Journal of Mathematical Research & Exposition
Sept., 2009, Vol. 29, No. 5, pp. 945–950
DOI:10.3770/j.issn:1000-341X.2009.05.022
Http://jmre.dlut.edu.cn

Improved Upper Bounds for the Largest Eigenvalue of Unicyclic Graphs

HU Sheng Biao
(Department of Mathematics, Qinghai Nationalities College, Qinghai 810007, China)
(E-mail: shengbiaohu@yahoo.com.cn)

Abstract Let $G(V,E)$ be a unicyclic graph, C_m be a cycle of length m and $C_m \subset G$, and $u_i \in V(C_m)$. The $G-E(C_m)$ are m trees, denoted by T_i, $i=1,2,\ldots,m$. For $i=1,2,\ldots,m$, let e_{u_i} be the excentricity of u_i in T_i and

$$e_c = \max\{e_{u_i} : i=1,2,\ldots,m\}.$$

Let $k=e_c+1$. For $j=1,2,\ldots,k-1$, let

$$\delta_{ij} = \max\{d_v : \mathrm{dist}(v,u_i)=j, v\in T_i\},$$

$$\delta_j = \max\{\delta_{ij} : i=1,2,\ldots,m\},$$

$$\delta_0 = \max\{d_{u_i} : u_i \in V(C_m)\}.$$

Then

$$\lambda_1(G) \le \max\{\max_{2\le j\le k-2}(\sqrt{\delta_{j-1}-1}+\sqrt{\delta_j-1}), 2+\sqrt{\delta_0-2}, \sqrt{\delta_0-2}+\sqrt{\delta_1-1}\}.$$

If $G\cong C_n$, then the equality holds, where $\lambda_1(G)$ is the largest eigenvalue of the adjacency matrix of G.

Keywords unicyclic graph; adjacency matrix; largest eigenvalue.

Document code A
MR(2000) Subject Classification 05C50; 05C05
Chinese Library Classification O157.5

1. Introduction

Let $G=(V,E)$ be a simple undirected connected graph. Let $A(G)$ be the adjacency matrix of G, which is real symmetric matrix. Let $\lambda_1(G)$ be the largest eigenvalue of $A(G)$. Let d_v denote the degree of $v\in V$ and Δ denote the largest vertex degree of G.

Let T be a tree with largest vertex degree Δ. In [1], Godsil proved that

$$\lambda_1(T) < 2\sqrt{\Delta-1}. \tag{1}$$

For (1), Stevanović proved (1) again in [2, Theorem1, p.36] in a different way.

Received date: 2007-06-04; **Accepted date**: 2008-07-07
Foundation item: the National Natural Science Foundation of China (No. 10861009).

Definition 1 *Let T be a tree. If x and y are nonadjacent vertices of T, then $T+xy$ is obtained from T by joining x to y. $T+xy$ just contains one cycle and is called unicyclic graph.*

In [3], Hu proved that: If G is a unicyclic graph and Δ is the maximum vertex degree, then

$$\lambda_1(G) \leq 2\sqrt{\Delta - 1}. \tag{2}$$

The equality holds if and only if $G \cong C_n$.

We use already mentioned fact that if H is a subgraph of G, then $\lambda_1(H) \leq \lambda_1(G)$. If $G = T + xy$ is a unicyclic graph, then T is a subgraph of G. Thus we have that (1) is a corollary of (2). We recall that the *excentricity* of a vertex u is the largest distance from u to any other vertex of the graph. The *excentricity* of the vertex u is denoted by e_u.

In [4], Rojo gave an improvement of the bound (1). Let T be a tree with largest vertex degree Δ. Let u be a vertex of T such that $d_u = \Delta$. Let $k = e_u + 1$. For $j = 1, 2, \ldots, k-1$, let $\delta_j = \max\{d_v : d(v, u) = j\}$. Then

$$\lambda_1(T) < \max\{\max_{2\leq j\leq k-2}(\sqrt{\delta_j - 1} + \sqrt{\delta_{j-1} - 1}), \sqrt{\delta_1 - 1} + \sqrt{\Delta}\}, \tag{3}$$

where $d(u, v)$ denotes the distance between u and v.

In [5], Hu obtained another improvement of the bound (1). Let $w \in V$ such that $d_w = 1$. Let k=e_w+1. For $j = 1, 2, \ldots, k-2$, let $\delta'_j = \max\{d_v : \mathrm{dist}(v, w) = j\}$. Then

$$\lambda_1(T) < \max_{1\leq j\leq k-2}\{\sqrt{\delta'_j - 1} + \sqrt{\delta'_{j-1} - 1}\}, \tag{4}$$

where δ'_0=2.

In this paper, we improve the upper bounds for the largest eigenvalue of unicyclic graphs. For terminology and notation not introduced here, we refer to [7].

2. Preliminaries

Let T be a rooted tree such that in each level the vertices have equal degree. We agree that the root vertex is at level 1 and that T has k level. Thus the vertices in the level k have degree 1.

Definition 2 *Let T_i $(i = 1, 2, \ldots, m)$ be m copies of the rooted tree T and u_i denote the root vertex of the T_i. Let G_k be a graph obtained from T_i $(i = 1, 2, \ldots, m)$, adding an edge between the vertex u_i and the vertex u_{i+1} $(i = 1, 2, \ldots, m, m+1 \equiv 1 \bmod m)$. Then G_k is a rooted graph with a cycle which is regarded as a root of G_k and we call G_k a cycle-rooted graph.*

Obviously, as a root of G_k, the induced subgraph $C_m = [\{u_1, u_2, \ldots, u_m\}]$ is called *cycle − rooted* of G_k.

Thus the Level 1 of G_k is a cycle C_m with m vertices and the level j $(1 \leq j \leq k)$ of G_k is the level j of $\bigcup_{i=1}^m T_i$.

Let G be a unicyclic graph. Then G has an induced subgraph as cycle, denoted by C_m. Let $u_i \in V(C_m)$. Obviously, $G - E(C_m)$ has m trees, denoted by $T_1, T_2, \ldots, T_m$. For $i = 1, 2, \ldots, m$, let e_{u_i} be excentricity of u_i in T_i. Let $e_c = \max\{e_{u_i} : i = 1, 2, \ldots, m\}$. We call e_c the excentricity of the cycle C_m in G. Let $k = e_c+1$. For $j = 1, 2, \ldots, k-1$, let

$$\delta_{ij} = \max\{d_v : \mathrm{dist}(v, u_i) = j, v \in T_i\},$$

$$\delta_j = \max\{\delta_{ij} : i = 1, 2, \ldots, m\}, \quad \delta_0 = \max\{d_{u_i} : u_i \in V(C_m)\}.$$

Let G_k be a cycle-rooted graph of k levels such that C_m is the cycle-rooted. For $j = 1, 2, \ldots, k$, the vertices in level j have degree δ_{j-1}. Observe that δ_{k-1}=1. Clearly, the unicyclic graph G is an induced subgraph of G_k.

Example 1 Let G_3 be the cycle-rooted graph, as shown in Figure 1.

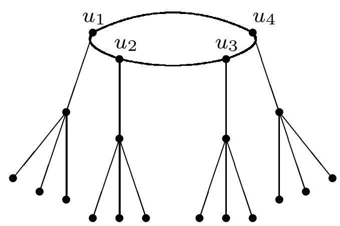

Figure 1 G_3

We see that this graph has 3 levels. The cycle-rooted is $C_4 = [\{u_1, u_2, u_3, u_4\}]$, where e_c=2, $k = e_c$+1=3, δ_0=3, δ_1=4, δ_2=1.

Definition 3[6] *For a given graph $G = (V, E)$, let $V_1, V_2, \ldots, V_k$ be a partition of V. Then $V_1, V_2, \ldots, V_k$ are said to be equitable if for each $i, j = 1, 2, \ldots, k$, there is a constant c_{ij} such that for each $v \in V_i$ there are exactly c_{ij} edges joining v to the vertices in V_j. Given an equitable partition $P = (V_1, V_2, \ldots, V_k)$ of a graph G, we now define the quotient G/P of G. We know that c_{ij} is the number of edges which join a fixed vertex in V_i to vertices in V_j. Then G/P is the directed graph with the cells $V_i (i = 1, 2, \ldots, k)$ of P as its vertices, and with c_{ij} going from V_i to V_j. Thus the adjacency matrix $A(G/P)$ is the $k \times k$ matrix with (i, j) entry equal to c_{ij}.*

For the cycle-rooted graph G_k, let

$$V_1 = V(C_m) \ \ \text{(the vertices of level 1 in } G),$$

$$V_j = \{\text{the vertices of level } j \text{ in } G, j = 2, 3, \ldots k\}.$$

Then $P = (V_1, V_2, \ldots, V_k)$ is an equitable partition. The adjacency matrix of quotient G/P is

$$A(G/P) = \begin{pmatrix} 2 & \delta_0 - 2 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \delta_1 - 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \delta_2 - 1 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 0 & \delta_{k-2} - 1 \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}. \tag{5}$$

Lemma 1 *Let $A_k = A(G/P)$ and*

$$B_k = \begin{pmatrix} 2 & \sqrt{\delta_0 - 2} & 0 & 0 & \dots & 0 & 0 \\ \sqrt{\delta_0 - 2} & 0 & \sqrt{\delta_1 - 1} & 0 & \dots & 0 & 0 \\ 0 & \sqrt{\delta_1 - 1} & 0 & \sqrt{\delta_2 - 1} & \dots & 0 & 0 \\ 0 & 0 & \sqrt{\delta_2 - 1} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & \sqrt{\delta_{k-2} - 1} \\ 0 & 0 & 0 & 0 & \dots & \sqrt{\delta_{k-2} - 1} & 0 \end{pmatrix}. \quad (6)$$

Then A_k and B_k have the same spectra.

Proof Clearly

$$\det(\lambda I_k - A_k) = \lambda \det(\lambda I_{k-1} - A_{k-1}) + (1 - \delta_{k-2}) \det(\lambda I_{k-2} - A_{k-2}),$$

and

$$\det(\lambda I_k - B_k) = \lambda \det(\lambda I_{k-1} - B_{k-1}) + (1 - \delta_{k-2}) \det(\lambda I_{k-2} - B_{k-2}).$$

By induction, it is easy to get

$$\det(\lambda I_k - A_k) = \det(\lambda I_k - B_k).$$

Thus A_k and B_k have the same spectra.

Lemma 2[6] *Let P be an equitable partition of the connected graph G. Then $A(G)$ and $A(G/P)$ have the same spectral radius.*

3. Main results

Theorem 1 *Let $G(V, E)$ be a unicyclic graph, C_m be a cycle of length m and $C_m \subset G$, and $u_i \in V(C_m)$. The $G - E(C_m)$ are m trees and denoted by T_i, $i = 1, 2, \dots, m$. For $i = 1, 2, \dots, m$, let e_{u_i} be the excentricity of u_i in T_i and*

$$e_c = \max\{e_{u_i} : i = 1, 2, \dots, m\}.$$

Let $k = e_c + 1$. For $j = 1, 2, \dots, k - 1$, let

$$\delta_{ij} = \max\{d_v : \mathrm{dist}(v, u_i) = j, v \in T_i\},$$

$$\delta_j = \max\{\delta_{ij} : i = 1, 2, \dots, m\}, \quad \delta_0 = \max\{d_{u_i} : u_i \in V(C_m)\}.$$

Then

$$\lambda_1(G) \leq \max\{\max_{2 \leq j \leq k-2} (\sqrt{\delta_{j-1} - 1} + \sqrt{\delta_j - 1}), 2 + \sqrt{\delta_0 - 2}, \sqrt{\delta_0 - 2} + \sqrt{\delta_1 - 1}\}. \quad (7)$$

If $G \cong C_n$, then the equality holds.

Proof Let G_k be a cycle-rooted graph with cycle-rooted C_m. For $j = 1, 2, \dots, k$, the vertices in level j have degree δ_{j-1}. Let

$$V_1 = V(C_m) \text{ (the vertices of level 1 in } G_k),$$

$$V_j = \text{the vertices of level j in } G_k,\ \text{j=2,3,}\ldots,\ \text{k}.$$

Then $P = (V_1, V_2, \ldots, V_k)$ is an equitable partition of G_k. The adjacency matrix $A(G/P)$ of quotient G/P of G_k is (5). By Lemma 1, $A(G/P)$ and B_k have the same spectra. For (6), from the *Geršhgorin* theorem and Lemma 2, we obtain

$$\lambda_1(G_k) \leq \max\{\max_{2\leq j\leq k-2}(\sqrt{\delta_{j-1}-1}+\sqrt{\delta_j-1}), 2+\sqrt{\delta_0-2}, \sqrt{\delta_0-2}+\sqrt{\delta_1-1}\}.$$

Since G is an induced subgraph of G_k, we have

$$\lambda_1(G) \leq \lambda_1(G_k).$$

Thus the upper bound (7) follows.

If $G \cong C_n$, then $\lambda_1(G) = \lambda_1(C_n) = 2$. For $j = 1, 2, \ldots, k-2$, δ_j does not exist and $\delta_0 = 2$. Then there holds the following equality

$$\max\{\max_{2\leq j\leq k-2}(\sqrt{\delta_{j-1}-1}+\sqrt{\delta_j-1}), 2+\sqrt{\delta_0-2}, \sqrt{\delta_0-2}+\sqrt{\delta_1-1}\}$$

$$= 2+\sqrt{\delta_0-2} = 2. \qquad \square$$

Since a tree T is a subgraph of a unicyclic graph G, we have

Corollary 2 *Let T be a tree. Then*

$$\lambda_1(T) < \max\{\max_{2\leq j\leq k-2}(\sqrt{\delta_{j-1}-1}+\sqrt{\delta_j-1}), 2+\sqrt{\delta_0-2}, \sqrt{\delta_0-2}+\sqrt{\delta_1-1}\}, \tag{8}$$

where the root of T is a path with m vertices, denoted by $P_m = u_1u_2\ldots u_m$. $\delta_0 = \max\{d_{u_1}+1, d_{u_2}, d_{u_3}, \ldots, d_{u_{m-1}}, d_{u_m}+1\}$.

Now, we observe that, except for the case when $\delta_0 = \Delta$=3, we have

$$\max\{\max_{2\leq j\leq k-2}(\sqrt{\delta_{j-1}-1}+\sqrt{\delta_j-1}), 2+\sqrt{\delta_0-2}, \sqrt{\delta_0-2}+\sqrt{\delta_1-1}\} \leq 2\sqrt{\Delta-1}.$$

Since

$$\sqrt{\delta_{j-1}-1}+\sqrt{\delta_j-1} \leq 2\sqrt{\Delta-1} \ \text{ for } \ j = 2, 3, \ldots, k-2,$$

$$\sqrt{\delta_0-2}+\sqrt{\delta_1-1} < 2\sqrt{\Delta-1}.$$

For $\Delta \geq 4$, we have

$$2+\sqrt{\delta_0-2} \leq 2+\sqrt{\Delta-2} < 2\sqrt{\Delta-1}.$$

When Δ=3, if δ_0=2, then $2+\sqrt{\delta_0-2} < 2\sqrt{\Delta-1}$.

When Δ=2, δ_0=2 and for $j = 1, 2, \ldots, k-1$, δ_j is non-existent, $G \cong C_n$,

$$\lambda_1(G) = 2+\sqrt{\delta_0-2} = 2\sqrt{\Delta-1} = 2.$$

Consequently, the new bounds (7) and (8) give better results than the bounds (2) and (1) except for the case when $\delta_0 = \Delta$=3.

Example 2 Figure 2

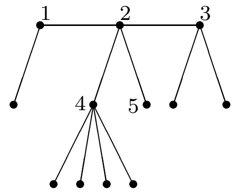

Figure 2 T

Let the induced subgraph $G[\{1,2,3\}] \cong P_3$ be the root of T. Then e_c=2, $k = e_c$+1=3, δ_0=4, δ_1=5, δ_2=1. From (8) we have that

$$\lambda_1(T) < \max\{(\sqrt{\delta_1 - 1} + \sqrt{\delta_2 - 1}), 2 + \sqrt{\delta_0 - 2}, \sqrt{\delta_0 - 2} + \sqrt{\delta_1 - 1}\}$$
$$= \max\{2, 2 + \sqrt{2}\} \doteq 3.414.$$

From (1)

$$\lambda_1(T) < 2\sqrt{\Delta - 1} = 2\sqrt{4} = 4.$$

Let the vertex 4 be a root vertex of T. Since $d_4 = 5 = \Delta$, we have e_4=3, $k = e_4$+1=4, δ_1=4, δ_2=3, δ_3=1. From (3) we have that

$$\lambda_1(T) < \max\{\sqrt{\delta_2 - 1} + \sqrt{\delta_1 - 1}, \sqrt{\delta_3 - 1} + \sqrt{\delta_2 - 1}, \sqrt{\delta_1 - 1} + \sqrt{\Delta}\}$$
$$= \max\{\sqrt{2} + \sqrt{3}, \sqrt{2}, \sqrt{3} + \sqrt{5}\} \doteq 3.968.$$

Let the vertex 5 be a root vertex of T. Since d_5=1, we have e_5=3, $k = e_5$+1=4, δ'_1=4, δ'_2=5, δ'_3=1, and we know δ'_0=2[5]. From (4) we have that

$$\lambda_1(T) < \max_{1 \le j \le k-2}\{\sqrt{\delta'_j - 1} + \sqrt{\delta'_{j-1} - 1}\}$$
$$= \max\{\sqrt{3} + 1, 2 + \sqrt{3}, 2\} \doteq 3.732.$$

For this example, the upper bound (8) is better than the upper bounds (1), (3) and (4).

References

[1] GODSIL C D. *Spectra of Trees* [M]. North-Holland Math. Stud., 87, North-Holland, Amsterdam, 1984.

[2] STEVANOVIĆ D. *Bounding the largest eigenvalue of trees in terms of the largest vertex degree* [J]. Linear Algebra Appl., 2003, **360**: 35–42.

[3] HU Shengbiao. *The largest eigenvalue of unicyclic graphs* [J]. Discrete Math., 2007, **307**(2): 280–284.

[4] ROJO O. *Improved bounds for the largest eigenvalue of trees* [J]. Linear Algebra Appl., 2005, **404**: 297–304.

[5] HU Shengbiao. *A note on an upper bound for the largest eigenvalue of trees* [J]. Acta Math. Sinica (Chin. Ser.), 2007, **50**(1): 145–148. (in Chinese)

[6] GODSIL C D. *Algebraic Combinatorics* [M]. Chapman & Hall, New York, 1993.

[7] BONDY J A, MURTY U S R. *Graph Theory with Applications* [M]. American Elsevier Publishing Co., Inc., New York, 1976.