, mixed norm spaces and Bergman Spaces. Zhang[10] considered the boundedness of T_g between $\beta_{(1-r^2)^p}$ and $\beta_{(1-r^2)^q}$ for $0 < p,\ q < \infty$. The purpose of this work is to obtain the necessary and sufficient conditions on $g \in H(B)$, such that the operator $T_g : \beta_\mu \to \beta_\nu$ (respectively $\beta_{\mu,0} \to \beta_{\nu,0}$) is bounded or compact, which generalizes the result in [6]. The main results are followings:

Theorem A *Let μ, ν be normal and $g \in H(B)$. Then*

(1) $T_g : \beta_\mu \to \beta_\nu$ is bounded if and only if

$$\sup_{z\in B} \nu(|z|)|Rg(z)|\Big(1 + \int_0^{|z|} \frac{1}{\mu(t)}\mathrm{d}t\Big) < \infty,$$

(2) $T_g : \beta_{\mu,0} \to \beta_{\nu,0}$ is bounded if and only if

$$g \in \beta_{\nu,0} \quad \text{and} \quad \sup_{z\in B} \nu(|z|)|Rg(z)|\Big(1 + \int_0^{|z|} \frac{1}{\mu(t)}\mathrm{d}t\Big) < \infty.$$

Theorem B *Let μ, ν be normal and $g \in H(B)$. Then*

(1) The following statements are equivalent:

(i) $T_g : \beta_\mu \to \beta_\nu$ is compact;

(ii) $T_g : \beta_{\mu,0} \to \beta_{\nu,0}$ is compact;

(iii) $$\lim_{|z|\to 1-} \nu(|z|)|Rg(z)|\Big(1 + \int_0^{|z|} \frac{1}{\mu(t)}\mathrm{d}t\Big) = 0. \tag{1.1}$$

(2) $T_g : \beta_\mu \to \beta_\nu$ is compact if and only if $g \in \beta_\nu$ and for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$\nu(|z|)|Rg(z)|\Big(1 + \int_0^{|z|} \frac{1}{\mu(t)}\mathrm{d}t\Big) < \varepsilon \tag{1.2}$$

as $\nu(|z|)|Rg(z)| < \delta$.

2. Some lemmas

In what follows c, c_1, c_2 will stand for positive constants not depending on the functions being condidered, but whose values may change from line to line. The expression $a \simeq b$ means $c_1 a \le b \le c_2 a$.

Lemma 2.1 *Let $f \in H(B)$, $z \in B$, $y \in \partial B$, $\langle z, y\rangle = 0$. Define*

$$J_y f(z) = \langle \nabla f(z), \overline{y}\rangle.$$

If

$$|f(z)| \le \frac{c}{\mu(|z|)}, \tag{2.1}$$

then

$$|J_y f(z)| \le \frac{c}{\mu(|z|)\sqrt{1-|z|^2}}. \tag{2.2}$$

Proof Fix $r \in (0,1)$ and define

$$h(\lambda) = f(re_1 + \lambda e_2)$$

for all $\lambda \in C$, with $|\lambda|^2 < 1 - r^2$. Here $e_1 = (e_1^1, e_1^2, \dots, e_1^n)$, $e_2 = (e_2^1, e_2^2, \dots, e_2^n)$ are a pair of orthogonal unit vectors which could be replaced by any pair of orthogonal unit vectors in C^n. Take $z = re_1 + \lambda e_2$, where $z_i = re_1^i + \lambda e_2^i$. By (2.1), we see

$$|h(\lambda)| \le \frac{c}{\mu(\sqrt{r^2 + |\lambda|^2})}.$$

Let Γ be the circle on which $|\lambda|^2 = \frac{1-r^2}{4}$. Applying the Cauchy formula gives

$$\begin{aligned} |h'(0)| &\le \frac{1}{2\pi}\int_\Gamma \frac{|h(\lambda)|}{|\lambda^2|}|\mathrm{d}\lambda| \le \frac{c}{\mu(\sqrt{r^2+|\lambda|^2})\sqrt{1-r^2}} \\ &= \frac{c}{\mu(|z|)\sqrt{1-|z|^2+|\lambda|^2}}. \end{aligned}$$

Since

$$\begin{aligned} h'(0) &= \frac{\partial f(z)}{\partial \lambda}|_{\lambda=0} = \Big(\frac{\partial f(z)}{\partial z_1}\frac{\partial z_1}{\partial \lambda} + \cdots + \frac{\partial f(z)}{\partial z_n}\frac{\partial z_n}{\partial \lambda}\Big)\Big|_{\lambda=0} \\ &= \frac{\partial f(z)}{\partial z_1}e_2^1 + \cdots + \frac{\partial f(z)}{\partial z_n}e_2^n \\ &= \langle \nabla f(z), \overline{e_2}\rangle, \end{aligned}$$

without loss of generality, we take $e_2 = y$ and obtain

$$\begin{aligned} |J_y f(z)| &= |\langle \nabla f(z), \overline{y}\rangle| = |h'(0)| \\ &\le \frac{c}{\mu(|z|)\sqrt{1-|z|^2+|\lambda|^2}} \le \frac{c}{\mu(|z|)\sqrt{1-|z|^2}}. \end{aligned}$$

Lemma 2.2 *Let μ, ν be normal and $f \in H(B)$. Then*

(1) $f \in \beta_\mu \Leftrightarrow \sup_{z\in B}\mu(|z|)|Rf(z)| < \infty$ and $\|f\|_{\beta_\mu} \approx |f(0)| + \sup_{z\in B}\mu(|z|)|Rf(z)|$;

(2) $f \in \beta_{\mu,0} \Leftrightarrow \lim_{|z|\to 1-}\mu(|z|)|Rf(z)| = 0$.

Proof (1) Suppose $f \in \beta_\mu$. We have

$$\mu(|z|)|Rf(z)| = \mu(|z|)|\langle \nabla f(z), \overline{z}\rangle| \le \mu(|z|)|\nabla f(z)||z|.$$

So

$$\sup_{z\in B}\mu(|z|)|Rf(z)| \le \sup_{z\in B}\mu(|z|)|\nabla f(z)||z| \le \sup_{z\in B}\mu(|z|)|\nabla f(z)|.$$

On the other hand, suppose

$$\sup_{z\in B}\mu(|z|)|Rf(z)| \le c < \infty.$$

It is easy to see

$$|Rf(z)| \le \frac{c}{\mu(|z|)}. \tag{2.3}$$

Notice that all holomorphic functions are bounded in any compact subset of B. So

$$\sup_{|z|\le\frac{1}{2}} \mu(|z|)|\nabla f(z)| < \infty.$$

Now for $|z| > \frac{1}{2}$, there exist unit vectors $u_2, \ldots, u_n$ belonging to $\{y : y \in C^n,\ \langle z, y\rangle = 0\}$ such that $\frac{z}{|z|}, u_2, \ldots, u_n$ is a base of C^n. Hence

$$\begin{aligned}|\nabla f(z)|^2 &= |\langle \nabla f(z), \frac{\overline{z}}{|z|}\rangle|^2 + |\langle \nabla f(z), \overline{u_2}\rangle|^2 + \cdots + |\langle \nabla f(z), \overline{u_n}\rangle|^2 \\ &= \frac{|Rf(z)|^2}{|z|^2} + |J_{u_2} f(z)|^2 + \cdots + |J_{u_n} f(z)|^2. \end{aligned} \tag{2.4}$$

By(2.2), (2.3), when $y \in \partial B$, $\langle z, y\rangle = 0$, we obtain

$$|J_y Rf(z)| \le \frac{c}{\mu(|z|)\sqrt{1-|z|^2}}.$$

Because $J_y f(z) = \int_0^1 J_y Rf(tz)\mathrm{d}t$, with the property of $\mu(r)$, we have

$$\begin{aligned}|J_y f(z)| &\le \int_0^1 |J_y Rf(tz)|\mathrm{d}t < c\int_0^1 \frac{1}{\mu(|tz|)\sqrt{1-|tz|^2}}\mathrm{d}t \\ &= c\frac{1}{\mu(|z|)}\int_0^{|z|} \frac{\mu(|z|)}{\mu(s)\sqrt{1-s^2}}\mathrm{d}s < \frac{c}{\mu(|z|)}. \end{aligned} \tag{2.5}$$

Combining (2.3), (2.4) and (2.5), we obtain

$$|\nabla f(z)|^2 \le \frac{c}{\mu^2(|z|)}.$$

Hence

$$\sup_{z\in B} \mu(|z|)|\nabla f(z)| < \infty.$$

Moreover, there is $\|f\|_{\beta_\mu} \approx |f(0)| + \sup_{z\in B}\mu(|z|)|Rf(z)|$.

(2) The proof is similar to (1) and is omitted here.

Lemma 2.3[11] *Let μ be normal, $\mu(r_s) = 2^{-s}$ $(s = 1, \ldots)$, and $n_s = [\frac{1}{1-r_s}]$, where the symbol $[x]$ means the greatest integer not exceeding x. Write*

$$\varphi(\xi) = 1 + \sum_{s=1}^{\infty} 2^s \xi^{n_s}, \quad \xi \in D. \tag{2.6}$$

Then

(1) $\varphi(r)$ is increasing on $[0, 1)$ and

$$\inf_{r\in[0,1)} \mu(r)\varphi(r) \ge c, \quad \sup_{\xi\in D} \mu(|\xi|)|\varphi(\xi)| \le c.$$

(2) For all $z = (z_1, \ldots, z_n) \in B$ and $\frac{1}{\sqrt{2}} < r < 1$, we have

$$\Big|\int_0^{rz_1} \varphi(t)\mathrm{d}t\Big| \le \int_0^r \varphi(t)\mathrm{d}t \le c\Big\{\varphi(\frac{1}{2}) + \int_0^{r^2} \varphi(t)\mathrm{d}t\Big\}.$$

Lemma 2.4[11] *Let μ be normal.*

(1) If $f \in \beta_\mu$, then

$$|f(z)| \le c\Big(1 + \int_0^{|z|} \frac{1}{\mu(t)}\mathrm{d}t\Big)\|f\|_{\beta_\mu}, \tag{2.7}$$

(2) If $f \in \beta_{\mu,0}$ and $\int_0^1 \frac{1}{\mu(t)}\mathrm{d}t = \infty$, then

$$\lim_{|z|\to 1-} \frac{|f(z)|}{\int_0^{|z|} \frac{1}{\mu(t)}\mathrm{d}t} = 0. \tag{2.8}$$

Lemma 2.5 *Let μ,ν be normal and $g \in H(B)$. Then $T_g : \beta_\mu \to \beta_\nu$ is compact if and only if for any sequence $\{f_j\}$ in β_μ which converges to 0 uniformly on any compact subset of B, we have $\|T_g f_j\|_{\beta_\nu} \to 0$ $(j \to \infty)$.*

Proof It can be proved by Montel theorem and Lemma 2.4. The details are omitted here.

Lemma 2.6 *Let μ,ν be normal, $\int_0^1 \frac{1}{\mu(t)} = \infty$ and $g \in H(B)$. If for any $\varepsilon > 0$, there exists $\delta > 0$, such that*

$$\nu(|z|)|Rg(z)|\Big(1 + \int_0^{|z|} \frac{1}{\mu(t)}\mathrm{d}t\Big) < \varepsilon, \quad \text{as} \;\; \nu(|z|)|Rg(z)| < \delta,$$

then $g \in \beta_{\nu,0}$.

Proof Suppose $g \overline{\in} \beta_{\nu,0}$. There exist $\eta > 0$ and some sequence $\{z^k\} \subset B$ such that $\lim_{k\to\infty} |z^k| = 1$, but for $k = 1, 2, \ldots$

$$\nu(|z^k|)|Rg(z^k)| = \eta > 0.$$

Take $\varepsilon_0 > 0$. There exists $\delta > 0$ such that $\eta \in (0, \delta)$. By condition, when $\nu(|z|)|Rg(z)| = \eta < \delta$, we have

$$\nu(|z|)|Rg(z)|\Big(1 + \int_0^{|z|} \frac{1}{\mu(t)}\mathrm{d}t\Big) < \varepsilon_0.$$

So

$$\nu(|z^k|)|Rg(z^k)|\Big(1 + \int_0^{|z^k|} \frac{1}{\mu(t)}\mathrm{d}t\Big) < \varepsilon_0.$$

On the other hand

$$\nu(|z^k|)|Rg(z^k)|\Big(1 + \int_0^{|z^k|} \frac{1}{\mu(t)}\mathrm{d}t\Big) = \eta\Big(1 + \int_0^{|z^k|} \frac{1}{\mu(t)}\mathrm{d}t\Big) \to \infty, \quad k \to \infty,$$

this is a contradiction. Hence $g \in \beta_{\nu,0}$. □

Lemma 2.7[11] *Let μ be normal and $\int_0^1 \frac{1}{\mu(t)}\mathrm{d}t < \infty$. If sequence $\{f_j\}$is a bounded sequence in β_μ satisfying $f_j \to 0$ uniformly on any compact subset of B. Then $\lim_{j\to\infty} \sup_{z\in B} |f_j(z)| = 0$.*

3. The boundedness of T_g

Proof of Theorem A (1) Sufficiency. Suppose $\sup_{z\in B} \nu(|z|)|Rg(z)|(1 + \int_0^{|z|} \frac{1}{\mu(t)}\mathrm{d}t) < \infty$. We need to prove $T_g : \beta_\mu \to \beta_\nu$ is bounded.

For any $f \in \beta_\mu$, by (2.7) and the property $R[T_g f] = |f(z)||Rg(z)|(z)$[7], we obtain

$$\nu(|z|)|R[T_g f](z)| = \nu(|z|)|f(z)||Rg(z)|$$

$$\leq c\nu(|z|)|Rg(z)|\Big(1+\int_0^{|z|}\frac{1}{\mu(t)}\mathrm{d}t\Big)\|f\|_{\beta_\mu}.$$

Together with $T_gf(0)=0$, we obtain

$$\begin{aligned}\|T_gf\|_{\beta_\nu}&=|T_gf(0)|+\sup_{z\in B}\nu(|z|)|R[T_gf](z)|\\&\leq c\sup_{z\in B}\nu(|z|)|Rg(z)|\Big(1+\int_0^{|z|}\frac{1}{\mu(t)}\mathrm{d}t\Big)\|f\|_{\beta_\mu}.\end{aligned}$$

Hence $\|T_g\|\leq c\sup_{z\in B}\nu(|z|)|Rg(z)|(1+\int_0^{|z|}\frac{1}{\mu(t)}\mathrm{d}t)$. Sufficiency is proved.

Necessity. Suppose $T_g:\beta_\mu\to\beta_\nu$ is bounded. We need to prove

$$\sup_{z\in B}\nu(|z|)|Rg(z)|\Big(1+\int_0^{|z|}\frac{1}{\mu(t)}\mathrm{d}t\Big)<\infty.$$

It is easy to see $g\in\beta_\nu$ from the fact that $g(z)=g(0)+\int_0^1\frac{Rg(tz)}{t}\mathrm{d}t=g(0)+(T_g1)(z)\in\beta_\nu$.

Fix $w\in B$ and take the test function

$$h_w(z)=1+\int_0^{\langle z,w\rangle}\varphi(t)\mathrm{d}t,\quad\forall w\in B.$$

By Lemma 2.3

$$\mu(|z|)|\nabla h_w(z)|=\mu(|z|)|\varphi(\langle z,w\rangle)||w|\leq\mu(|\langle z,w\rangle|)|\varphi(\langle z,w\rangle)|\leq c. \tag{3.1}$$

Hence $h_w\in\beta_\mu$. Therefore

$$c\|T_g\|\geq\|T_gh_w\|_{\beta_\mu}\geq\nu(|w|)|Rg(w)||h_w(w)|=\nu(|w|)|Rg(w)|\Big(1+\int_0^{|w|^2}\varphi(t)\mathrm{d}t\Big).$$

By Lemma 2.3, when $|w|>\frac{1}{\sqrt{2}}$, we have

$$\begin{aligned}\nu(|w|)|Rg(w)|\Big(1+\int_0^{|w|}\frac{1}{\mu(t)}\mathrm{d}t\Big)&\leq\nu(|w|)|Rg(w)|\Big(1+c\int_0^{|w|}\varphi(t)\mathrm{d}t\Big)\\&\leq\nu(|w|)|Rg(w)|\Big\{1+c\Big(\varphi(\frac{1}{2})+\int_0^{|w|^2}\varphi(t)\mathrm{d}t\Big)\Big\}\\&\leq c\|g\|_{\beta_\nu}+c\nu(|w|)|Rg(w)|\Big(1+\int_0^{|w|^2}\varphi(t)\mathrm{d}t\Big)\\&\leq c<\infty\end{aligned}$$

and when $|w|\leq\frac{1}{\sqrt{2}}$, we have

$$\nu(|w|)|Rg(w)|\Big(1+\int_0^{|w|}\frac{1}{\mu(t)}\mathrm{d}t\Big)\leq\|g\|_{\beta_\nu}\Big(1+\int_0^{\frac{1}{\sqrt{2}}}\frac{1}{\mu(t)}\mathrm{d}t\Big)<\infty.$$

Hence

$$\begin{aligned}&\sup_{z\in B}\nu(|z|)|Rg(z)|\Big(1+\int_0^{|z|}\frac{1}{\mu(t)}\mathrm{d}t\Big)\\&\leq\Big(\sup_{|z|\leq\frac{1}{\sqrt{2}}}+\sup_{|z|>\frac{1}{\sqrt{2}}}\Big)\nu(|z|)|Rg(z)|\Big(1+\int_0^{|z|}\frac{1}{\mu(t)}\mathrm{d}t\Big)<\infty.\end{aligned}$$

The proof is completed. □

(2) Sufficiency. Suppose $g \in \beta_{\nu,0}$ and $\sup_{z\in B}\nu(|z|)|Rg(z)|(1+\int_0^{|z|}\frac{1}{\mu(t)}\mathrm{d}t) < \infty$. By (1), we know $T_g : \beta_\mu \to \beta_\nu$ is bounded. So for any $f \in \beta_{\mu,0}$, we need to prove $T_g f \in \beta_{\nu,0}$. Let $A = \sup_{z\in B}\nu(|z|)|Rg(z)|(1+\int_0^{|z|}\frac{1}{\mu(t)}\mathrm{d}t)$.

i) If $\int_0^1 \frac{1}{\mu(t)}\mathrm{d}t = \infty$, by Lemma 2.4, we have $\lim_{|z|\to 1-}\frac{|f(z)|}{\int_0^{|z|}\frac{1}{\mu(t)}\mathrm{d}t} = 0$. That is to say for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\frac{|f(z)|}{\int_0^{|z|}\frac{1}{\mu(t)}\mathrm{d}t} < \varepsilon \quad \text{as} \quad \delta_1 < |z| < 1.$$

Hence

$$\begin{aligned}\nu(|z|)|R[T_g f](z)| &= \nu(|z|)|Rg(z)||f(z)| < c\nu(|z|)|Rg(z)|\Big(1+\int_0^{|z|}\frac{1}{\mu(t)}\mathrm{d}t\Big)\frac{|f(z)|}{\int_0^{|z|}\frac{1}{\mu(t)}\mathrm{d}t}\\ &\le cA\varepsilon.\end{aligned}$$

Therefore $T_g f \in \beta_{\nu,0}$.

ii) If $\int_0^1 \frac{1}{\mu(t)}\mathrm{d}t < \infty$, by $g \in \beta_{\nu,0}$, we know for the above $\varepsilon > 0$ there exists $\delta_2 \in (0,1)$, such that

$$\nu(|z|)|Rg(z)| < \varepsilon \quad \text{as} \quad \delta_2 < |z| < 1.$$

Hence by Lemma 2.4, we obtain

$$\nu(|z|)|R[T_g f](z)| \le c\nu(|z|)|Rg(z)|\Big(1+\int_0^{|z|}\frac{1}{\mu(t)}\mathrm{d}t\Big)\|f\|_{\beta_\mu} \le c\varepsilon.$$

Therefore $T_g f \in \beta_{\nu,0}$.

Combining i) and ii) gives $T_g f \in \beta_{\nu,0}$. Sufficiency is proved.

Necessity. Suppose $T_g : \beta_{\mu,0} \to \beta_{\nu,0}$ is bounded. It is easy to obtain $g \in \beta_{\nu,0}$. On the other hand, the test function h_w in Theorem A belongs to $\beta_{\mu,0}$ as well. Thus, the proof is similar to that in theorem A and is omitted here. The proof is completed. □

4. The compactness of T_g

Proof of Theorem B (1) (i)⇒ (ii). It is obvious and omitted here.

(ii)⇒(iii). Suppose $T_g : \beta_{\mu,0} \to \beta_{\nu,0}$ is compact. It is clear that $g \in \beta_{\nu,0}$. If (1.1) did not hold, there would exist $\varepsilon_0 > 0$ and a sequence $\{z^j\} \subset B$ such that $\lim_{j\to\infty}|z^j| = 1$, but for j=1,2,...

$$\nu(|z^j|)|Rg(z^j)|\Big(1+\int_0^{|z^j|}\frac{1}{\mu(t)}\mathrm{d}t\Big) \ge 2\varepsilon_0. \tag{4.1}$$

We may assume $\{z^j\}$ satisfies $|z^{j+1}| \ge |z^j|$ and $z^j \to z^0 \in \partial B$. Because $\lim_{j\to\infty}\nu(|z^j|)|Rg(z^j)| = 0$, for any j, there exists $k_j > j$ such that

$$\nu(|z^{k_j}|)|Rg(z^{k_j})|\Big(1+\int_0^{|z^j|}\frac{1}{\mu(t)}\mathrm{d}t\Big) < \varepsilon_0. \tag{4.2}$$

Combining (4.1) and (4.2), we obtain

$$\nu(|z^{k_j}|)|Rg(z^{k_j})|\int_{|z^j|}^{|z^{k_j}|}\frac{1}{\mu(t)}\mathrm{d}t>\varepsilon_0. \tag{4.3}$$

Let $F_j(z)=\int_0^{\langle z,\frac{|z^j|}{z^{k_j}}\rangle}\varphi(\zeta)\mathrm{d}\zeta$, $H_j(z)=\int_0^{\langle z,\frac{|z^{k_j}|}{z^{k_j}}\rangle}\varphi(\zeta)\mathrm{d}\zeta$, $G(z)=\int_0^{\langle z,z^0\rangle}\varphi(\zeta)\mathrm{d}\zeta$, where

$$\Big\langle z,\frac{|z^j|}{z^{k_j}}\Big\rangle=\sum_{i=1}^n z_i\frac{|z^j|}{z_i^{k_j}},\ z=(z_1,\dots,z_n),z^{k_j}=(z_1^{k_j},\dots,z_n^{k_j})\in B.$$

It is easy to see $F_j(0)=H_j(0)=G(0)=0$ and the sequences $\{F_j\}$, $\{H_j\}$ converge to $G(z)$ uniformly on any compact subset of B.

Fix j. Then

$$\mu(|z|)|\nabla F_j(z)|=\mu(|z|)\Big|\varphi\Big(\langle z,\frac{|z^j|}{z^{k_j}}\rangle\Big)\Big|\Big|\frac{|z^j|}{z^{k_j}}\Big|\le\mu(|z|)\Big|\varphi\Big(\langle z,\frac{|z^j|}{z^{k_j}}\rangle\Big)\Big|\to 0$$

as $|z|\to 1-$.

It is easy to check that $\mu(|z|)|\nabla H_j(z)|\to 0$ as $|z|\to 1-$. Hence for any j, $F_j,H_j\in\beta_{\mu,0}$, and $\{F_j\}$ and $\{H_j\}$ are bounded sequences on $\beta_{\mu,0}$. Because $T_g:\beta_{\mu,0}\to\beta_{\nu,0}$ is compact, $\{T_gF_j\}$ and $\{T_gH_j\}$ have convergent subsequences. We may assume $\{T_gF_j\}$ and $\{T_gH_j\}$ converge to Φ and Ψ respectively. Then Φ, $\Psi\in\beta_{\nu,0}$.

For any $z\in B$, we have

$$\begin{aligned}(T_gF_j)(z)&=\int_0^1 F_j(tz)Rg(tz)\frac{\mathrm{d}t}{t}=\int_0^1\int_0^{\langle tz,\frac{|z^j|}{z^{k_j}}\rangle}\varphi(\zeta)\mathrm{d}\zeta Rg(tz)\frac{\mathrm{d}t}{t}\\&\to\int_0^1\int_0^{\langle tz,z^0\rangle}\varphi(\zeta)\mathrm{d}\zeta Rg(tz)\frac{\mathrm{d}t}{t}=(T_gG)(z).\end{aligned}$$

Hence $\Phi=T_gG$. It is easy to check that $\Psi=T_gG$. Therefore

$$\|T_gF_j-T_gH_j\|_{\beta_\nu}\le\|T_gF_j-\Phi\|_{\beta_\nu}+\|T_gH_j-\Psi\|_{\beta_\nu}\to 0,\quad j\to\infty. \tag{4.4}$$

On the other hand, by (4.3) we obtain

$$\begin{aligned}\|T_gF_j-T_gH_j\|_{\beta_\nu}&=\sup_{z\in B}\nu(|z|)|Rg(z)||F_j(z)-H_j(z)|\\&\ge\nu(|z^{k_j}|)|Rg(z^{k_j})||F_j(z^{k_j})-H_j(z^{k_j})|\\&\ge c\nu(|z^{k_j}|)|Rg(z^{k_j})|\int_{|z^j|}^{|z^{k_j}|}\frac{1}{\mu(t)}\mathrm{d}t\\&>c\varepsilon_0.\end{aligned}$$

This is a contradiction to (4.4). Hence

$$\lim_{|z|\to 1-}\nu(|z|)|Rg(z)|\Big(1+\int_0^{|z|}\frac{1}{\mu(t)}\mathrm{d}t\Big)=0.$$

iii)⇒ i). Suppose (1.1) holds. For any $\varepsilon>0$, ther exists $\delta_1>0$ such that

$$\nu(|z|)|Rg(z)|\Big(1+\int_0^{|z|}\frac{1}{\mu(t)}\mathrm{d}t\Big)<\varepsilon$$

as $\delta_1 < |z| < 1$. For any $f \in \beta_\mu$,

$$\nu(|z|)|R[T_g f](z)| \le c\nu(|z|)|Rg(z)|\Big(1 + \int_0^{|z|} \frac{1}{\mu(t)}\mathrm{d}t\Big)\|f\|_{\beta_\mu} < c\varepsilon.$$

Hence $T_g f \in \beta_{\nu,0}$. Especially, $g \in \beta_{\nu,0}$ when $f \equiv 1$.

Next we need to prove $T_g : \beta_\mu \to \beta_{\nu,0}$ is compact. Let $\{f_j\}$ be a bounded sequence in β_μ, say $\|f\|_{\beta_\mu} \le 1$ and $f_j(z) \to 0$ uniformly on any compact subset of B.

For the above $\varepsilon > 0$, there exists $\delta_2 > 0$. Take $\delta = \max\{\delta_1, \delta_2\}$ such that

$$\nu(|z|)|R[T_g f_j](z)| \le c\nu(|z|)|Rg(z)|\Big(1 + \int_0^{|z|} \frac{1}{\mu(t)}\mathrm{d}t\Big)\|f_j\|_{\beta_\mu} < c\varepsilon \tag{4.5}$$

as $|z| > \delta$.

Moreover, when $|z| \le \delta$,

$$\nu(|z|)|R[T_g f_j](z)| \le \|g\|_{\beta_\nu}|f_j(z)| < c\varepsilon. \tag{4.6}$$

Combining (4.5) and (4.6) yields

$$\|T_g f_j\|_{\beta_\nu} \to 0, \quad j \to \infty.$$

By Lemma 2.5, we know $T_g : \beta_\mu \to \beta_{\nu,0}$ is compact. The proof is completed. □

(2) Sufficiency.

i) If $\int_0^1 \frac{1}{\mu(t)}\mathrm{d}t = \infty$, by Lemma 2.6 and the condition we know $g \in \beta_{\nu,0}$. So for the δ of the condition, there exists δ' such that

$$\nu(|z|)|Rg(z)| < \delta \quad \text{as} \quad \delta' < |z| < 1.$$

Hence $\lim_{|z|\to 1} \nu(|z|)|Rg(z)|(1 + \int_0^{|z|} \frac{1}{\mu(t)}\mathrm{d}t) = 0$. By result of (1) we know $T_g : \beta_\mu \to \beta_{\nu,0} \subset \beta_\nu$ is compact.

ii) If $\int_0^1 \frac{1}{\mu(t)} < \infty$, by $g \in \beta_\nu$ we know $\sup_{z\in B} \nu(|z|)|Rg(z)|(1 + \int_0^{|z|} \frac{1}{\mu(t)}\mathrm{d}t) < \infty$. Hence by the result of (1) of Theorem A, we obtain that for any $f \in \beta_\mu$, $T_g f \in \beta_\nu$. Let $\{f_j\}$ be a bounded sequence in β_μ, say $\|f_j\|_{\beta_\mu} \le 1$ and $f_j(z) \to 0$ uniformly on any compact subset of B. By Lemma 2.7 we obtain

$$\lim_{j\to\infty} \sup_{z\in B} |f_j(z)| = 0.$$

Hence

$$\|T_g f_j\|_{\beta_\nu} \le \sup_{z\in B} \nu(|z|)|Rg(z)| \sup_{z\in B} |f_j(z)| \to 0, \quad j \to \infty.$$

Therefore $T_g : \beta_\mu \to \beta_\nu$ is compact.

Necessity. Suppose $T_g : \beta_\mu \to \beta_\nu$ is compact. It is easy to check that $g \in \beta_\nu$. If (1.2) did not hold, there would exist $\varepsilon_0 > 0$ and $\{z^j\} \subset B$ such that

$$\nu(|z^j|)|Rg(z^j)| < \frac{1}{j}, \tag{4.7}$$

but

$$\nu(|z^j|)|Rg(z^j)|\Big(1 + \int_0^{|z^j|} \frac{1}{\mu(t)}\mathrm{d}t\Big) \ge 2\varepsilon_0. \tag{4.8}$$

By (4.7) and (4.8) we know $1+\int_0^{|z^j|}\frac{1}{\mu(t)}\mathrm{d}t \geq 2j\varepsilon_0$, so $\lim_{j\to\infty}|z^j| = 1$. We may assume $|z^{j+1}| \geq |z^j|$ and $z^j \to z^0 \in \partial B$. By(4.7) we know for any j, there exists $k_j > j$ such that

$$\nu(|z^{k_j}|)|Rg(z^{k_j})|\Big(1+\int_0^{|z^j|}\frac{1}{\mu(t)}\mathrm{d}t\Big) < \varepsilon_0. \tag{4.9}$$

Combining (4.8) and (4.9) gives

$$\nu(|z^{k_j}|)|Rg(z^{k_j})|\int_{|z^j|}^{|z^{k_j}|}\frac{1}{\mu(t)}\mathrm{d}t > \varepsilon_0.$$

Take functions $F_j(z)$, $H_j(z)$ and $G_j(z)$. The details of the rest are similar to those in the proof of (1) of Theorem B and omitted here.

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