Journal of Mathematical Research & Exposition
Nov., 2009, Vol. 29, No. 6, pp. 961–973
DOI:10.3770/j.issn:1000-341X.2009.06.003
Http://jmre.dlut.edu.cn

Two-Parameter Weak Hopf Algebras Corresponding to Borcherds-Cartan Matrix

AI Chun Rui[1], *YANG Shi Lin*[2]
(1. Department of Mathematics, Henan Normal University, Henan 453007, China;
2. College of Applied Sciences, Beijing University of Technology, Beijing 100124, China)
(E-mail: slyang@bjut.edu.cn)

Abstract In this paper a class of two-parameter weak Hopf algebras $w^{\tau}_{r,s}(\mathfrak{g})$ corresponding to Borcherds-Cartan matrix is constructed. This class consists of noncommutative and noncocommutative weak Hopf algebras but not Hopf algebras. It can be viewed as a generalized class of one-parameter weak Hopf algebras $wU_q(\mathfrak{g})$.

Keywords Kac-Moody algebras; weak antipode; Borcherds-Cartan matrix.

Document code A
MR(2000) Subject Classification 17B67; 17B37
Chinese Library Classification O152.5

1. Introduction

In the early 1990s, many authors investigated two-parameter or multi-parameter quantum groups. Recall that Du, Parshall and Wang's work[1], Dobrev and Parashar's work[2] focused on quantized function algebras and quantum enveloping algebras for the type A case. In 2005, Bergeron, Gao and Hu[3] obtained two-parameter quantum groups of other types and proved that they are just the Drinfel'd doubles. On the other hand, many mathematicians and physicians are interested in the generalization of Hopf algebra, a typical way is to introduce a kind of weak co-product such that $\triangle(1) \neq 1 \otimes 1$ into an algebra[4]. The face algebra[5] and generalized Kac algebra[6] are examples of this class of weak Hopf algebras.

In fact, one can define a weak antipode on a given bialgebra by replacing the antipode of Hopf algebra[7]. By definition, a bialgebra $(H, \mu, \eta, \triangle, \varepsilon)$ is called a weak Hopf algebra if there exists an anti-homomorphism $T \in \mathrm{Hom}_K(H, H)$ such that $T * \mathrm{id}_H * T = \mathrm{id}_H$ and $\mathrm{id}_H * T * \mathrm{id}_H = T$, where id_H is the identity map and $*$ is the convolution product. Yang[8] constructed a class of weak Hopf algebras in this sense based on the quantized enveloping algebras $U_q(\mathfrak{g})$. Thanks to the definition of quantized enveloping algebra $U^{\tau}_q(\mathfrak{g})$ associated with a generalized Kac-Moody algebra $\mathfrak{g}$[9], Wu[10] introduced a generator J such that $J^m = J$ for some integer $m \geq 3$ and constructed a new class of weak Hopf algebra $wU_q(\mathfrak{g})$ by weakening the group-likes of $U^{\tau}_q(\mathfrak{g})$ motivated by the paper[8].

Received date: 2009-02-16; **Accepted date**: 2009-05-16
Foundation item: the National Natural Science Foundation of China (Nos. 10671016; 10771014); the Foundation of Beijing Educational Committee (No. KM200710005013).

A generalized Kac-Moody algebra can be regarded as a Kac-Moody algebra with imaginary simple roots. It is determined by a Borcherds-Cartan matrix $A = (a_{ij})_{i,j\in I}$, where either $a_{ii} = 2$, or $a_{ii} \le 0$. If $a_{ii} \le 0$, then the index i is called imaginary and the corresponding simple root α_i is called imaginary root. Along the line of two-parameter quantum groups[3], it is interesting to construct a class of two-parameter weak Hopf algebras $w^{\tau}_{r,s}(\mathfrak{g})$. Following the idea of Yang[8] and Wu[10] and basing on the two-parameter quantum group $U_{r,s}(\mathfrak{g})$ associated with a Borcherds-Cartan matrix, in this paper we construct a class of two-parameter weak Hopf algebras. Let $\tau = (\{\kappa_i\}_{i\in I}|\ \{\bar{\kappa}'_i)\}_{i\in I})$ be an admissible type and E_i, F_i are of type τ. The algebra $w^{\tau}_{r,s}(\mathfrak{g})$ is generated by E_i, F_i, K_i, $\bar{K}_i$, $K'_i, \bar{K}'_i$ $(i \in I)$ and J with a series of relations, which is associated with the generalized Kac-Moody algebra $\mathfrak{g}$. In this paper, it is shown that $(w^{\tau}_{r,s}(\mathfrak{g}), \mu, \eta, \triangle, \varepsilon)$ is a noncommutative and noncocommutative weak Hopf algebra with the weak antipode T, but not a Hopf algebra. Some properties of this class of weak Hopf algebras are also investigated.

We organize the paper as follows. In Section 2 we give some notations and recall some basic facts. In Section 3 we define a two-parameter quantum algebra $w^{\tau}_{r,s}(\mathfrak{g})$. In Section 4 a weak Hopf algebra structure is equipped with $w^{\tau}_{r,s}(\mathfrak{g})$ and a basic fact for $w^{\tau}_{r,s}(\mathfrak{g})$ is described. And furthermore we give a special example in the case of $m = 2$.

2. Notations and preliminaries

We fix some notations and review some fundamental results about generalized Kac-Moody algebras.

Let $I = \{1, \ldots, n\}$ or the set of positive integers and $A = (a_{ij})_{i,j\in I}$ a Borcherds-Cartan matrix. That is, the matrix A satisfies:

(1) $a_{ii} = 2$ or $a_{ii} \le 0$ for all $i \in I$,

(2) $a_{ij} \le 0$ for all $i \ne j$,

(3) $a_{ij} \in \mathbb{Z}$,

(4) $a_{ij} = 0$ if and only if $a_{ji} = 0$. An index i is real if $a_{ii} = 2$ and imaginary if $a_{ii} \le 0$. Let $I^+ = \{i \in I | a_{ii} = 2\}$ and $I^{im} = I - I^+$. In addition, we assume that all $a_{ii} \in 2\mathbb{Z}$ and $a_{ii} \ne 0$.

Let $A = (a_{ij})_{i,j\in I}$ be a symmetrizable Borcherds-Cartan matrix. This means that there exist a set of mutual prime positive integers $\{d_i | i \in I\}$ such that $d_i a_{ij} = d_j a_{ji}$ for all i, $j \in I$.

Let $\alpha = (a_i)_{i\in I} \in \mathbb{Z}^{|I|}$ and $\beta = (b_i)_{i\in I} \in \mathbb{Z}^{|I|}$, where a_i, b_i almost all but finite are zero. For example, $\alpha_i = (a_i)_{i\in I}$ where $a_i = 1$ and $a_j = 0$ with $j \ne i$. We define

$$\langle\, \alpha, \beta\, \rangle = \sum_{i\in I} d_i a_i b_i - \sum_{i<j} a_{ij} d_j a_i b_j.$$

Set

$$(\alpha, \beta) = \langle\, \alpha, \beta\, \rangle + \langle\, \beta, \alpha\, \rangle.$$

It is well known that both $\langle\, -, -\, \rangle$ and $(-, -)$ are well defined. The bilinear forms $\langle\, -, -\, \rangle$ and

$(-,-)$ are called Euler form and symmetric Euler form, respectively. In particular, we have

$$\langle i,j \rangle := \langle \alpha_i, \alpha_j \rangle = \begin{cases} -d_i a_{ij}, & i<j, \\ d_i, & i=j, \\ 0, & i>j. \end{cases}$$

Let $\mathbb{Q}(r,s)$ be the function field in two variables r, s over the field $\mathbb{Q}$ of rational numbers. Let $r_i = r^{d_i}$, $s_i = s^{d_i}$ for $i \in I$. For an indeterminant u, v and an integer n, we define

$$(n)_v = \frac{v^n - 1}{v-1},$$
$$(n)_v! = (n)_v \cdots (2)_v (1)_v \quad \text{and} \quad (0)_v! = 1,$$
$$\binom{n}{k}_v = \frac{(n)_v!}{(k)_v!(n-k)_v!}.$$

Definition 2.1 *The two-parameter quantum group $U_{r,s}(\mathfrak{g})$ associated with symmetrizable Borcherds-Cartan matrix $A=(a_{ij})_{i,j\in I}$ is a unital associative algebra over $\mathbb{Q}(r,s)$ generated by e_i, f_i, $k_i^{\pm 1}$, $k_i^{'\pm 1}$, $i \in I$, subject to the following relations:*

$$k_i k_i^{-1} = k_i^{'} k_i^{'-1} = 1, \tag{2.1}$$
$$k_i k_j = k_j k_i, \quad k_i k_j^{-1} = k_j^{-1} k_i, \quad k_i^{-1} k_j^{-1} = k_j^{-1} k_i^{-1}, \tag{2.2}$$
$$k_i^{'} k_j^{'} = k_j^{'} k_i^{'} \quad k_i^{'} k_j^{'-1} = k_j^{'-1} k_i^{'} \quad k_i^{'-1} k_j^{'-1} = k_j^{'-1} k_i^{'-1}, \tag{2.3}$$
$$k_j e_i = r^{\langle i,j \rangle} s^{-\langle j,i \rangle} e_i k_j, \quad k_j^{'} e_i = r^{-\langle j,i \rangle} s^{\langle i,j \rangle} e_i k_j^{'}, \tag{2.4}$$
$$k_j f_i = r^{-\langle i,j \rangle} s^{\langle j,i \rangle} f_i k_j, \quad k_j^{'} f_i = r^{\langle j,i \rangle} s^{-\langle i,j \rangle} f_i k_j^{'}, \tag{2.5}$$
$$e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{'}}{r_i - s_i}, \tag{2.6}$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_i c(i,j,k) e_i^{1-a_{ij}-k} e_j e_i^k = 0 \quad \textit{if} \quad a_{ii}=2,\ i \neq j, \tag{2.7}$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_i c(i,j,k) f_i^k f_j f_i^{1-a_{ij}-k} = 0 \quad \textit{if} \quad a_{ii}=2,\ i \neq j, \tag{2.8}$$
$$e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0, \quad \textit{if} \quad a_{ij} = 0, \tag{2.9}$$

where

$$c(i,j,k) = (r_i s_i^{-1})^{\frac{k(k-1)}{2}} r^{k\langle j,i \rangle} s^{-k\langle i,j \rangle}, \quad i \neq j,$$
$$\binom{1-a_{ij}}{k}_i := \binom{1-a_{ij}}{k}_{r_i s_i^{-1}}.$$

Proposition 2.1 *The algebra $U_{r,s}(\mathfrak{g})$ is a Hopf algebra with the co-multiplication, the counit and the antipode given by*

$$\triangle(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1}, \quad \triangle(k_i^{'\pm 1}) = k_i^{'\pm 1} \otimes k_i^{'\pm 1};$$
$$\triangle(e_i) = 1 \otimes e_i + e_i \otimes k_i, \quad \triangle(f_i) = f_i \otimes 1 + k_i^{'} \otimes f_i;$$
$$\varepsilon(k_i^{\pm 1}) = \varepsilon(k_i^{'\pm 1}) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0;$$
$$S(k_i^{\pm 1}) = k_i^{\mp 1}, \quad S(k_i^{'\pm 1}) = k_i^{'\mp 1};$$

$$S(e_i)=-e_ik_i^{-1},\quad S(f_i)=-k_i^{'-1}f_i.$$

Proof It suffices to prove that $\triangle$, ε are algebra homomorphisms and S is an antipode. One can refer to the proof of Theorem in Section 4. We only prove that S is the antipode of $U_{r,s}(\mathfrak{g})$. It is easy to see that S keeps relations (2.1), (2.2), (2.3) and (2.9),

$$S(e_i)S(k_j)=-e_ik_i^{-1}k_j^{-1}=-r^{\langle i,j\rangle}s^{-\langle j,i\rangle}k_j^{-1}e_ik_i^{-1}=-r^{\langle i,j\rangle}s^{-\langle j,i\rangle}S(k_j)S(e_i).$$

Similarly, we have $S(e_i)S(k_j^{'})=-r^{-\langle j,i\rangle}s^{\langle i,j\rangle}S(k_j^{'})S(e_i)$,

$$\begin{aligned}S(f_j)S(e_i)-S(e_i)S(f_j)&=k_j^{'-1}f_je_ik_i^{-1}-e_ik_i^{-1}k_j^{'-1}f_j\\&=\delta_{ij}k_j^{'-1}\frac{k_i^{'}-k_i}{r_i-s_i}k_i^{-1}=\delta_{ij}\frac{T(k_i)-T(k_i^{'})}{r_i-s_i}.\end{aligned}$$

The map S also keeps quantum Serre relations. Indeed, we let $s=1-a_{ij}$

$$\begin{aligned}&\sum_{k=0}^{s}(-1)^k\binom{s}{k}_i c(i,j,k)S(e_i)^kS(e_j)S(e_i)^{s-k}\\&=\sum_{k=0}^{s}(-1)^k\binom{s}{k}_i c(i,j,k)(-e_ik_i^{-1})^k(-e_jk_j^{-1})(-e_ik_i^{-1})^{s-k}\\&=(-1)^{s+1}(r_i^{-1}s_i)^{k(s-k)}r^{-k\langle j,i\rangle-(s-k)\langle i,j\rangle}s^{k\langle i,j\rangle+(s-k)\langle j,i\rangle}\times\\&\quad\Big(\sum_{k=0}^{s}(-1)^k\binom{s}{k}_i c(i,j,k)e_i^ke_je_i^{s-k}\Big)(k_i^{-1})^sk_j^{-1}=0.\end{aligned}$$

For the relation (2.8) the proof is similar.

It remains to prove that the following relation

$$\sum_{(x)}x'S(x'')=\sum_{(x)}S(x')x''=\varepsilon(x)$$

holds when x is any of the generators $e_i,f_i,k_i^{\pm1},k_i^{'\pm1},i\in I$. For the generator e_i, we have

$$\begin{aligned}\sum_{(e_i)}e_i^{'}S(e_i^{''})&=\mu(\mathrm{id}\otimes S)\triangle(e_i)=\mu(\mathrm{id}\otimes S)(1\otimes e_i+e_i\otimes k_i)\\&=S(e_i)+e_iS(k_i)=-e_ik_i^{-1}+e_ik_i^{-1}=0\end{aligned}$$

and

$$\begin{aligned}\sum_{(e_i)}S(e_i^{'})e_i^{''}&=\mu(S\otimes\mathrm{id})\triangle(e_i)=\mu(S\otimes\mathrm{id})(1\otimes e_i+e_i\otimes k_i)\\&=e_i+S(e_i)k_i=e_i-e_ik_i^{-1}k_i=0.\end{aligned}$$

Since $\varepsilon(e_i)=0$, we have $\sum_{(e_i)}e_i^{'}S(e_i^{''})=\sum_{(e_i)}S(e_i^{'})e_i^{''}=\varepsilon(e_i)$. For other generators the proof is similar. This completes the proof. □

3. The τ-type weak quantum algebras $w_{r,s}^{\tau}(\mathfrak{g})$

Let $m\geq 2$ be a fixed positive integer. Let us introduce generators J, K_i, $\bar{K}_i$, $K_i^{'}$ and $\bar{K}_i^{'}$

with $i \in I$, which satisfy the following relations:

$$J^m = J, \quad J = K_i \bar{K}_i = \bar{K}_i K_i = K_i^{'} \bar{K}_i^{'} = \bar{K}_i^{'} K_i^{'}. \tag{3.1}$$

We suppose K_i and $\bar{K}_i$ are not zero divisors and

$$K_i J^{m-1} = J^{m-1} K_i = K_i, \quad \bar{K}_i J^{m-1} = J^{m-1} \bar{K}_i = \bar{K}_i. \tag{3.2}$$

$$K_i^{'} J^{m-1} = J^{m-1} K_i^{'} = K_i^{'}, \quad \bar{K}_i^{'} J^{m-1} = J^{m-1} \bar{K}_i^{'} = \bar{K}_i^{'}. \tag{3.3}$$

An element E_i is said to be of type $m-1$ if it satisfies

$$\begin{aligned} K_j E_i &= r^{\langle i,j \rangle} s^{-\langle j,i \rangle} E_i K_j, \quad \bar{K}_j E_i = r^{-\langle i,j \rangle} s^{\langle j,i \rangle} E_i \bar{K}_j; \\ K_j^{'} E_i &= r^{-\langle j,i \rangle} s^{\langle i,j \rangle} E_i K_j^{'}, \quad \bar{K}_j^{'} E_i = r^{\langle j,i \rangle} s^{-\langle i,j \rangle} E_i \bar{K}_j^{'}. \end{aligned} \tag{3.4}$$

Similarly, if

$$\begin{aligned} K_j F_i &= r^{-\langle i,j \rangle} s^{\langle j,i \rangle} F_i K_j, \quad \bar{K}_j F_i = r^{\langle i,j \rangle} s^{-\langle j,i \rangle} F_i \bar{K}_j; \\ K_j^{'} F_i &= r^{\langle j,i \rangle} s^{-\langle i,j \rangle} F_i K_j^{'}, \quad \bar{K}_j^{'} F_i = r^{-\langle j,i \rangle} s^{\langle i,j \rangle} F_i \bar{K}_j^{'}, \end{aligned} \tag{3.5}$$

then F_i is said to be of type $m-1$.

To define other relations, it is convenient to set $J^0 = J^{m-1}$. Suppose

$$K_j E_i J^t \bar{K}_j = r^{\langle i,j \rangle} s^{-\langle j,i \rangle} E_i J^{t+1}, \quad K_j^{'} E_i J^t \bar{K}_j^{'} = r^{-\langle j,i \rangle} s^{\langle i,j \rangle} E_i J^{t+1}, \quad E_i J^{m-1} = E_i \tag{3.6}$$

for some $0 \leq t \leq m-2$. Then we say that E_i is of type t. Similarly, F_i is of type t if it satisfies the following

$$K_j F_i J^t \bar{K}_j = r^{-\langle i,j \rangle} s^{\langle j,i \rangle} F_i J^{t+1}, \quad K_j^{'} F_i J^t \bar{K}_j^{'} = r^{\langle j,i \rangle} s^{-\langle i,j \rangle} F_i J^{t+1}, \quad F_i J^{m-1} = F_i. \tag{3.7}$$

Lemma 3.1 *(1) If E_i is of type t for $0 \leq t \leq m-2$, then $E_i J^{t+1}$ is of type $m-1$.*

(2) If F_i is of type t for $0 \leq t \leq m-2$, then $F_i J^{t+1}$ is of type $m-1$.

Proof If E_i is of type t, we have

$$K_j E_i J^{t+1} = K_j E_i J^t J = K_j E_i J^t \bar{K}_j K_j = r^{\langle i,j \rangle} s^{-\langle j,i \rangle} E_i J^{t+1} K_j$$

and

$$\bar{K}_j E_i J^{t+1} = r^{-\langle i,j \rangle} s^{\langle j,i \rangle} \bar{K}_j K_j E_i J^t \bar{K}_j = r^{-\langle i,j \rangle} s^{\langle j,i \rangle} J E_i J^t \bar{K}_j = r^{-\langle i,j \rangle} s^{\langle j,i \rangle} E_i J^{t+1} \bar{K}_j.$$

Similarly,

$$K_j^{'} E_i J^{t+1} = K_j^{'} E_i J^t J = K_j^{'} E_i J^t \bar{K}_j^{'} K_j^{'} = r^{-\langle j,i \rangle} s^{\langle i,j \rangle} E_i J^{t+1} K_j^{'}$$

and

$$\bar{K}_j^{'} E_i J^{t+1} = r^{\langle j,i \rangle} s^{-\langle i,j \rangle} \bar{K}_j^{'} K_j^{'} E_i J^t \bar{K}_j^{'} = r^{\langle j,i \rangle} s^{-\langle i,j \rangle} J E_i J^t \bar{K}_j^{'} = r^{\langle j,i \rangle} s^{-\langle i,j \rangle} E_i J^{t+1} \bar{K}_j{'}.$$

So $E_i J^{t+1}$ is of type $m-1$ by definition.

The proof of (2) is similar to that of (1). □

Proposition 3.1 *(1) E_i is of type t for $0 \leq t \leq m-2$ if and only if E_i is of type $m-1$ and $E_i J^{m-1} = E_i$.*

(2) F_i is of type t for $0 \le t \le m-2$ if and only if F_i is of type $m-1$ and $F_iJ^{m-1} = F_i$.

Proof (1) If E_i is of type t for $0 \le t \le m-2$, then $K_jE_iJ^{t+1} = r^{\langle i,j \rangle}s^{-\langle j,i \rangle}E_iJ^{t+1}K_j$ by Lemma 3.1. So we have

$$K_jE_i = K_jE_iJ^{m-1} = K_jE_iJ^{t+1}J^{m-t-2} = r^{\langle i,j \rangle}s^{-\langle j,i \rangle}E_iJ^{t+1}K_jJ^{m-t-2} = r^{\langle i,j \rangle}s^{-\langle j,i \rangle}E_iK_j$$

and

$$\bar{K}_jE_i = \bar{K}_jE_iJ^{m-1} = \bar{K}_jE_iJ^{t+1}J^{m-t-2} = r^{-\langle i,j \rangle}s^{\langle j,i \rangle}E_iJ^{t+1}\bar{K}_jJ^{m-t-2} = r^{-\langle i,j \rangle}s^{\langle j,i \rangle}E_i\bar{K}_j.$$

Similarly,

$$K'_jE_i = K'_jE_iJ^{m-1} = K'_jE_iJ^{t+1}J^{m-t-2} = r^{-\langle j,i \rangle}s^{\langle i,j \rangle}E_iJ^{t+1}K'_jJ^{m-t-2} = r^{-\langle j,i \rangle}s^{\langle i,j \rangle}E_iK'_j$$

and

$$\bar{K}'_jE_i = \bar{K}'_jE_iJ^{m-1} = \bar{K}'_jE_iJ^{t+1}J^{m-t-2} = r^{\langle j,i \rangle}s^{-\langle i,j \rangle}E_iJ^{t+1}\bar{K}'_jJ^{m-t-2} = r^{\langle j,i \rangle}s^{-\langle i,j \rangle}E_i\bar{K}'_j.$$

Hence E_i is of type $m-1$.

Conversely, if E_i is of type $m-1$ and $E_iJ^{m-1} = E_i$, then

$$K_jE_iJ^t\bar{K}_j = r^{\langle i,j \rangle}s^{-\langle j,i \rangle}E_iK_jJ^t\bar{K}_j = r^{\langle i,j \rangle}s^{-\langle j,i \rangle}E_iJ^{t+1},$$
$$K'_jE_iJ^t\bar{K}'_j = r^{-\langle j,i \rangle}s^{\langle i,j \rangle}E_iK'_jJ^t\bar{K}'_j = r^{-\langle j,i \rangle}s^{\langle i,j \rangle}E_iJ^{t+1}.$$

This means that E_i is of type t.

(2) The proof of (2) is similar to (1). □

Remark 3.1 Note that if E_i is of type t, then $JE_i = E_iJ$. Indeed, if E_i is of type $m-1$ for example, then

$$JE_i = K_j\bar{K}_jE_i = r^{-\langle i,j \rangle}s^{\langle i,j \rangle}K_jE_i\bar{K}_j = E_iK_j\bar{K}_j = E_iJ.$$

Similarly, $JF_i = F_iJ$ if F_i is of type t.

The types of E_i and F_i are denoted by κ_i, $\bar{\kappa}'_i$, respectively. Let $\tau = (\{\kappa_i\}_{i\in I}|\ \{\bar{\kappa}'_i)\}_{i\in I})$ and the τ is called admissible if it satisfies the following condition:

(1) If $k_i = t$, then $\bar{k}_i = t$ for $1 \le t \le m-2$;

(2) If $k_i = 0$, then $\bar{k}_i = 0,\ m-1$;

(3) If $k_i = m-1$, then $\bar{k}_i = 0,\ m-1$.

In the sequel, we always assume that τ is admissible and $m \ge 2$.

Definition 3.2 *The algebra $w^{\tau}_{r,s}(\mathfrak{g})$ over $\mathbb{Q}(r,s)$ is generated by E_i, F_i, K_i, $\bar{K}_i$, K'_i, $\bar{K}'_i$ $(i \in I)$ and J, which satisfy the following relations:*

$$J = K_i\bar{K}_i = K'_i\bar{K}'_i \quad \text{for all} \quad i \in I, \tag{3.8}$$
$$J^{m-1}x = xJ^{m-1} = x, \ \text{for } x = K_i, \bar{K}_i, K'_i, \bar{K}'_i, \tag{3.9}$$
$$K_i\bar{K}_j = \bar{K}_jK_i, \quad K_iK_j = K_jK_i, \quad \bar{K}_i\bar{K}_j = \bar{K}_j\bar{K}_i, \tag{3.10}$$
$$K'_i\bar{K}'_j = \bar{K}'_jK'_i, \quad K'_iK'_j = K'_jK'_i, \quad \bar{K}_i{}'\bar{K}_j{}' = \bar{K}'_j\bar{K}'_i, \tag{3.11}$$
$$K'_i\bar{K}_j = \bar{K}_jK'_i, \quad K_iK'_j = K'_jK_i, \quad \bar{K}'_iK_j = K_j\bar{K}'_i, \quad \bar{K}'_i\bar{K}_j = \bar{K}_j\bar{K}'_i, \tag{3.12}$$

$$E_i\, F_i \text{ are of admissible type } \tau, \tag{3.13}$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{'}}{r_i - s_i}, \tag{3.14}$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_i c(i,j,k)E_i^{1-a_{ij}-k}E_jE_i^k = 0 \;\; \text{if} \;\; a_{ii}=2, i\neq j, \tag{3.15}$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_i c(i,j,k)F_i^kF_jF_i^{1-a_{ij}-k} = 0 \;\; \text{if} \;\; a_{ii}=2, i\neq j, \tag{3.16}$$

$$E_iE_j - E_jE_i = F_iF_j - F_jF_i = 0, \; \text{if } a_{ij}=0, \tag{3.17}$$

where

$$c(i,j,k) = (r_is_i^{-1})^{\frac{k(k-1)}{2}} r^{k\langle j,i\rangle}s^{-k\langle i,j\rangle}, \quad i\neq j,$$
$$\binom{1-a_{ij}}{k}_i := \binom{1-a_{ij}}{k}_{r_is_i^{-1}}.$$

The algebra $w_{r,s}^{\tau}(\mathfrak{g})$ is said to be a τ-type weak quantum algebra associated with the generalized Kac-Moody algebra $\mathfrak{g}$.

It is straightforward to check by induction that E_i (respectively F_i) is of type $m-1$ or type t for $0 \le t \le m-2$, the following relations hold in $w_{r,s}^{\tau}(\mathfrak{g})$

$$E_i^mK_j^n = r^{-mn\langle i,j\rangle}s^{mn\langle j,i\rangle}K_j^nE_i^m, \quad F_i^mK_j^n = r^{mn\langle i,j\rangle}s^{-mn\langle j,i\rangle}K_j^nF_i^m.$$
$$E_i^mK_j^{'n} = r^{mn\langle j,i\rangle}s^{-mn\langle i,j\rangle}K_j^{'n}E_i^m, \quad F_i^mK_j^{'n} = r^{-mn\langle j,i\rangle}s^{mn\langle i,j\rangle}K_j^{'n}F_i^m.$$

In particular, we have

$$E_i^mK_i^n = r_i^{-mn}s_i^{mn}K_i^nE_i^m, \quad F_i^mK_i^n = r_i^{mn}s_i^{-mn}K_i^nF_i^m,$$
$$E_i^mK_i^{'n} = r_i^{mn}s_i^{-mn}K_i^{'n}E_i^m, \quad F_i^mK_i^{'n} = r_i^{-mn}s_i^{mn}K_i^{'n}F_i^m.$$

By Remark 3.1, J^t is a central element for all $0 \le t \le m-1$ in $w_{r,s}^{\tau}(\mathfrak{g})$.

4. The weak Hopf algebras structure of $w_{r,s}^{\tau}(\mathfrak{g})$

To make the τ-type algebra $w_{r,s}^{\tau}(\mathfrak{g})$ become a weak Hopf algebra, we define three maps

$$\triangle: \quad w_{r,s}^{\tau}(\mathfrak{g}) \longrightarrow w_{r,s}^{\tau}(\mathfrak{g})\otimes w_{r,s}^{\tau}(\mathfrak{g})$$
$$\varepsilon: \quad w_{r,s}^{\tau}(\mathfrak{g}) \longrightarrow K$$
$$T: \quad w_{r,s}^{\tau}(\mathfrak{g}) \longrightarrow w_{r,s}^{\tau}(\mathfrak{g})$$

as follows

$$\Delta(K_i) = K_i\otimes K_i, \quad \Delta(\bar{K}_i) = \bar{K}_i\otimes\bar{K}_i, \tag{4.1}$$

$$\Delta(K_i^{'}) = K_i^{'}\otimes K_i^{'}, \quad \Delta(\bar{K}_i^{'}) = \bar{K}_i^{'}\otimes\bar{K}_i^{'}, \tag{4.2}$$

$$\Delta(E_i) = J^{m-1-t}\otimes E_i + E_i\otimes K_iJ^t, \quad E_i \text{ is of type } t. \tag{4.3}$$

If $t=0$, then $\Delta(E_i) = J^{m-1}\otimes E_i + E_i\otimes K_i$. If $t = m-1$, then $\Delta(E_i) = 1\otimes E_i + E_i\otimes K_i$. Similarly,

$$\Delta(F_i) = F_i\otimes J^{m-1-t} + K_i^{'}J^t\otimes F_i, \quad F_i \text{ is of type } t, \tag{4.4}$$

$$\varepsilon(K_i) = \varepsilon(\bar{K}_i) = 1, \quad \varepsilon(K_i^{'}) = \varepsilon(\bar{K}_i^{'}) = 1, \quad \varepsilon(J) = 1, \tag{4.5}$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0. \tag{4.6}$$

The map T is defined as follows

$$T(1) = 1, \quad T(K_i) = \bar{K}_i J^{m-2}, \quad T(\bar{K}_i) = K_i J^{m-2}, \tag{4.7}$$

$$T(J) = J^{m-2}, \quad T(K_i^{'}) = \bar{K}_i^{'} J^{m-2}, \quad T(\bar{K}_i^{'}) = K_i^{'} J^{m-2}, \tag{4.8}$$

$$T(E_i) = -E_i \bar{K}_i J^{m-2}, \quad T(F_i) = -\bar{K}_i^{'} F_i J^{m-2}. \tag{4.9}$$

We use μ, η to denote the multiplication and unit of $w_{r,s}^{\tau}(\mathfrak{g})$. The main theorem is as follows.

Theorem *$(w_{r,s}^{\tau}(\mathfrak{g}), \mu, \eta, \triangle, \varepsilon)$ is a noncommutative and noncocommutative weak Hopf algebra with the weak antipode T, but not a Hopf algebra.*

The Theorem follows from Lemmas 4.1 and 4.2 below.

Lemma 4.1 *$w_{r,s}^{\tau}(\mathfrak{g})$ is a bialgebra with a co-multiplication $\triangle$ and co-unit ε.*

Proof It is easy to check that the following relations hold:

$$\triangle(J) = \triangle(K_i) \triangle (\bar{K}_i) = \triangle(K_i^{'}) \triangle (\bar{K}_i^{'}),$$

$$\triangle(K_i) \triangle (\bar{K}_j) = \triangle(\bar{K}_j) \triangle (K_i), \quad \triangle(K_i^{'}) \triangle (\bar{K}_j^{'}) = \triangle(\bar{K}_j^{'}) \triangle (K_i^{'}),$$

$$\triangle(K_i J^{m-1}) = \triangle(K_i), \quad \triangle(\bar{K}_i J^{m-1}) = \triangle(\bar{K}_i),$$

$$\triangle(K_i^{'} J^{m-1}) = \triangle(K_i^{'}), \quad \triangle(\bar{K}_i^{'} J^{m-1}) = \triangle(\bar{K}_i^{'}).$$

If E_i is of type $m-1$, then

$$\begin{aligned}\Delta(K_j)\Delta(E_i) &= (K_j \otimes K_j)((1 \otimes E_i + E_i \otimes K_i) = K_j \otimes K_j E_i + K_j E_i \otimes K_j K_i \\ &= r^{\langle i,j \rangle} s^{-\langle j,i \rangle} \Delta(E_i)\Delta(K_j).\end{aligned}$$

Similarly, we can prove $\Delta(K_j^{'})\Delta(E_i) = r^{-\langle j,i \rangle} s^{\langle i,j \rangle} \Delta(E_i)\Delta(K_j^{'})$.

If E_i is of type t for $0 \le t \le m-2$, then

$$\begin{aligned}\Delta(K_j)\Delta(E_i)\Delta(J)^{t+1} &= (K_j \otimes K_j)(J^{m-1-t} \otimes E_i + E_i \otimes K_i J^t)(J^t \otimes J^t)(\bar{K}_j \otimes \bar{K}_j) \\ &= K_j J^{m-1-t} J^t \bar{K}_j \otimes K_j E_i J^t \bar{K}_j + K_j E_i J^t \bar{K}_j \otimes K_j K_i J^t J^t \bar{K}_j \\ &= r^{\langle i,j \rangle} s^{-\langle j,i \rangle} (J^m \otimes E_i J^{t+1} + E_i J^{t+1} \otimes K_i J^{2t+1}) \\ &= r^{\langle i,j \rangle} s^{-\langle j,i \rangle} \Delta(E_i)\Delta(J)^{t+1}\end{aligned}$$

and

$$\Delta(E_i)\Delta(J) = (J^{m-t} \otimes E_i J + E_i J \otimes K_i J^{t+1}) = \Delta(J)\Delta(E_i).$$

Similarly, we can prove

$$\Delta(K_j^{'})\Delta(E_i)\Delta(J)^t \Delta(\bar{K}_j^{'}) = r^{-\langle j,i \rangle} s^{\langle i,j \rangle} \Delta(E_i)\Delta(J)^{t+1}.$$

For the F_i cases, all relations can be proved in a similar way.

To show that

$$\Delta(E_i)\Delta(F_j)-\Delta(F_j)\Delta(E_i)=\delta_{ij}\frac{\Delta(K_i)-\Delta(K_i^{'})}{r_i-s_i}, \tag{4.10}$$

we assume that E_i and F_j are of type p, q, respectively, where $0\le p,q\le m-2$. We have

$$\begin{aligned}\Delta(E_i)\Delta(F_j)-\Delta(F_j)\Delta(E_i)=&(J^{m-1-p}\otimes E_i+E_i\otimes K_iJ^p)(F_j\otimes J^{m-1-q}+K_j^{'}J^q\otimes F_j)-\\&(F_j\otimes J^{m-1-q}+K_j^{'}J^q\otimes F_j)(J^{m-1-p}\otimes E_i+E_i\otimes K_iJ^p)\\=&J^{m-1-p+q}K_j^{'}\otimes(E_iF_j-F_jE_i)+(E_iF_j-F_jE_i)\otimes K_iJ^{m-1-p+q}\\=&\delta_{ij}\frac{\Delta(K_i)-\Delta(K_i^{'})}{r_i-s_i}.\end{aligned}$$

Suppose E_i is of type $m-1$ and F_j is of type q for $q=0,\ m-1$. We have

$$\begin{aligned}\Delta(E_i)\Delta(F_j)-\Delta(F_j)\Delta(E_i)=&(1\otimes E_i+E_i\otimes K_i)(F_j\otimes J^{m-1-q}+K_j^{'}J^q\otimes F_j)-\\&(F_j\otimes J^{m-1-q}+K_j^{'}J^q\otimes F_j)(1\otimes E_i+E_i\otimes K_i)\\=&K_j^{'}J^q\otimes(E_iF_j-F_jE_i)+(E_iF_j-F_jE_i)\otimes K_iJ^{m-1-q}\\=&\delta_{ij}\frac{\Delta(K_i)-\Delta(K_i^{'})}{r_i-s_i}.\end{aligned}$$

Similarly, we can prove that

$$\Delta(E_i)\Delta(F_j)-\Delta(F_j)\Delta(E_i)=\delta_{ij}\frac{\Delta(K_i)-\Delta(K_i^{'})}{r_i-s_i}$$

if E_i is of type p for $p=0,m-1$ and F_j is of type $m-1$. Therefore (3.14) holds for all $i,\ j$.

Finally, we have to prove that $\triangle$ satisfies the relations (3.15)–(3.17). The direct calculation shows that

$$\Delta(E_i)\Delta(E_j)-\Delta(E_j)\Delta(E_i)=0,\quad \Delta(F_i)\Delta(F_j)-\Delta(F_j)\Delta(F_i)=0.$$

Hence $\triangle$ satisfies the relation (3.17).

To show that $\triangle$ satisfies relation (3.15), the following cases should be considered:

(1) E_i is of type t, E_j is of type p. $a_{ii}=2$, where $0\le t,\ p\le m-2$;

(2) E_i is of type $m-1$, E_j is of type p. $a_{ii}=2$, where $0\le p\le m-2$;

(3) E_i is of type t, E_j is of type $m-1$. $a_{ii}=2$, where $0\le t\le m-2$;

(4) E_i and E_j are of type $m-1$.

We will show that $\triangle$ keeps the relation (3.14) for the case (1). The proof for the rest cases are more or less the same as the case of $U_q(g)$[11, pp 67-68].

Let $r=1-a_{ij}$ and

$$u_{ij}=\sum_{a=0}^{r}(-1)^a\binom{r}{a}_i(\frac{r_i}{s_i})^{\frac{a-(a-1)}{2}}r^{a\langle j,i\rangle}s^{-a\langle i,j\rangle}E_i^{r-a}E_jE_i^a.$$

Since $\triangle(E_i)=J^{(m-1-t)}\otimes E_i+E_i\otimes K_iJ^t$ and

$$(E_i\otimes K_iJ^t)(J^{m-1-t}\otimes E_i)=r_is_i^{-1}(J^{(m-1-t)}\otimes E_i)(E_i\otimes K_iJ^t),$$

we have

$$\triangle(E_i)^a=\sum_{\beta=0}^{a}\binom{a}{\beta}_{r_is_i^{-1}}J^{(m-1-t)\beta}E_i^{a-\beta}\otimes E_i^{\beta}K_i^{a-\beta}J^{t(a-\beta)}.$$

This implies that

$$\sum_{a=0}^{r}(-1)^a\binom{r}{a}_i(\frac{r_i}{s_i})^{\frac{a-(a-1)}{2}}r^{a\langle j,i\rangle}s^{-a\langle i,j\rangle}\triangle(E_i)^{r-a}\triangle(E_j)\triangle(E_i)^a$$

$$=J^{(m-1-t)r+(m-1-p)}\otimes u_{ij}+u_{ij}\otimes J^{tr+p}K_i^rK_j+\sum_{\varepsilon=1}^{r}J^{(m-1-t)\varepsilon+(m-1-p)}E_i^{r-\varepsilon}\otimes X_\varepsilon+$$

$$\sum_{l,n}J^{(m-1-t)(r-l-n)}E_i^lE_jE_i^n\otimes Y_{l,n}$$

with suitable X_ε and $Y_{l,n}$. The last sum is over the integers $l,n\geq 0$ with $l+n<r$. We have to show that all X_ε and $Y_{l,n}$ are equal to zero. Since

$$K_i^{r-a-\zeta}E_jE_i^{\varepsilon-\zeta}=r^{(r-a-\zeta)\langle j,i\rangle}s^{-(r-a-\zeta)\langle i,j\rangle}(r_is_i^{-1})^{(r-a-\zeta)(\varepsilon-\zeta)}E_jE_i^{\varepsilon-\zeta}K_i^{r-a-\zeta},$$

X_ε (with $1\leq\varepsilon\leq r$) is just the following identity

$$X_\varepsilon=\sum_{a=0}^{r}(-1)^a\binom{r}{a}_i(r_is_i^{-1})^{\frac{a(a-1)}{2}}r^{a\langle j,i\rangle}s^{-a\langle i,j\rangle}\sum_{\zeta}\binom{r-a}{\zeta}_iE_i^{\zeta}K_i^{r-a-\zeta}E_j\binom{a}{\varepsilon-\zeta}_i$$
$$E_i^{\varepsilon-\zeta}K_i^{a-\varepsilon+\zeta}J^{t(r-\varepsilon)}$$
$$=\sum_{a=0}^{r}\sum_{\zeta}(-1)^a\binom{r}{a}_i\binom{r-a}{\zeta}_i\binom{a}{\varepsilon-\zeta}_i(r_is_i^{-1})^{\frac{a(a-1)}{2}+(r-a-\zeta)(\varepsilon-\zeta)}$$
$$r^{(r-\zeta)\langle j,i\rangle}s^{-(r-\zeta)\langle i,j\rangle}E_i^{\zeta}E_jE_i^{\varepsilon-\zeta}K_i^{(r-\varepsilon)}J^{t(r-\varepsilon)}=0.$$

Since $K_i^{r-a-l}K_jE_i^n=(r_is_i^{-1})^{n(r-a-l)}r^{n\langle i,j\rangle}s^{-n\langle j,i\rangle}E_i^nK_i^{r-a-l}K_j$ for all l,n as above the term $Y_{l,n}$ is equal to

$$Y_{l,n}=\sum_{a=n}^{r-l}(-1)^a\binom{r}{a}_i(r_is_i^{-1})^{\frac{a(a-1)}{2}}r^{a\langle j,i\rangle}s^{-a\langle i,j\rangle}\binom{r-a}{l}_iE_i^lK_i^{r-a-l}K_j$$
$$\binom{a}{n}_iE_i^nK_i^{a-n}J^{t(r-l-n)+p}$$
$$=\sum_{a=n}^{r-l}(-1)^a\binom{r}{a}_i\binom{r-a}{l}_i\binom{a}{n}_i(r_is_i^{-1})^{\frac{a(a-1)}{2}+n(r-a-l)}$$
$$r^{a\langle j,i\rangle+n\langle i,j\rangle}s^{-a\langle i,j\rangle-n\langle j,i\rangle}E_i^lE_i^nK_i^{(r-l-n)}K_jJ^{t(r-l-n)+p}=0.$$

Hence, $\triangle$ keeps the relation (3.15). Similarly, we can prove that $\triangle$ satisfies the relation (3.16). Therefore, $\triangle$ is an algebra homomorphism. On the other hand, it is easy to see that

$$(\triangle\otimes\mathrm{id})\triangle(x)=(\mathrm{id}\otimes\triangle)\triangle(x)\text{ for any }x\in w_{r,s}^{\tau}(\mathfrak{g})$$

and that ε is a homomorphism from $w_{r,s}^{\tau}(\mathfrak{g})$ to $\mathbb{Q}(r,s)$ and enjoys the counit axioms. This proves $w_{r,s}^{\tau}(\mathfrak{g})$ is a bialgebra. □

Lemma 4.2 *T is a weak antipode of the bialgebra $w_{r,s}^{\tau}(\mathfrak{g})$.*

Proof The following relations hold

$$T(K_i)T(\bar{K}_j)=T(\bar{K}_j)T(K_i),\quad T(K_i^{'})T(\bar{K}_j^{'})=T(\bar{K}_j^{'})T(K_i^{'}),$$
$$T(K_i^{'})T(\bar{K}_j)=T(\bar{K}_j)T(K_i^{'}),\quad T(\bar{K}_i^{'})T(K_j)=T(K_j)T(\bar{K}_i^{'}),$$
$$T(J^{m-1})T(\bar{K}_i)=T(\bar{K}_i),\quad T(J^{m-1})T(\bar{K}_i^{'})=T(\bar{K}_i^{'}),$$
$$T(J^{m-1})T(K_i)=T(K_i),\quad T(J^{m-1})T(K_i^{'})=T(K_i^{'}),$$
$$T(E_i)T(E_j)=T(E_j)T(E_i),\quad T(F_i)T(F_j)=T(F_j)T(F_i),\quad \text{if}\quad a_{ij}=0.$$

If E_i is of type $m-1$, then

$$\begin{aligned}T(E_i)T(K_j)&=-E_i\bar{K}_iJ^{m-2}\bar{K}_jJ^{m-2}=-r^{\langle i,j\rangle}s^{-\langle j,i\rangle}\bar{K}_jJ^{m-2}E_i\bar{K}_iJ^{m-2}\\&=r^{\langle i,j\rangle}s^{-\langle j,i\rangle}T(K_j)T(E_i).\end{aligned}$$

If E_i is of type t for $0\le t\le m-2$, then

$$\begin{aligned}T(\bar{K}_j)T(J)^tT(E_i)T(K_j)&=-K_jJ^{m-2}J^{(m-2)t}E_i\bar{K}_iJ^{m-2}\bar{K}_jJ^{m-2}\\&=-r^{\langle i,j\rangle}s^{-\langle j,i\rangle}J^{(m-2)(t+1)}E_i\bar{K}_iJ^{m-2}\\&=r^{\langle i,j\rangle}s^{-\langle j,i\rangle}T(J)^{t+1}T(E_i).\end{aligned}$$

The following relation can be proved in a similar way

$$T(\bar{K}_j^{'})T(J)^tT(E_i)T(K_j^{'})=r^{-\langle j,i\rangle}s^{\langle i,j\rangle}T(J)^{t+1}T(E_i).$$

Similarly, we can prove

$$T(K_j)T(F_i)=r^{-\langle i,j\rangle}s^{\langle j,i\rangle}T(F_i)T(K_j),T(K_j^{'})T(F_i)=r^{\langle j,i\rangle}s^{-\langle i,j\rangle}T(F_i)T(K_j^{'})$$

and

$$T(\bar{K}_j)T(J)^tT(F_i)T(K_j)=r^{-\langle i,j\rangle}s^{\langle j,i\rangle}T(J)^{t+1}T(F_i),$$
$$T(\bar{K}_j^{'})T(J)^tT(F_i)T(K_j^{'})=r^{\langle j,i\rangle}s^{-\langle i,j\rangle}T(J)^{t+1}T(F_i)$$

if F_i is of type t for $0\le t\le m-2$. Moreover,

$$\begin{aligned}T(F_j)T(E_i)-T(E_i)T(F_j)&=\bar{K}_j^{'}F_jJ^{m-2}E_i\bar{K}_iJ^{m-2}-E_i\bar{K}_iJ^{m-2}\bar{K}_j^{'}F_jJ^{m-2}\\&=J^{2(m-2)}r^{-\langle j,i\rangle}s^{\langle i,j\rangle}(\bar{K}_j^{'}F_jE_i\bar{K}_i-\bar{K}_j^{'}E_i\bar{K}_iF_j)\\&=J^{2(m-2)}(\bar{K}_j^{'}F_jE_i\bar{K}_i-\bar{K}_j^{'}E_iF_j\bar{K}_i)\\&=\delta_{ij}\frac{T(K_i)-T(K_i^{'})}{r_i-s_i}.\end{aligned}$$

T also satisfies the quantum Serre relation. Indeed, suppose $a_{ii}=2$ and $s=1-a_{ij}$. We have

$$\begin{aligned}&\sum_{k=0}^{s}(-1)^k\binom{s}{k}_i c(i,j,k)T(E_i)^kT(E_j)T(E_i)^{s-k}\\&=(-1)^{s+1}J^{(m-2)(s+1)}\sum_{k=0}^{s}(-1)^k\binom{s}{k}_i c(i,j,k)(E_i\bar{K}_i)^k(E_j\bar{K}_j)(E_i\bar{K}_i)^{s-k}\\&=(-1)^{s+1}J^{(m-2)(s+1)}(r_i^{-1}s_i)^{k(s-k)}r^{-k\langle j,i\rangle-(s-k)\langle i,j\rangle}s^{k\langle i,j\rangle+(s-k)\langle j,i\rangle}\end{aligned}$$

$$\Big(\sum_{k=0}^{s}(-1)^k\binom{s}{k}_i c(i,j,k)E_i^kE_jE_i^{s-k}\Big)\bar{K}_j\bar{K}_i^s=0.$$

The argument for F_i is similar. It follows that T can be extended to an anti-automorphism of $w_{r,s}^{\tau}(\mathfrak{g})$. Now, we define the convolution product in the bialgebra $(w_{r,s}^{\tau}(\mathfrak{g}),\mu,\eta,\triangle,\varepsilon)$ as

$$(f*g)(x)=\mu(f*g)\triangle(x)$$

for all $f,\ g\in \mathrm{Hom}(w_{r,s}^{\tau}(\mathfrak{g}),\ w_{r,s}^{\tau}(\mathfrak{g}))$ and $x\in w_{r,s}^{\tau}(\mathfrak{g})$. We have to prove that T satisfies the antipode axioms such that $\mathrm{id}*T*\mathrm{id}=\mathrm{id}$ and $T*\mathrm{id}*T=T$. In fact it can be proved as in[14,16] that $T*\mathrm{id}*T(x)=T(x)$ and $\mathrm{id}*T*\mathrm{id}(x)=\mathrm{id}(x)$ for any $x\in w_{r,s}^{\tau}(\mathfrak{g})$. Therefore, T is a weak antipode of $w_{r,s}^{\tau}(\mathfrak{g})$. □

Assume that $w_{r,s}^{\tau}(\mathfrak{g})$ is a Hopf algebra and there exists an algebra anti-morphism $S: w_{r,s}^{\tau}(\mathfrak{g})\longrightarrow w_{r,s}^{\tau}(\mathfrak{g})$. Then S must satisfy $(S*\mathrm{id})(J)=\mu S(J)$ and $S(J)J=1$ and J is invertible. It is impossible since $J(J^{m-1}-1)=0$. The proof of the theorem is finished. □

Let $\bar{J}=\frac{1}{m-1}\sum_{r=1}^{m-1}J^r$, $W_{r,s}=w_{r,s}^{\tau}(\mathfrak{g})\bar{J}$ and $\bar{W}_{r,s}=w_{r,s}^{\tau}(\mathfrak{g})(1-\bar{J})$. It is easy to see that $\bar{J}$ is a central idempotent element and that $w_{r,s}^{\tau}(\mathfrak{g})=W_{r,s}\oplus\bar{W}_{r,s}$ and $W_{r,s}\cong U_{r,s}(\mathfrak{g})$ as algebras but not as Hopf algebras.

Example 4.1 $W_{r,s}\cong U_{r,s}(\mathfrak{g})$ as Hopf algebras if $m=2$.

To see this, we have $\bar{J}=J$ since $m=2$. It is easy to see that

$$w_{r,s}^{\tau}(\mathfrak{g})=W_{r,s}\oplus\bar{W}_{r,s}$$

as algebras. Note that $W_{r,s}$ is generated by E_iJ, F_iJ, K_iJ, $\bar{K}_iJ$, $K_i^{'}J$, $\bar{K}_i^{'}J$, and J subject to the relations (3.8)–(3.11) and

$$K_j(E_iJ)=r^{\langle i,j\rangle}s^{-\langle j,i\rangle}(E_iJ)K_j,\quad K_j^{'}(E_iJ)=r^{-\langle j,i\rangle}s^{\langle i,j\rangle}(E_iJ)K_j^{\prime},\tag{4.11}$$

$$K_j(F_iJ)=r^{-\langle i,j\rangle}s^{\langle j,i\rangle}(F_iJ)K_j,\quad K_j^{'}(F_iJ)=r^{\langle j,i\rangle}s^{-\langle i,j\rangle}(F_iJ)K_j^{'},\tag{4.12}$$

$$(E_iJ)(F_jJ)-(F_jJ)(E_iJ)=\delta_{ij}\frac{K_iJ-K_i^{'}J}{r_i-s_i},\tag{4.13}$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_i c(i,j,k)(E_iJ)^{1-a_{ij}-k}(E_jJ)(E_iJ)^k=0\quad\text{if}\quad a_{ii}=2,i\neq j,\tag{4.14}$$

$$\sum_{k=0}^{1-a_{ij}}(-1)^k\binom{1-a_{ij}}{k}_i c(i,j,k)(F_iJ)^k(F_jJ)(F_iJ)^{1-a_{ij}-k}=0\quad\text{if}\quad a_{ii}=2,i\neq j,\tag{4.15}$$

$$(E_iJ)(E_jJ)-(E_jJ)(E_iJ)=(F_iJ)(F_jJ)-(F_jJ)(F_iJ)=0\quad\text{if}\quad a_{ij}=0,\tag{4.16}$$

where

$$c(i,j,k)=(r_is_i^{-1})^{\frac{k(k-1)}{2}}r^{k\langle j,i\rangle}s^{-k\langle i,j\rangle},\quad i\neq j.$$

Here J can be viewed as the identity of $W_{r,s}$. At this point of view $W_{r,s}$ is a Hopf algebra. The comultiplication $\triangle$ is

$$\triangle(E_iJ)=J\otimes E_iJ+E_iJ\otimes K_i,$$

$$\triangle(F_iJ)=F_iJ\otimes J+K_i^{'}\otimes F_iJ,$$

$$\triangle(K_iJ) = K_iJ \otimes K_iJ, \quad \triangle(\bar{K}_iJ) = \bar{K}_iJ \otimes \bar{K}_iJ,$$
$$\triangle(K_i^{'}J) = K_i^{'}J \otimes K_i^{'}J, \quad \triangle(\bar{K}_i^{'}J) = \bar{K}_i^{'}J \otimes \bar{K}_i^{'}J.$$

The counit ε is

$$\varepsilon(E_iJ) = \varepsilon(F_iJ) = 0,$$
$$\varepsilon(K_iJ) = \varepsilon(\bar{K}_iJ) = 1, \quad \varepsilon(K_i^{'}J) = \varepsilon(\bar{K}_i^{'}J) = 1.$$

The antipode S is

$$S(E_iJ) = -(E_iJ)\bar{K}_i, \quad S(F_iJ) = -\bar{K}_i^{'}(F_iJ),$$
$$S(K_iJ) = \bar{K}_iJ, \quad S(\bar{K}_iJ) = K_iJ,$$
$$S(K_i^{'}J) = \bar{K}_{i'}J, \quad S(\bar{K}_i^{'}J) = K_i^{'}J.$$

Let ψ be the algebra morphism from $U_{r,s}(\mathfrak{g})$ to $W_{r,s}$, defined by

$$\psi(e_i) = E_iJ, \quad \psi(f_i) = F_iJ, \quad \psi(k_i) = K_iJ, \quad \psi(k_i^{-1}) = \bar{K}_iJ,$$
$$\psi(k_i^{'}) = K_i^{'}J, \quad \psi(k_i^{'-1}) = \bar{K}_i^{'}J, \quad \psi(1) = J.$$

It is straightforward to check that ψ is a Hopf algebra isomorphism. □

5. Remarks

In this paper a class of two-parameter weak Hopf algebras $w_{r,s}^{\tau}(\mathfrak{g})$ corresponding to Borcherds-Cartan matrix is constructed. One observation is possible to extend the construction to a quantized superalgebra. All the two-parameter weak Hopf algebras given in this paper have non-cocommutative coproducts. This implies existence of universal R-matrices that could give new solutions of quantum Yang-Baxter equations. A future work is to investigate the form of such R-matrices. We expect the expressions would not be that different from those of the original Hopf algebra.

References

[1] DU Jie, PARSHALL B, WANG Jianpan. *Two-parameter quantum linear groups and the hyperbolic invariance of q-Schur algebras* [J]. J. London Math. Soc. (2), 1991, **44**(3): 420–436.
[2] DOBREV V K, PARASHAR P. *Duality for multiparametric quantum* GL(n) [J]. J. Phys. A, 1993, **26**(23): 6991–7002.
[3] BERGERON N, GAO Yun, HU Naihong. *Drinfel'd doubles and Lusztig's symmetries of two-parameter quantum groups* [J]. J. Algebra, 2006, **301**(1): 378–405.
[4] BÖHM G, NILL F, SZLACHÁNYI K. *Weak Hopf algebras. I. Integral theory and C^*-structure* [J]. J. Algebra, 1999, **221**(2): 385–438.
[5] HAYASHI T. *An algebra related to the fusion rules of Wess-Zumino-Witten models* [J]. Lett. Math. Phys., 1991, **22**(4): 291–296.
[6] YAMANOUCHI T. *Duality for generalized Kac algebras and a characterization of finite groupoid algebras* [J]. J. Algebra, 1994, **163**(1): 9–50.
[7] LI Fang. *Weak Hopf algebras and some new solutions of the quantum Yang-Baxter equation* [J]. J. Algebra, 1998, **208**(1): 72–100.
[8] YANG Shilin. *Weak Hopf algebras corresponding to Cartan matrices* [J]. J. Math. Phys., 2005, **46**(7): 073502.
[9] BORCHERDS R E. *Generalised Kac-Moody algebras* [J]. J. Algebra, 1988, **115**(2): 501–512.
[10] WU Zhixiang. *A class of weak Hopf algebras related to a Borcherds-Cartan matrix* [J]. J. Phys. A, 2006, **39**(47): 14611–14626.
[11] JANTZEN J C. *Lectures on Quantum Group* [M]. American Mathematical Society, Providence, RI, 1996.