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Green's Relations on a Kind of Semigroups of Linear Transformations

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Abstract Let V be a linear space over a field F with finite dimension, $L(V)$ the semigroup, under composition, of all linear transformations from V into itself. Suppose that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ is a direct sum decomposition of V, where $V_1, V_2, \ldots, V_m$ are subspaces of V with the same dimension. A linear transformation $f \in L(V)$ is said to be sum-preserving, if for each i $(1 \le i \le m)$, there exists some j $(1 \le j \le m)$ such that $f(V_i) \subseteq V_j$. It is easy to verify that all sum-preserving linear transformations form a subsemigroup of $L(V)$ which is denoted by $L^{\oplus}(V)$. In this paper, we first describe Green's relations on the semigroup $L^{\oplus}(V)$. Then we consider the regularity of elements and give a condition for an element in $L^{\oplus}(V)$ to be regular. Finally, Green's equivalences for regular elements are also characterized.

Keywords linear spaces; linear transformations; semigroups; Green's equivalence; regular semigroups.

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1. Introduction and preliminaries

Let X be an arbitrary set, $\mathcal{T}_X$ the full transformation semigroup on the set X and E be an equivalence relation on X. The first author observed in [6] a class of transformation semigroups determined by the equivalence E, namely

$$T_E(X) = \{f \in \mathcal{T}_X : \forall\,(a,b) \in E, (f(a), f(b)) \in E\}.$$

$T_E(X)$ is obviously a subsemigroup of $\mathcal{T}_X$. The common nature of all elements in $T_E(X)$ is that they preserve the decomposition induced by the equivalence E. In other words, all $f \in T_E(X)$ satisfy the condition that for each E-class A there exists some E-class B such that $f(A) \subseteq B$. In recent years, some properties for $T_E(X)$ are investigated in many papers. For example, [7] considered the Green's equivalences, [9] and [10] discussed some subsemigroups of $T_E(X)$ inducing certain lattices of equivalences on the set X, and [8] investigated the rank of $T_E(X)$ for a special case of X and E.

In this paper we examine a related semigroup defined as follows. Let V be a linear space over a field F and $L(V)$ be the semigroup, under composition, of all linear transformations on the

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linear space V. Suppose that $V = \oplus\{V_i : i \in I\}$, where each V_i is a subspace of V with $|I| \geq 2$ and $\dim V_i \geq 2$ for each i. A linear transformation $f \in L(V)$ is called sum-preserving if for each $i \in I$, there exists some $j \in I$ such that $f(V_i) \subseteq V_j$. It is not hard to verify that if f and g are sum-preserving, then so is fg. Consequently, all sum-preserving linear transformations form a subsemigroup of $L(V)$ which will be denoted by $L^{\oplus}(V)$.

We notice that many conclusions for $\mathcal{T}_X$ have their parallelism for $L(V)$. For example, in 1966, Howie[2] characterized the transformations in $\mathcal{T}_X$ that can be written as a product of finite number idempotents in $\mathcal{T}_X$. Since then Erdos[3] and Dawlings[4] gave different proofs of the result that when V is finite-dimensional, $\alpha \in L(V)$ is a finite product of proper idempotents in $L(V)$ if and only if $\dim(\alpha(V)) < \dim V$. Later in 1985, Reynolds and Sullivan[5] investigated the case of infinite-dimensional spaces and obtained the results similar to Howie's.

We may compare the elements in $L^{\oplus}(V)$ with that in $T_E(X)$ and find that all they are transformations of a set (or a linear space) preserving some decomposition. Therefore, $L^{\oplus}(V)$ can be regarded as the linear transformation version of the semigroup $T_E(X)$.

In this paper, we are going to consider a special case for the direct sum decomposition, namely, we assume $\dim V_i = n \geq 2$ for each $i \in I = \{1, 2, \ldots, m\}$ with $m \geq 2$ while

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_m, \quad \dim V_i = n \ (1 \leq i \leq m).$$

Here we focus our attention to Green's equivalence relations and the regularity for the semigroup $L^{\oplus}(V)$. Accordingly, in Section 2, we describe five Green's relations and conclude that $\mathcal{D} = \mathcal{J}$. In Section 3, we consider the condition for an element $f \in L^{\oplus}(V)$ to be regular. By the way, we describe the Green's relations for regular elements in the semigroup $L^{\oplus}(V)$.

In order to avoid repeat, in the remainder of the paper, the symbols $V_i, V_j, V_l, V_{j_s}, \ldots$ will always denote certain subspaces in the direct sum decomposition $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$ without further mention. In addition, if we have defined a number of linear mappings $f_i : V_i \to V_{i'}$ where $i, i' \in I$, then there exists a unique linear transformation $f \in L^{\oplus}(V)$ satisfying $f|V_i = f_i$. Finally, for convenience, we do not distinguish the zero vector 0 and the singleton set $\{0\}$. As we have seen previously, we write $f(V_i) = 0$ to mean $f(V_i) = \{0\}$.

For standard concepts and notations in semigroup theory one can consult [1].

2. Green's relations

In this section, we focus our attention on Green's relations for the semigroup $L^{\oplus}(V)$. We begin with the relation $\mathcal{L}$. Before stating the result, we need some notations.

Let $f \in L^{\oplus}(V)$ with $V_j \cap f(V) \neq 0$. Denote $W_j = \oplus\{V_i : 0 \neq f(V_i) \subseteq V_j\}$. Then it is easy to see that $f(W_j) = V_j \cap f(V)$. Suppose that all the subspaces V_j such that $V_j \cap f(V) \neq 0$ are $V_{j_1}, V_{j_2}, \ldots, V_{j_t}$. Denote $K(f) = \{W_{j_1}, \ldots, W_{j_t}\}$. Denote by $\ker(f)$ the kernel of f, that is, $\ker(f) = \{x \in V : f(x) = 0\}$.

Theorem 2.1 *Let $f, g \in L^{\oplus}(V)$. Then $f\mathcal{L}g$ if and only if $\ker(f) = \ker(g)$ and $K(f) = K(g)$.*

Proof Suppose $f\mathcal{L}g$. Then there exist $u, v \in L^{\oplus}(V)$, such that $uf = g$ and $vg = f$. Hence

$$g(\ker(f)) = uf(\ker(f)) = u(0) = 0.$$

Thus, $\ker(f) \subseteq \ker(g)$. Similarly, $\ker(g) \subseteq \ker(f)$ and $\ker(f) = \ker(g)$. Suppose that

$$K(f) = \{W_{j_1}, \dots, W_{j_t}\} \text{ and } K(g) = \{U_{l_1}, \dots, U_{l_s}\}.$$

Without loss of generality, we may assume that $u(V_{j_1}) \subseteq V_{l_1}$. So

$$g(W_{j_1}) = uf(W_{j_1}) \subseteq u(V_{j_1}) \subseteq V_{l_1}.$$

Clearly, $g(V_i) \neq 0$ for each $V_i \subseteq W_{j_1}$, since $\ker(f) = \ker(g)$. Thus $W_{j_1} \subseteq U_{l_1}$. Assume $f(U_{l_1}) = vg(U_{l_1}) \subseteq v(V_{l_1}) \subseteq V_p$ for some p. Notice that $f = vg = vuf$, $f(W_{j_1}) \subseteq V_{j_1}$ and

$$f(W_{j_1}) = vuf(W_{j_1}) \subseteq vu(V_{j_1}) \subseteq v(V_{l_1}) \subseteq V_p,$$

we have $V_p = V_{j_1}$ and $f(U_{l_1}) \subseteq V_{j_1}$. By $\ker(f) = \ker(g)$ again, $f(V_i) \neq 0$ for each $V_i \subseteq U_{l_1}$. Consequently, $U_{l_1} \subseteq W_{j_1}$ and $W_{j_1} = U_{l_1}$ holds. Similarly, one can verify that each $W \in K(f)$ is equal to some $U \in K(g)$ and $s = t$. Therefore, $K(f) = K(g)$ and the necessity follows.

In order to show the sufficiency, suppose $\ker(f) = \ker(g)$ and $K(f) = K(g)$. We must find some $u, v \in L^{\oplus}(V)$ satisfying $uf = g$ and $vg = f$. Denote $f_i = f|V_i$ and $g_i = g|V_i$ $(1 \leq i \leq m)$. Then $\ker f_i = \ker g_i$. While for each $W \in K(f) = K(g)$, $f|W$ and $g|W$ are linear mappings and

$$\ker(f|W) = \ker(g|W). \tag{2.1.1}$$

If $V_j \cap f(V) \neq 0$, then there exists some $W \in K(f) = K(g)$ such that $f(W) = V_j \cap f(V)$, $g(W) = V_l \cap g(V)$. Let $f(W) = V_j' \subseteq V_j$ and $g(W) = V_l' \subseteq V_l$. From (2.1.1), V_j' and V_l' have the same dimension. Without loss of generality, we may assume $W = V_1 \oplus V_2 \oplus \cdots \oplus V_t$. Take a basis $e_1, \dots, e_{r_1}, e_{r_1+1}, \dots, e_n$ for V_1, a basis $\alpha_1, \dots, \alpha_{r_2}, \alpha_{r_2+1}, \dots, \alpha_n$ for V_2, …, a basis $\beta_1, \dots, \beta_{r_t}, \beta_{r_t+1}, \dots, \beta_n$ for V_t, where $e_{r_1+1}, \dots, e_n$ is a basis for $\ker(f_1)$, $\alpha_{r_2+1}, \dots, \alpha_n$ is a basis for $\ker(f_2)$, …, $\beta_{r_t+1}, \dots, \beta_n$ is a basis for $\ker(f_t)$. Then $\{e_i\} \cup \{\alpha_i\} \cup \cdots \cup \{\beta_i\}$ is a basis for W. While in the subspace V_j', $f(e_1), \dots, f(e_{r_1})$ are linearly independent, and so also are $f(\alpha_1), \dots, f(\alpha_{r_2}), \dots$, and $f(\beta_1), \dots, f(\beta_{r_t})$. It is not difficult to see that

$$V_j' = \langle f(e_1), \dots, f(e_{r_1}), f(\alpha_1), \dots, f(\alpha_{r_2}), \dots, f(\beta_1), \dots, f(\beta_{r_t})\rangle.$$

Now we extend $f(e_1), \dots, f(e_{r_1})$ to obtain a basis for V_j' by adding some $f(\alpha_s)$ $(1 \leq s \leq r_2), \dots$, and $f(\beta_k)$ $(1 \leq k \leq r_t)$. Without loss of generality, we assume the basis is

$$f(e_1), \dots, f(e_{r_1}), f(\alpha_1), \dots, f(\alpha_p), \dots, f(\beta_1), \dots, f(\beta_q). \tag{2.1.2}$$

We claim that

$$g(e_1), \dots, g(e_{r_1}), g(\alpha_1), \dots, g(\alpha_p), \dots, g(\beta_1), \dots, g(\beta_q) \tag{2.1.3}$$

are linearly independent. Otherwise, suppose

$$\sum_{i=1}^{r_1} a_i g(e_i) + \sum_{j=1}^{p} b_j g(\alpha_j) + \cdots + \sum_{k=1}^{q} c_k g(\beta_k) = 0$$

for some $a_i, b_j, c_k \in F$. Let

$$\xi = a_1e_1 + \cdots + a_{r_1}e_{r_1} + b_1\alpha_1 + \cdots + b_p\alpha_p + \cdots + c_1\beta_1 + \cdots + c_q\beta_q \in W.$$

Then $g(\xi) = 0$ and $\xi \in W \cap \ker(g) = W \cap \ker(f)$. Hence

$$0 = f(\xi) = \sum_{i=1}^{r_1} a_i f(e_i) + \sum_{j=1}^{p} b_j f(\alpha_j) + \cdots + \sum_{k=1}^{q} c_k f(\beta_k).$$

Notice that (2.1.2) is linearly independent, the above equation implies that

$$a_1 = \cdots = a_{r_1} = b_1 = \cdots = b_p = \cdots = c_1 = \cdots = c_q = 0.$$

Thus, (2.1.3) are linearly independent, while being a basis for V_l'.

Extend (2.1.2) to a basis B for V_j and define a linear mapping $u_j : V_j \to V_l$ such that

$$u_j(f(e_1)) = g(e_1), \ldots, u_j(f(e_{r_1})) = g(e_{r_1}),$$
$$u_j(f(\alpha_1)) = g(\alpha_1), \ldots, u_j(f(\alpha_p)) = g(\alpha_p),$$
$$\cdots$$
$$u_j(f(\beta_1)) = g(\beta_1), \ldots, u_j(f(\beta_q)) = g(\beta_q),$$

and for each $\eta \in B$ out of (2.1.2), let $u_j(\eta) = 0$. For each V_i, if $V_i \cap f(V) \neq 0$, then define u_i on V_i as above. If $V_i \cap f(V) = 0$, then let $u_i(x) = 0$ for each $x \in V_i$. Thus, these u_i uniquely determine a linear transformation u on the linear space V. Obviously, $u \in L^{\oplus}(V)$.

Now we verify that $uf = g$. For each V_i and $x \in V_i$, if $f(x) = 0$, then $g(x) = 0$ since $\ker(f) = \ker(g)$, and $uf(x) = g(x)$ in this case. If $f(x) \neq 0$, then there exists some $W \in K(f)$ such that $V_i \subseteq W$. Without loss of generality, we assume

$$W = V_1 \oplus V_2 \oplus \cdots \oplus V_t,$$

then $f(x) \in f(W) = V_j' \subseteq V_j$. As above, we assume (2.1.2) to be a basis for V_j'. Then

$$f(x) = \sum_{i=1}^{r_1} a_i f(e_i) + \sum_{j=1}^{p} b_j f(\alpha_j) + \cdots + \sum_{k=1}^{q} c_k f(\beta_k) = f(\xi),$$

where

$$\xi = a_1e_1 + \cdots + a_{r_1}e_{r_1} + b_1\alpha_1 + \cdots + b_p\alpha_p + \cdots + c_1\beta_1 + \cdots + c_q\beta_q.$$

Since $\ker(f) = \ker(g)$, we have $g(x) = g(\xi)$. By the definition of u,

$$uf(x) = u(\sum_{i=1}^{r_1} a_i f(e_i) + \sum_{j=1}^{p} b_j f(\alpha_j) + \cdots + \sum_{k=1}^{q} c_k f(\beta_k)) = g(\xi) = g(x).$$

Thus, $uf(x) = g(x)$ holds for every $x \in V_i$. Consequently, $uf(x) = g(x)$ holds for every $x \in V$ and $uf = g$. Similarly, one can find $v \in L^{\oplus}(V)$ such that $vg = f$. Therefore, $f\mathcal{L}g$ holds. □

Before describing the relation $\mathcal{R}$ on $L^{\oplus}(V)$ some notations should be introduced. Let $f \in L^{\oplus}(V)$. If $V_j \cap f(V) \neq 0$, then there exists some V_i such that $0 \neq f(V_i) \subseteq V_j$. Denote

$$P_j(f) = \{f(V_i) : 0 \neq f(V_i) \subseteq V_j\}$$

and define a partial order $\leq$ on $P_j(f)$ by letting $A \leq B$ if and only if $A \subseteq B$. Denote by $M_j(f)$ the collection of all maximal elements in $P_j(f)$. Then for each i with $0 \neq f(V_i) \subseteq V_j$, there exists some s such that $f(V_i) \subseteq f(V_s) \in M_j(f)$.

Now we can state and prove the conclusion for the relation $\mathcal{R}$.

Theorem 2.2 *Let $f, g \in L^{\oplus}(V)$. Then the following statements are equivalent:*

(1) $f\mathcal{R}g$.

(2) For each i $(1 \leq i \leq m)$ there exist j, k such that $f(V_i) \subseteq g(V_j)$ and $g(V_i) \subseteq f(V_k)$.

(3) $f(V) = g(V)$ and $M_j(f) = M_j(g)$ holds for each j with $V_j \cap f(V) \neq 0$.

Proof (1)$\Longrightarrow$(2) Suppose $f\mathcal{R}g$. Then there exist $u,\ v \in L^{\oplus}(V)$ such that $fu = g$ and $gv = f$. For each i, there exists some j such that $v(V_i) \subseteq V_j$. Consequently, $f(V_i) = gv(V_i) \subseteq g(V_j)$. Similarly, there exists some k such that $g(V_i) \subseteq f(V_k)$ holds.

(2)$\Longrightarrow$(3) It is not difficult to see from (2) that $f(V) \subseteq g(V)$ and $g(V) \subseteq f(V)$, so $f(V) = g(V)$. Suppose $V_j \cap f(V) \neq 0$ and $f(V_i) \in M_j(f)$. Then there exist $i_1,\ i_2$ such that $f(V_i) \subseteq g(V_{i_1}) \subseteq f(V_{i_2})$. From $f(V_i) \subseteq V_j \cap f(V_{i_2})$, we see that $f(V_{i_2}) \subseteq V_j$. Since $f(V_i) \in M_j(f)$ and $f(V_i) \subseteq f(V_{i_2})$, we have $f(V_{i_2}) = g(V_{i_1}) = f(V_i)$. Take $g(V_{i_3}) \in M_j(g)$ such that $g(V_{i_1}) \subseteq g(V_{i_3})$. By (2) again, there exists i_4 such that $g(V_{i_3}) \subseteq f(V_{i_4})$. Thus,

$$f(V_i) \subseteq g(V_{i_1}) \subseteq g(V_{i_3}) \subseteq f(V_{i_4}) \subseteq V_j$$

which implies that $f(V_{i_4}) = f(V_i) = g(V_{i_3}) \in M_j(g)$ and that $M_j(f) \subseteq M_j(g)$. By symmetry, we have $M_j(g) \subseteq M_j(f)$ and therefore $M_j(f) = M_j(g)$ holds.

(3)$\Longrightarrow$(1) Suppose that $f(V) = g(V)$ and $M_j(f) = M_j(g)$ holds for each j with $V_j \cap f(V) \neq 0$. We first look for some $h \in L^{\oplus}(V)$ such that $fh = g$. For each V_i, if $g(V_i) = 0$, then define $h(x) = 0$ for each $x \in V_i$. If there is some j such that $0 \neq g(V_i) \subseteq V_j$, then there is some $A \in M_j(g) = M_j(f)$ such that $g(V_i) \subseteq A$. Denote $g_i = g|V_i$ and assume $A = f(V_s) = g(V_t)$. Take a basis $e_1, \ldots, e_r, e_{r+1}, \ldots, e_n$ for V_i where $e_{r+1}, \ldots, e_n$ is a basis for $\ker(g_i)$. Then $g(e_1), g(e_2), \ldots, g(e_r)$ are linearly independent. Let $f_s = f|V_s : V_s \to V_j$. Choose $e'_1, e'_2, \ldots, e'_r \in V_s$ such that

$$f_s(e'_1) = g(e_1), f_s(e'_2) = g(e_2), \ldots, f_s(e'_r) = g(e_r).$$

Then $e'_1, e'_2, \ldots, e'_r$ are linearly independent. Define a linear mapping $h_i : V_i \to V_s$ such that

$$h_i(e_1) = e'_1, \ldots, h_i(e_r) = e'_r, \quad h_i(e_{r+1}) = 0, \ldots, h_i(e_n) = 0.$$

Then for each vector $x = a_1e_1 + \cdots + a_re_r + a_{r+1}e_{r+1} + \cdots + a_ne_n \in V_i$, we have

$$\begin{aligned} fh_i(x) &= f(a_1h_i(e_1) + \cdots + a_rh_i(e_r)) = f(a_1e'_1 + \cdots + a_re'_r) \\ &= a_1f(e'_1) + \cdots + a_rf(e'_r) = a_1g(e_1) + \cdots + a_rg(e_r) \\ &= g(x). \end{aligned}$$

These h_i defined on each V_i determine a linear transformation h on V. It is obvious that $h \in L^{\oplus}(V)$ and $fh = g$. By symmetry, there exists $k \in L^{\oplus}(V)$ such that $gk = f$ holds. Therefore, $f\mathcal{R}g$. □

As an immediate consequence of Theorems 2.1 and 2.2, we have the following

Theorem 2.3 *Let $f, g \in L^{\oplus}(V)$. Then the following statements are equivalent:*

(1) $(f, g) \in \mathcal{H}$.

(2) $\ker(f) = \ker(g)$, $K(f) = K(g)$ *and for each* i $(1 \le i \le m)$, *there exist* j, k *such that* $f(V_i) \subseteq g(V_j)$, $g(V_i) \subseteq f(V_k)$.

Let $f \in L^{\oplus}(V)$ and assume that all the subspaces V_i with $f(V) \cap V_i \ne 0$ are $V_{i_1}, V_{i_2}, \ldots, V_{i_s}$. Denote $V'_{i_t} = f(V) \cap V_{i_t}$ $(1 \le t \le s)$. Then one easily verifies that

$$f(V) = V'_{i_1} \oplus V'_{i_2} \oplus \cdots \oplus V'_{i_s}.$$

The following concept will be useful in describing the relations $\mathcal{D}$ and $\mathcal{J}$ on $L^{\oplus}(V)$.

Definition 2.4 *Let U and W be two subspaces of V where*

$$U = V'_{i_1} \oplus V'_{i_2} \oplus \cdots \oplus V'_{i_k} \quad \text{and} \quad W = V'_{j_1} \oplus V'_{j_2} \oplus \cdots \oplus V'_{j_k}$$

and each V'_{i_s} is a non-zero subspace of V_{i_s} while each V'_{j_s} is a non-zero subspace of V_{j_s}. If $\phi : U \to W$ is an isomorphism such that for each s $(1 \le s \le k)$ there exists a unique r $(1 \le r \le k)$ such that $\phi(V'_{i_s}) = V'_{j_r}$, then ϕ is called a sum-preserving isomorphism.

Suppose that $f, g \in L^{\oplus}(V)$ and $\phi : f(V) \to g(V)$ is a sum-preserving isomorphism satisfying $\phi(V_i \cap f(V)) = V_j \cap g(V)$. If for each $A \in M_j(g)$, there exists $B \in M_i(f)$ such that $\phi(B) = A$, while for each $C \in M_i(f)$ there exists $D \in M_j(g)$ such that $\phi(C) = D$, then we write $\phi(M_i(f)) = M_j(g)$.

Next we consider the condition for two elements in $L^{\oplus}(V)$ to be $\mathcal{D}$ equivalent.

Theorem 2.5 *Let $f, g \in L^{\oplus}(V)$. Then $f\mathcal{D}g$ if and only if there exists a sum-preserving isomorphism $\phi : f(V) \to g(V)$ such that for each i with $f(V) \cap V_i \ne 0$, there exists some j such that $\phi(f(V) \cap V_i) = g(V) \cap V_j$ and $\phi(M_i(f)) = M_j(g)$.*

Proof Suppose $f\mathcal{D}g$. Then there exists $h \in L^{\oplus}(V)$ such that $f\mathcal{L}h$ and $h\mathcal{R}g$. From Theorems 2.1 and 2.2, we have $\ker(f) = \ker(h)$, $K(f) = K(h)$, $h(V) = g(V)$ and $M_j(h) = M_j(g)$ holds for each j with $h(V) \cap V_j \ne 0$.

We first establish the isomorphism ϕ from $f(V)$ onto $h(V)$. Suppose $f(V) \cap V_i \ne 0$. Take $W \in K(f) = K(h)$ such that $f(W) = f(V) \cap V_i$. Then there is some j such that $h(W) = h(V) \cap V_j$. Since $\ker(f) = \ker(h)$, we have $\ker(f|W) = \ker(h|W)$ and $\dim f(W) = \dim h(W)$ which implies that $f(W)$ and $h(W)$ are isomorphic. Take a basis $e_1, e_2, \ldots, e_r$ for $f(W) = f(V) \cap V_i$ and choose $w_1, w_2, \ldots, w_r \in W$ such that

$$f(w_1) = e_1, f(w_2) = e_2, \ldots, f(w_r) = e_r.$$

Then $w_1, w_2, \ldots, w_r$ are linearly independent.

Let

$$e'_1 = h(w_1), e'_2 = h(w_2), \ldots, e'_r = h(w_r).$$

Then $e'_1, e'_2, \ldots, e'_r$ are linearly independent while being a basis for $h(W)$. Define a linear mapping

$\phi_i : f(V) \cap V_i \to h(V) \cap V_j$ such that $\phi_i(e_t) = e'_t$, $t = 1, 2, \ldots, r$. Then ϕ_i is an isomorphism and $\phi_i f(x) = h(x)$ for each $x \in W$. Suppose

$$M_i(f) = \{f(V_{i_1}), f(V_{i_2}), \ldots, f(V_{i_s})\}.$$

By virtue of $\ker(f) = \ker(h)$, one routinely verifies that

$$M_j(h) = \{h(V_{i_1}), h(V_{i_2}), \ldots, h(V_{i_s})\}.$$

Besides, since $V_{i_1}, V_{i_2}, \ldots, V_{i_s}$ are contained in W and $\phi_i f = h$ on W, we have

$$\phi_i(f(V_{i_1})) = h(V_{i_1}), \ldots, \phi_i(f(V_{i_s})) = h(V_{i_s})$$

which implies that $\phi_i(M_i(f)) = M_j(h)$. Notice that $h(V) = g(V)$ and $M_j(h) = M_j(g)$, it is evident that $\phi_i : f(V) \cap V_i \to g(V) \cap V_j$ is an isomorphism satisfying $\phi_i(M_i(f)) = M_j(g)$. Furthermore, we obtain the isomorphism ϕ from $f(V)$ onto $g(V)$ determined by these ϕ_i on $f(V) \cap V_i$. Clearly, ϕ is a sum-preserving isomorphism as required.

Conversely, suppose that there exists a sum-preserving isomorphism $\phi : f(V) \to g(V)$ satisfying the condition of the theorem. Let $h = \phi f$. Then $h \in L^{\oplus}(V)$, $h(V) = g(V)$ and $\ker(f) = \ker(h)$. Assume $W \in K(f)$ with $f(W) = f(V) \cap V_i \neq 0$. Then there exists j such that

$$h(W) = \phi f(W) = \phi(f(V) \cap V_i) = g(V) \cap V_j = h(V) \cap V_j \subseteq V_j.$$

Notice that $f(V_i) \neq 0$ for every $V_i \subseteq W$ and that $\ker(f) = \ker(h)$, it readily follows that $h(V_i) \neq 0$ for every $V_i \subseteq W$. Denote $W' = \oplus\{V_i : 0 \neq h(V_i) \subseteq V_j\}$. Then $W' \in K(h)$ and $W \subseteq W'$. Hence $K(f)$ refines $K(h)$. Take $W^* \in K(h)$. Then there exists some s, such that

$$\phi f(W^*) = h(W^*) = h(V) \cap V_s = g(V) \cap V_s.$$

Since ϕ is a sum-preserving isomorphism, there exists some t such that

$$\phi(f(W^*)) = g(V) \cap V_s = \phi(f(V) \cap V_t).$$

It follows that $f(W^*) = f(V) \cap V_t$ and that W^* is contained in some $W \in K(f)$. So $K(h)$ refines $K(f)$ as well and $K(f) = K(h)$. Consequently, $f\mathcal{L}h$ holds.

Finally we verify that $h\mathcal{R}g$. As we have seen above that $h(V) = g(V)$. Now for each V_i with $g(V) \cap V_i \neq 0$, there exists some j such that $\phi(f(V) \cap V_j) = g(V) \cap V_i$ and $\phi(M_j(f)) = M_i(g)$. Then

$$h(V) \cap V_i = \phi f(V) \cap V_i = g(V) \cap V_i = \phi(f(V) \cap V_j),$$

which together with $\ker(f) = \ker(h)$ and $K(f) = K(h)$ implies that $M_i(h) = \phi(M_j(f)) = M_i(g)$ and $h\mathcal{R}g$. Consequently, $f\mathcal{D}g$ follows and the proof is completed. □

Now we consider the final Green relation $\mathcal{J}$ on the semigroup $L^{\oplus}(V)$.

Theorem 2.6 *Let $f, g \in L^{\oplus}(V)$. Then $f\mathcal{J}g$ if and only if there exist sum-preserving isomorphisms*

$$\phi : f(V) \to g(V) \quad \text{and} \quad \psi : g(V) \to f(V),$$

such that for each i, there exist p, q such that $f(V_i) \subseteq \psi(g(V_p)), g(V_i) \subseteq \phi(f(V_q))$.

Proof Suppose $f\mathcal{J}g$. Then there exist $h, k, u, v \in L^{\oplus}(V)$ such that $hfk = g$ and $ugv = f$. Thus, $uhfkv(V) = f(V)$. Since $fkv(V)$ is a subspace of $f(V)$ and

$$\dim f(V) = \dim uhfkv(V) \leq \dim fkv(V) \leq \dim f(V),$$

we have $\dim fkv(V) = \dim f(V)$ and $fkv(V) = fk(V) = f(V)$. Similarly, $g(V) = gv(V)$. Consequently, from $hf(V) = h(fk(V)) = g(V)$ we see that $\dim g(V) \leq \dim f(V)$. By symmetry, $\dim f(V) \leq \dim g(V)$. Thus, $\dim f(V) = \dim g(V)$ and $f(V)$ is isomorphic to $g(V)$. Let $\phi = h|f(V)$ and $\psi = u|g(V)$. Then $\phi : f(V) \to g(V)$ and $\psi : g(V) \to f(V)$ are isomorphisms. Next we verify that both ϕ and ψ are sum-preserving. Suppose

$$f(V) = V'_{i_1} \oplus V'_{i_2} \oplus \cdots \oplus V'_{i_t} \text{ and } g(V) = V'_{j_1} \oplus V'_{j_2} \oplus \cdots \oplus V'_{j_s}$$

where $V'_{i_p} = f(V) \cap V_{i_p}$, $1 \leq p \leq t$ and $V'_{j_q} = g(V) \cap V_{j_q}, 1 \leq q \leq s$. Since h is sum-preserving, for each p there exists a unique q such that $\phi(V'_{i_p}) \subseteq V'_{j_q}$. Notice that ϕ is surjective, it must be the case that $t \geq s$. By symmetry, $s \geq t$ and $t = s$. Thus, $\phi(V'_{i_p}) = V'_{j_q}$ and ϕ maps different V'_{i_p} into different V'_{j_q} isomorphically. Hence ϕ is a sum-preserving isomorphism. Similarly, ψ is sum-preserving isomorphism as well.

Now for each i, there exists some p such that $v(V_i) \subseteq V_p$. Then $f(V_i) = ugv(V_i) \subseteq ug(V_p) = \psi(g(V_p))$. By symmetry, there exists q such that $g(V_i) \subseteq \phi(f(V_q))$, and the necessity follows.

Conversely, suppose the condition holds and we need to show that $f\mathcal{J}g$. We first look for some $h, k \in L^{\oplus}(V)$ such that $hfk = g$. For each i, if $g(V_i) = 0$, then define $k(x) = 0$ for every $x \in V_i$. If $g(V_i) \neq 0$, choose a basis $e_1, \ldots, e_r, e_{r+1}, \ldots, e_n$ for V_i such that $g(e_{r+1}) = 0, \ldots, g(e_n) = 0$ and $g(e_1), \ldots, g(e_r)$ are linearly independent. By hypothesis, there exists V_q such that $g(V_i) \subseteq \phi(f(V_q))$. Take linearly independent vectors $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$ in V_q such that

$$g(e_1) = \phi f(\varepsilon_1), g(e_2) = \phi f(\varepsilon_2), \ldots, g(e_r) = \phi f(\varepsilon_r).$$

Define a linear mapping k from V_i into V_q such that

$$k(e_1) = \varepsilon_1, k(e_2) = \varepsilon_2, \ldots,\ k(e_r) = \varepsilon_r, k(e_{r+1}) = 0, \ldots, k(e_n) = 0.$$

One easily verifies that $g(x) = \phi fk(x)$ holds for each $x \in V_i$. Thus, these k defined on each V_i determine uniquely a linear transformation k of V. Clearly, $k \in L^{\oplus}(V)$ and $g(x) = \phi fk(x)$ for each $x \in V$.

Now we define the linear transformation h. For each V_j with $V_j \cap f(V) = 0$, define $h(x) = 0$ for every $x \in V_j$. For those V_j with $f(V) \cap V_j \neq 0$, since ϕ is sum-preserving, there exists some l such that $\phi(f(V) \cap V_j) = g(V) \cap V_l$. Take a basis $e_1, \ldots, e_r$ for $f(V) \cap V_j$ and extend this to a basis

$$e_1, \ldots, e_r, e_{r+1}, \ldots, e_n$$

for V_j. Define a linear mapping h from V_j into V_l such that

$$h(e_1) = \phi(e_1), \ldots, h(e_r) = \phi(e_r), h(e_{r+1}) = 0, \ldots, h(e_n) = 0.$$

Then one routinely verifies that $h|(f(V) \cap V_j) = \phi|(f(V) \cap V_j)$. Consequently, there exists a unique linear transformation h on V determined by these linear mappings h defined on each

V_j. Clearly, $h \in L^{\oplus}(V)$, $h|f(V) = \phi$ and $g(x) = \phi f k(x) = hfk(x)$ holds for arbitrary $x \in V$. Consequently, $g = hfk$. By symmetry, there exist $u,\ v \in L^{\oplus}(V)$ such that $ugv = f$ and it follows that $f\mathcal{J}g$. □

It is well-known that $\mathcal{D} \subseteq \mathcal{J}$ for every semigroup. In what follows, we will soon see that $\mathcal{D} = \mathcal{J}$ for the semigroups $L^{\oplus}(V)$.

Suppose $f, g \in L^{\oplus}(V)$ and $f\mathcal{J}g$. Assume that

$$f(V) = V'_{i_1} \oplus V'_{i_2} \oplus \cdots \oplus V'_{i_s}, \quad g(V) = V'_{j_1} \oplus V'_{j_2} \oplus \cdots \oplus V'_{j_s}$$

and $\phi : f(V) \to g(V)$, $\psi : g(V) \to f(V)$ are both sum-preserving isomorphisms satisfying the condition in Theorem 2.6. Then we have the following two lemmas.

Lemma 2.7 *There exists a positive integer r such that $(\psi\phi)^r : f(V) \to f(V)$ is a sum-preserving isomorphism such that*

$$(\psi\phi)^r(V'_{i_k}) = V'_{i_k} \ \ \textit{and} \ \ (\psi\phi)^r(M_{i_k}(f)) = M_{i_k}(f)$$

holds for each k $(1 \le k \le s)$.

Proof It is clear that $\psi\phi : f(V) \to f(V)$ is a sum-preserving isomorphism and for each i_k, there exists a unique i'_k such that

$$\psi\phi(V'_{i_k}) = V'_{i'_k}, \quad k = 1, 2, \ldots, s.$$

Thus, $\psi\phi$ induces a permutation ρ of the set $\{i_1, i_2, \ldots, i_s\}$ where

$$\rho = \begin{pmatrix} i_1 & i_2 & \cdots & i_s \\ i'_1 & i'_2 & \cdots & i'_s \end{pmatrix}.$$

By the property of permutations, there exists a positive integer r such that ρ^r is the identity permutation of the set $\{i_1, i_2, \ldots, i_s\}$. Let $\xi = (\psi\phi)^r$. Then $\xi : f(V) \to f(V)$ is a sum-preserving isomorphism satisfying $\xi(V'_{i_k}) = V'_{i_k}$, $k = 1, 2, \ldots, s$.

In order to show the remainder, we assume $M_{i_1}(f) = M_1 \cup M_2 \cup \cdots \cup M_u$, where $M_r (1 \le r \le u)$ is the collection of those A in $M_{i_1}(f)$ with $\dim A = m_r$, and $m_1 > m_2 > \cdots > m_u \ge 1$. By Theorem 2.6, for each $A \in M_{i_1}(f)$ there is some p such that $A \subseteq \psi(g(V_p))$. While there is some q such that $g(V_p) \subseteq \phi(f(V_q))$. Hence $A \subseteq \psi\phi(f(V_q))$. Repeating the discussion, there exists some $p(A)$ $(1 \le p(A) \le m)$ such that

$$A \subseteq (\psi\phi)^r(f(V_{p(A)})) = \xi(f(V_{p(A)})). \tag{2.7.1}$$

Since ξ is sum-preserving and $\xi(V'_{i_1}) = V'_{i_1}$, one routinely verifies that $f(V_{p(A)}) \subseteq V'_{i_1}$.

We first verify

$$\{f(V_{p(A)}) : A \in M_1\} = M_1. \tag{2.7.2}$$

Suppose $A \in M_1$. Then $\dim f(V_{p(A)}) \le m_1$ since m_1 is the maximal dimension of the elements in $M_{i_1}(f)$. Now by (2.7.1), we have

$$\dim f(V_{p(A)}) \ge \dim A = m_1.$$

Therefore, $\dim f(V_{p(A)}) = m_1$ and $f(V_{p(A)}) \in M_1$. Thus, $\{f(V_{p(A)}) : A \in M_1\} \subseteq M_1$. From (2.7.1) it follows that $A = \xi(f(V_{p(A)}))$ for each $A \in M_1$. Notice that ξ is a sum-preserving isomorphism and that M_1 is a finite set, it is clear that (2.7.2) holds. Consequently, we have $\xi(M_1) = M_1$.

Next we verify that

$$\{f(V_{p(B)}) : B \in M_2\} = M_2. \tag{2.7.3}$$

Suppose $B \in M_2$. By (2.7.1) again, we have $\dim f(V_{p(B)}) \geq \dim B = m_2$. If $\dim f(V_{p(B)}) > m_2$, then there exists $A \in M_1$ such that $f(V_{p(B)}) \subseteq A$. Consequently,

$$B \subseteq \xi(f(V_{p(B)})) \subseteq \xi(A) \in M_1,$$

which contradicts the hypothesis that B is a maximal element in $P_{i_1}(f)$. Hence $\dim f(V_{p(B)}) = m_2$ and $B = \xi(f(V_{p(B)}))$. While $f(V_{p(B)})$ cannot be contained in any element of M_1. Consequently, $f(V_{p(B)}) \in M_2$ for each $B \in M_2$ and (2.7.3) follows. While we also have $\xi(M_2) = M_2$. Go on in this way, we can finally get

$$\{f(V_{p(A)}) : A \in M_i\} = M_i \text{ and } \xi(M_i) = M_i,\ i = 1, 2, \ldots, u.$$

Furthermore, $M_{i_1}(f) = \xi(M_{i_1}(f))$ holds. One similarly verifies that $M_{i_k}(f) = \xi(M_{i_k}(f))$ holds for $k = 2, \ldots, s$. The proof is completed. □

Lemma 2.8 *Let $\theta = \phi(\psi\phi)^{r-1}$. Then $\theta : f(V) \to g(V)$ is a sum-preserving isomorphism. Moreover, if $\theta(V'_{i_k}) = V'_{j_k}$, then $M_{j_k}(g) = \theta(M_{i_k}(f))$.*

Proof θ is clearly a sum-preserving isomorphism and $\xi = \psi\theta$. Denote

$$M_{i_k}(f) = M_1 \cup M_2 \cup \cdots \cup M_u \text{ and } M_{j_k}(g) = N_1 \cup N_2 \cup \cdots \cup N_v,$$

where $\dim B = m_r$ for each $B \in M_r$ $(1 \leq r \leq u)$ and $\dim A = n_t$ for each $A \in N_t$ $(1 \leq t \leq v)$ with $m_1 > m_2 > \cdots > m_u \geq 1$ and $n_1 > n_2 > \cdots > n_v \geq 1$. Suppose $\theta(V'_{i_k}) = V'_{j_k}$, then

$$\psi(V'_{j_k}) = \psi\theta(V'_{i_k}) = \xi(V'_{i_k}) = V'_{i_k}.$$

For each $A \in M_{j_k}(g)$ there exists some p such that $f(V_p) \subseteq V'_{i_k}$ and $A \subseteq \theta(f(V_p))$. Moreover, there exists some $B \in M_{i_k}(f)$ with $f(V_p) \subseteq B$. Consequently,

$$A \subseteq \theta(f(V_p)) \subseteq \theta(B). \tag{2.8.1}$$

By Theorem 2.6, for this B there exists some q such that $B \subseteq \psi(g(V_q))$ and it is clear that $g(V_q) \subseteq V'_{j_k}$. Thus there is $A' \in M_{j_k}(g)$ such that $g(V_q) \subseteq A'$. Hence we have

$$B \subseteq \psi(g(V_q)) \subseteq \psi(A'). \tag{2.8.2}$$

Suppose $A \in N_1$. Then $\dim A = n_1$. By (2.8.1) and (2.8.2), we have

$$n_1 = \dim A \leq \dim B \leq \dim A' \leq n_1$$

and $\dim B = n_1 = \dim A'$. Notice that $B \in M_{i_k}(f)$, so $\dim B \leq m_1$ and $n_1 \leq m_1$. Conversely, suppose $B \in M_{i_k}(f)$ and $\dim B = m_1$. From the discussion above, there exist q and some

$A' \in M_{j_k}(g)$ such that $B \subseteq \psi(g(V_q)) \subseteq \psi(A')$. Hence

$$m_1 = \dim B \leq \dim A' \leq n_1$$

and $m_1 = n_1$. Thus, (2.8.1) implies that $A = \theta(B)$ and that every element $A \in N_1$ is an image of some $B \in M_1$ under the isomorphism θ. Consequently, $|M_1| \geq |N_1|$. Similarly, from (2.8.2), for each $B \in M_1$ there exists $A' \in N_1$ such that $B = \psi(A')$, so $|M_1| \leq |N_1|$. Therefore, $|M_1| = |N_1|$ and $\theta(M_1) = N_1$.

Now suppose $A \in N_2$. By (2.8.1) again, there exists $B \in M_{i_k}(f)$ such that $A \subsetneq \theta(B)$. If $B \in M_1$, then there is some $A' \in N_1$ such that $A \subsetneq \theta(B) = A'$ which contradicts the fact that A is maximal. Thus, it must be the case that $B \notin M_1$ and $\dim(B) < m_1$. While from (2.8.2) we see that there exists some $A' \in M_{j_k}(g)$ such that $B \subseteq \psi(A')$. If $\dim A' = n_1$ $(= m_1)$, since $\theta(M_1) = N_1$, then there exists some $B' \in M_1$ such that $A' = \theta(B')$. Therefore there exists some $B'' \in M_1$ such that $B \subsetneq \psi(A') \subseteq \psi\theta(B') = B''$ holds, contradicting the fact that B is maximal. So $\dim A' < n_1$ $(= m_1)$ and

$$n_2 = \dim A \leq \dim B \leq \dim A' \leq n_2.$$

Consequently, $\dim B = n_2$, $A = \theta(B)$ and $n_2 = m_2$. Similarly, we have $|N_2| = |M_2|$ and $\theta(M_2) = N_2$. Repeating the discussion above, we finally obtain that

$$u = v,\ |N_i| = |M_i|,\ \theta(M_i) = N_i,\ n_i = m_i,\ i = 1, 2, \ldots, u.$$

Consequently, $M_{j_k}(g) = \theta(M_{i_k}(f))$ holds. The proof is completed. □

By Lemma 2.8 and Theorem 2.5, we can prove the following

Theorem 2.9 *In the semigroup $L^{\oplus}(V)$, $\mathcal{D} = \mathcal{J}$.*

Proof We only need to show that $\mathcal{J} \subseteq \mathcal{D}$. Suppose $(f, g) \in \mathcal{J}$. From Theorem 2.6, there exist sum-preserving isomorphisms $\phi : f(V) \to g(V)$ and $\psi : g(V) \to f(V)$ satisfying the condition in Theorem 2.6. Let $\xi = (\psi\phi)^r$. By Lemma 2.7, $\xi : f(V) \to f(V)$ is a sum-preserving isomorphism satisfying that $\xi(V'_{i_k}) = V'_{i_k}$, $\xi(M_{i_k}(f)) = M_{i_k}(f)$ $(1 \leq k \leq s)$. Denote $\theta = \phi(\psi\phi)^{r-1}$. By Lemma 2.8, $\theta : f(V) \to g(V)$ is a sum-preserving isomorphism and if $\theta(V'_{i_k}) = V'_{j_k}$, then $M_{j_k}(g) = \theta(M_{i_k}(f))$. Thus θ satisfies the condition of Theorem 2.5, hence $(f, g) \in \mathcal{D}$ and $\mathcal{J} = \mathcal{D}$ holds. □

3. Regular elements in $L^{\oplus}(V)$

In this section we consider the condition under which an element in $L^{\oplus}(V)$ is regular and when the semigroup $L^{\oplus}(V)$ is a regular semigroup. And then we investigate the Green's relations for regular elements in the semigroup $L^{\oplus}(V)$.

For $f \in L^{\oplus}(V)$, denote $\mathrm{Fix}(f) = \{x \in V : f(x) = x\}$. The following result is routinely verified and the proof is omitted.

Lemma 3.1 *Let $f \in L^{\oplus}(V)$. Then f is idempotent if and only if $f(V) = \mathrm{Fix}(f)$.*

Lemma 3.2 *Suppose $f \in L^{\oplus}(V)$ is an idempotent. Then for each $W \in K(f)$ there exits some*

$V_i \subseteq W$ *such that* $f(V_i) = f(W) = V_i \cap f(V)$.

Proof Suppose $f(W) = V_i \cap f(V)$. Then for each $x \in V_i \cap f(V)$, by Lemma 3.1, $x = f(x) \in f(V_i)$ which implies that $V_i \cap f(V) \subseteq f(V_i)$. Hence $0 \neq f(V_i) \subseteq V_j$ for some j. Notice that $V_i \cap f(V) \subseteq f(V_i)$ and $V_i \cap f(V) = f(V_i \cap f(V)) \subseteq V_j$, so $V_i = V_j$. Consequently, $f(V_i) \subseteq V_i$ and $f(V_i) = V_i \cap f(V)$. While $V_i \subseteq W$ follows from the definition of $K(f)$. □

Theorem 3.3 *Let* $f \in L^{\oplus}(V)$. *Then* f *is regular if and only if for each* i *with* $V_i \cap f(V) \neq 0$ *there exists some* j *such that* $f(V_j) = V_i \cap f(V)$.

Proof If f is regular, then there exists an idempotent g in $L^{\oplus}(V)$ such that $f\mathcal{L}g$. By Theorem 2.1 we have $\ker(f) = \ker(g)$ and $K(f) = K(g)$. Take a subspace V_i such that $V_i \cap f(V) \neq 0$. Then there exists $W \in K(f) = K(g)$ such that $f(W) = V_i \cap f(V)$. By Lemma 3.2, we can choose $V_j \subseteq W$ such that $g(V_j) = g(W) = V_j \cap g(V)$. Now $\ker(f) = \ker(g)$ and $g(V_j) = g(W)$ implies that $f(V_j) = f(W) = V_i \cap f(V)$ and the necessity holds.

Now suppose that f satisfies the condition and we shall find some idempotent g such that $f\mathcal{L}g$ which of course implies that f is regular. We first define g on each $W \in K(f)$. By hypothesis, there exist i and j such that $V_j \subseteq W$ and $f(V_j) = f(W) = V_i \cap f(V)$. Take a basis $\{e_u\}$ for $V_i \cap f(V)$ and choose $e'_u \in V_j$ such that $f(e'_u) = e_u$ for each u. Then $\{e'_u\}$ is linearly independent. Extend this to a basis $\{e_u\} \cup \{d_v\}$ for W. Then $f(d_v) = 0$ for each v. Now define a linear mapping $g : W \to V_j$ such that $g(e'_u) = e'_u$ for each u and $g(d_v) = 0$ for each v. For those V_i (if exists) with $f(V_i) = 0$, define $g(x) = 0$ for each $x \in V_i$. Thus, we have defined the linear transformation g of V. It is obvious that $g \in L^{\oplus}(V)$ and $g^2 = g$. By definition of g it readily follows that $K(f) = K(g)$ and $ker(f) = \ker(g)$. Consequently, $f\mathcal{L}g$ and f is regular in $L^{\oplus}(V)$. □

The following example tells us that the semigroup $L^{\oplus}(V)$ is not, in general, a regular semigroup.

Example Let $V = V_1 \oplus V_2 \oplus V_3$ where V_1 has a basis $e_1, e_2, \ldots, e_n$ $(n \geq 3)$, V_2 has a basis $\alpha_1, \alpha_2, \ldots, \alpha_n$ and V_3 has a basis $\beta_1, \beta_2, \ldots, \beta_n$. Define a linear transformation $f : V \to V$ such that

$$f(e_1) = f(\beta_1) = \alpha_1, f(\alpha_1) = f(e_i) = \alpha_2,\ f(\alpha_i) = f(\beta_i) = \alpha_3 \text{ (for } i \neq 1).$$

Then $f \in L^{\oplus}(V)$ and $V_2 \cap f(V) = \langle \alpha_1,\ \alpha_2,\ \alpha_3 \rangle$. However, $f(V_1) = \langle \alpha_1,\ \alpha_2 \rangle$, $f(V_2) = \langle \alpha_2,\ \alpha_3 \rangle$ and $f(V_3) = \langle \alpha_1,\ \alpha_3 \rangle$. It is clear that there is no j $(1 \leq j \leq 3)$ satisfying $V_2 \cap f(V) = f(V_j)$. By Theorem 3.3, f is not regular in the semigroup $f \in L^{\oplus}(V)$.

Next we investigate when the semigroup $L^{\oplus}(V)$ is a regular semigroup.

Theorem 3.4 *The semigroup* $L^{\oplus}(V)$ *is regular if and only if* $m = 1$ *or* $\dim V_i = 1$ *for each* i.

Proof If $m = 1$, then $V = V_1$ is an n dimensional space. Thus, $L^{\oplus}(V) = L(V)$ is a regular semigroup. If $\dim V_i = 1$ for each i, then V is a direct sum of m one dimensional spaces. Let $f \in L^{\oplus}(V)$. If $V_i \cap f(V) \neq 0$, then we have $V_i \cap f(V) = V_i$ since the subspace $V_i \cap f(V)$ must be one dimensional. Notice that there must be some j such that $0 \neq f(V_j) = V_i$, otherwise, we would conclude that $V_i \cap f(V) = 0$, a contradiction. Consequently, we have $f(V_j) = V_i \cap f(V) \neq 0$. By Theorem 3.3, f is regular and $L^{\oplus}(V)$ is a regular semigroup.

Conversely, suppose that $m > 1$ and $n \geq 2$. Take a basis $e_1, e_2, \ldots, e_n$ for V_1, a basis $g_1, g_2, \ldots, g_n$ for V_2. Define $f : V \to V$ such that $f(e_k) = e_1$, $f(g_k) = e_2$ for each k and $f(x) = 0$ for any $x \in V_s$ $(s \neq 1, 2)$. Clearly, $f \in L^{\oplus}(V)$ and $f(V) = \langle e_1, e_2 \rangle \subseteq V_1$. Thus, $V_1 \cap f(V) = \langle e_1, e_2 \rangle$. However, there is no j satisfying $V_1 \cap f(V) = f(V_j)$ which implies that f is not a regular element. Consequently, $L^{\oplus}(V)$ is not a regular semigroup. □

Finally, we describe Green's equivalences for regular elements in the semigroups $L^{\oplus}(V)$. We first make some observations.

Theorem 3.5 *Let $f, g \in L^{\oplus}(V)$ be regular. If $\ker(f) = \ker(g)$, then $K(f) = K(g)$.*

Proof Suppose

$$W = \oplus\{V_i : 0 \neq f(V_i) \subseteq V_j\} \in K(f).$$

Then $f(W) = V_j \cap f(V)$. Since f is regular, by Theorem 3.3, there exists some l such that $f(W) = V_j \cap f(V) = f(V_l)$. Suppose $0 \neq g(V_l) \subseteq V_k$ for some k. Denote

$$U = \oplus\{V_s : 0 \neq g(V_s) \subseteq V_k\}.$$

By Theorem 3.3 again, there exists some u such that $g(U) = V_k \cap g(V) = g(V_u)$. We claim that $W = U$. Actually, from $\ker(f) = \ker(g)$ one routinely verifies that, for each $V_i \subseteq W$, $f(V_i) \subseteq f(V_l)$ implies $0 \neq g(V_i) \subseteq g(V_l) \subseteq V_k$. Thus, $V_i \subseteq U$ and $W \subseteq U$ holds.

On the other hand, since $g(V_u) = V_k \cap g(V)$ and $g(V_l) \subseteq V_k$, we have $g(V_l) \subseteq g(V_u)$ which together with $\ker(f) = \ker(g)$ implies that $f(V_l) \subseteq f(V_u)$. Therefore,

$$f(V_l) = V_j \cap f(V) = f(V_u).$$

By $\ker(f) = \ker(g)$ again, we have $g(V_l) = g(V_u)$. Now for each $V_s \subseteq U$, we have $0 \neq g(V_s) \subseteq g(V_l)$. Hence $0 \neq f(V_s) \subseteq f(V_l)$. Thus, $V_s \subseteq W$ and $U \subseteq W$ holds. Consequently, $U = W$ and $K(f) \subseteq K(g)$. By symmetry, $K(g) \subseteq K(f)$, so $K(f) = K(g)$. □

Theorem 3.6 *Let f, $g \in L^{\oplus}(V)$ be regular elements. If $f(V) = g(V)$, then, for each i, there exist j, k such that $f(V_i) \subseteq g(V_j)$, $g(V_i) \subseteq f(V_k)$.*

Proof If $f(V_i) = 0$, then $f(V_i) \subseteq g(V_j)$ holds for arbitrary j. If $0 \neq f(V_i) \subseteq V_l$, then

$$V_l \cap g(V) = V_l \cap f(V) \neq 0.$$

Since g is regular, there exists j such that $V_l \cap g(V) = g(V_j)$. Consequently,

$$f(V_i) \subseteq V_l \cap f(V) = V_l \cap g(V) = g(V_j).$$

By symmetry, for each i, there exists k such that $g(V_i) \subseteq f(V_k)$. □

As an immediate consequence of Theorems 2.1, 2.2 and 3.3, we have the following result.

Theorem 3.7 *Let $f, g \in L^{\oplus}(V)$ be regular elements. Then*

(1) $f\mathcal{L}g$ if and only if $\ker(f) = \ker(g)$.

(2) $f\mathcal{R}g$ *if and only if* $f(V)=g(V)$.

Finally, we observe the relation $\mathcal{D}$ for regular elements.

Theorem 3.8 *Let* f, $g\in L^{\oplus}(V)$ *be regular elements. Then* $f\mathcal{D}g$ *if and only if there exists a sum-preserving isomorphism from* $f(V)$ *onto* $g(V)$.

Proof Suppose $f\mathcal{D}g$. Then there exists some $h\in L^{\oplus}(V)$ such that $f\mathcal{L}h$ and $h\mathcal{R}g$. By Theorem 2.1, $\ker(f)=\ker(h)$ and $K(f)=K(h)$. While by Theorem 2.2, $h(V)=g(V)$. Denote $K(f)=\{W_1,\dots,W_t\}=K(h)$. Denote $V'_{i_r}=f(W_r)=V_{i_r}\cap f(V)$ and $V'_{j_r}=h(W_r)=V_{j_r}\cap h(V)$, $1\le r\le t$. Then

$$f(V)=V'_{i_1}\oplus V'_{i_2}\oplus\cdots\oplus V'_{i_t},$$

$$h(V)=V'_{j_1}\oplus V'_{j_2}\oplus\cdots\oplus V'_{j_t}=g(V).$$

By the proof of Theorem 2.5, there exists a sum-preserving isomorphism from $f(V)$ onto $h(V)=g(V)$.

Conversely, if there exists a sum-preserving isomorphism ϕ from $f(V)$ onto $g(V)$, define $h:V\to V$ by $h=\phi f$. Then it is clear that $h\in L^{\oplus}(V)$, $\ker(f)=\ker(h)$ and $K(f)=K(h)$. By Theorem 2.1, $f\mathcal{L}h$. Hence h is also regular. While from the definition of h one easily verifies that $h(V)=g(V)$ and $h\mathcal{R}g$ follows from Theorem 3.7. Consequently, $f\mathcal{D}g$ holds. □

References

[1] HOWIE J M. *Fundamentals of Semigroup Theory* [M]. Oxford University Press, New York, 1995.

[2] HOWIE J M. *The subsemigroup generated by the idempotents of a full transformation semigroup* [J]. J. London Math. Soc., 1966, **41**: 707–716.

[3] ERDOS J A. *On products of idempotent matrices* [J]. Glasgow Math. J., 1967, **8**: 118–122.

[4] DAWLINGS R J H. *Products of idempotents in the semigroup of singular endomorphisms of a finite-dimensional vector space* [J]. Proc. Roy. Soc. Edinburgh Sect. A, 1981, **91**(1-2): 123–133.

[5] REYNOLDS M A, SULLIVAN R P. *Products of idempotent linear transformations* [J]. Proc. Roy. Soc. Edinburgh Sect. A, 1985, **100**(1-2): 123–138.

[6] PEI Huisheng. *Equivalences, α-semigroups and α-congruences* [J]. Semigroup Forum, 1994, **49**(1): 49–58.

[7] PEI Huisheng. *Regularity and Green's relations for semigroups of transformations that preserve an equivalence* [J]. Comm. Algebra, 2005, **33**(1): 109–118.

[8] PEI Huisheng. *On the rank of the semigroup $T_E(X)$* [J]. Semigroup Forum, 2005, **70**(1): 107–117.

[9] PEI Huisheng. *A regular α-semigroup inducing a certain lattice* [J]. Semigroup Forum, 1996, **53**(1): 98–113.

[10] PEI Huisheng. *Some α-semigroups inducing certain lattices* [J]. Semigroup Forum, 1998, **57**(1): 48–59.