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SZ-Derivations, PZ-Derivations and S-Derivations of a Matrix Algebra over Commutative Rings

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Abstract Let R be a commutative ring with identity, $N_n(R)$ the matrix algebra consisting of all $n \times n$ strictly upper triangular matrices over R with the usual product operation. An R-linear map $\phi : N_n(R) \to N_n(R)$ is said to be an SZ-derivation of $N_n(R)$ if $x^2 = 0$ implies that $\phi(x)x + x\phi(x) = 0$. It is said to be an S-derivation of $N_n(R)$ if $\phi(x^2) = \phi(x)x + x\phi(x)$ for any $x \in N_n(R)$. It is said to be a PZ-derivation of $N_n(R)$ if $xy = 0$ implies that $\phi(x)y + x\phi(y) = 0$. In this paper, by constructing several types of standard SZ-derivations of $N_n(R)$, we first characterize all SZ-derivations of $N_n(R)$. Then, as its application, we determine all S-derivations and PZ-derivations of $N_n(R)$, respectively.

Keywords SZ-derivations; S-derivations; PZ-derivations.

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1. Introduction

Let R be a commutative ring with identity. By an R-algebra (not necessarily associative) we simply mean an R-module $\mathcal{U}$ over R endowed with a bilinear operation $\mathcal{U} \times \mathcal{U} \to \mathcal{U}$, usually denoted by juxtaposition (unless $\mathcal{U}$ is a Lie algebra, in which case we always use the bracket). Recall that a linear map $\delta : \mathcal{U} \to \mathcal{U}$ is called a derivation of $\mathcal{U}$ if it satisfies the familiar product rule $\delta(xy) = x\delta(y) + \delta(x)y$. The problem of characterizing the derivations of matrix algebras and matrix Lie algebras has attracted the attention of some authors. For instance, Jϕndrup[1] characterized all derivations of the matrix ring $T_n(R)$, consisting of all upper triangular matrices over R. Wang[2] described all derivations of every parabolic Lie subalgebras of the general linear Lie algebra $\mathrm{gl}_n(R)$. Wang[3] determined all derivations of any intermediate Lie algebra between the Lie algebra of diagonal matrices and the Lie algebra $T_n(R)$ (with the usual bracket operation). Ou[4] characterized all derivations of the Lie algebra $N_n(R)$ (with the usual bracket operation). Benkovic[5] considered the Jordan derivations and anti-derivations on the R-algebra $T_n(R)$ (with the usual product operation). In the present article we intend to generalize the notion derivations

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to other more general cases.

Definition 1.1 *Let $\mathcal{A}$ be an associative R-algebra. A linear map $\phi : \mathcal{A} \to \mathcal{A}$ is said to be an SZ-derivation of $\mathcal{A}$ if $x^2 = 0$ implies that $\phi(x)x + x\phi(x) = 0$. It is said to be an S-derivation of $\mathcal{A}$ if $\phi(x^2) = \phi(x)x + x\phi(x)$ for any $x \in \mathcal{A}$. It is said to be a PZ-derivation of $\mathcal{A}$ if $xy = 0$ implies that $\phi(x)y + x\phi(y) = 0$.*

Remark 1.1 It should be pointed out that the notion of an S-derivation of $\mathcal{A}$ is commonly known as a Jordan derivation.

Remark 1.2 To determine all derivations of a given R-algebra $\mathcal{A}$ is an important task, since it is useful for us to learn more about the relationships between elements in $\mathcal{A}$ as well as the algebraic structure of $\mathcal{A}$. However, one easily sees that the condition for an R-linear map on $\mathcal{A}$ to be a derivation is much strong, so we try to relax such condition and define the so-called SZ-derivation of $\mathcal{A}$. Indeed, when one has determined all SZ-derivations on $\mathcal{A}$, then one can easily obtain all derivations of it. So the study of determining all SZ-derivations on R-algebras has significant applications.

It is easy to see that

$$\text{derivations of } \mathcal{A} \Rightarrow \text{PZ-derivations of } \mathcal{A} \Rightarrow \text{SZ-derivations of } \mathcal{A};$$
$$\text{derivations of } \mathcal{A} \Rightarrow \text{S-derivations of } \mathcal{A} \Rightarrow \text{SZ-derivations of } \mathcal{A}.$$

Now one might wonder:

1) Whether an SZ-derivation of $\mathcal{A}$ is a PZ-derivation of $\mathcal{A}$;
2) Whether an SZ-derivation of $\mathcal{A}$ is an S-derivation of $\mathcal{A}$;
3) Whether a PZ-derivation of $\mathcal{A}$ is a derivation of $\mathcal{A}$;
4) Whether an S-derivation of $\mathcal{A}$ is a derivation of $\mathcal{A}$.

The following two examples give negative answers.

Example 1.2 Let E_{ij} be the standard matrix units, $n \geq 4$, $a \in R$. We define $\phi :\ N_n(R) \to N_n(R)$, by

$$\sum_{1 \leq i < j \leq n} a_{ij} E_{ij} \mapsto a a_{13} E_{2n} - a a_{12} E_{3n} + b a_{n-2,n} E_{1,n-1} - b a_{n-1,n} E_{1,n-2}.$$

Then it is not difficult to verify that ϕ is an SZ-derivation of $N_n(R)$; it is a PZ-derivation if and only if $a = b = 0$ and it is an S-derivation if and only if $2a = 2b = 0$.

Example 1.3 Let $n \geq 4$, $a \in R$. We define $\phi : N_n(R) \to N_n(R)$ by

$$\sum_{1 \leq i < j \leq n} a_{ij} E_{ij} \mapsto \sum_{k=1}^{n-1} a(k-1) \sum_{j-i=k} a_{ij} E_{ij}.$$

Then it is verified that ϕ is a PZ-derivation of $N_n(R)$ but fails to be a derivation when $a \neq 0$.

Above two examples show that it is somewhat interesting to characterize all SZ-derivations, all S-derivations and PZ-derivations on certain R-algebras. As a maximal nilpotent subalgebra of the full matrix algebra, $N_n(R)$ is an interesting object of study.

2. Construction of standard SZ-derivations of $N_n(R)$

Let R be a commutative ring, R^* the set of all nonzero elements in R. Let n be a positive integer. We denote by $M_n(R)$ (resp., $N_n(R)$; resp., $D_n(R)$) the set of all $n \times n$ matrices (resp., strictly upper triangular matrices; resp., diagonal matrices) over R. We denote by E_{ij} the standard matrix unit whose (i,j)-entry is 1 and all other entries are 0. $N_n(R)$ has a basis $\{E_{ij} \mid 1 \le i < j \le n\}$, which consists of square-zero matrices. Let $\operatorname{Der} N_n(R)$ denote the derivation algebra of $N_n(R)$ and let $\operatorname{Dsz} N_n(R)$ (resp., $\operatorname{Ds} N_n(R)$, resp., $\operatorname{Dpz} N_n(R)$) denote the set consisting of all SZ-derivations (resp., S-derivations, resp., PZ-derivations) of $N_n(R)$. It is obvious that $\operatorname{Dsz} N_n(R)$, $\operatorname{Ds} N_n(R)$ and $\operatorname{Dpz} N_n(R)$ all form additive groups and

$$\operatorname{Der} N_n(R) \subseteq \operatorname{Dpz} N_n(R) \subseteq \operatorname{Dsz} N_n(R);$$

$$\operatorname{Der} N_n(R) \subseteq \operatorname{Ds} N_n(R) \subseteq \operatorname{Dsz} N_n(R).$$

We now construct several types of standard SZ-derivations of $N_n(R)$.

(1) Inner derivations

If $X \in N_n(R)$, then the map $\operatorname{ad}_X : N_n(R) \to N_n(R)$, $Y \mapsto [X,Y] = XY - YX$, is a derivation of $N_n(R)$, called the inner derivation of $N_n(R)$ induced by X.

(2) Diagonal derivations

If $H \in D_n(R)$, then the map $\operatorname{Dig}_H : N_n(R) \to N_n(R)$, $Y \mapsto [H,Y] = HY - YH$, is a derivation of $N_n(R)$, called the diagonal derivation of $N_n(R)$ induced by H.

(3) Central SZ-derivations

Let $n \ge 3$, $Y = \sum_{1 \le i < j \le n} y_{ij} E_{ij} \in N_n(R)$. We define $\eta_Y : N_n(R) \to N_n(R)$ by

$$\sum_{1 \le i < j \le n} a_{ij} E_{ij} \mapsto \Big(\sum_{1 \le i < j \le n} a_{ij} y_{ij}\Big) E_{1n}.$$

Then it is easy to check that $\eta_Y \in Dsz\ N_n(R)$, but generally η_Y fails to be a derivation of $N_n(R)$. η_Y is said to be a central SZ-derivation of $N_n(R)$.

(4) Extremal SZ-derivations

Suppose $n \ge 4$ and $e_1, e_2 \in R$, define $\chi^c_{e_1} : N_n(R) \to N_n(R)$ by

$$\sum_{1 \le i < j \le n} a_{ij} E_{ij} \mapsto e_1 a_{13} E_{2n} - e_1 a_{12} E_{3n},$$

and define $\chi^r_{e_2} : N_n(R) \to N_n(R)$ by

$$\sum_{1 \le i < j \le n} a_{ij} E_{ij} \mapsto e_2 a_{n-2,n} E_{1,n-1} - e_2 a_{n-1,n} E_{1,n-2}.$$

Then it is not difficult to check that $\chi^c_{e_1}$ and $\chi^r_{e_2}$ both are SZ-derivations of $N_n(R)$, called extremal SZ-derivations of $N_n(R)$.

(5) Extensible SZ-derivations

Suppose $n \ge 4$ and $f \in R$, define $\lambda_f : N_n(R) \to N_n(R)$ by

$$\sum_{1 \le i < j \le n} a_{ij} E_{ij} \mapsto \sum_{k=1}^{n-1} f(k-1)\Big(\sum_{j-i=k} a_{ij} E_{ij}\Big).$$

Then it is easy to check that λ_f is an SZ-derivation of $N_n(R)$, called an extensible SZ-derivation of $N_n(R)$.

With above standard SZ-derivations in hands, we can now describe all SZ-derivations, S-derivations and PZ-derivations of $N_n(R)$.

Theorem 2.1 *ϕ is an SZ-derivation of $N_n(R)$ if and only if*

1) $\phi = \mathrm{Dig}_H$, *when* $n = 2$;

2) $\phi = \mathrm{Dig}_H + \eta_Y$, *when* $n = 3$;

3) $\phi = \chi^c_{e_1} + \chi^r_{e_2} + \mathrm{ad}_X + \mathrm{Dig}_H + \lambda_f + \eta_Y$, *when* $n \geq 4$,

where ad_X, Dig_H, $\chi^c_{e_1}$, $\chi^r_{e_2}$, λ_f, *and* η_Y *are the inner derivation, diagonal derivation, extremal SZ-derivation, extensible SZ-derivation and central SZ-derivation of* $N_n(R)$*, respectively.*

Theorem 2.2 *ϕ is a PZ-derivation of $N_n(R)$ if and only if*

1) $\phi = \mathrm{Dig}_H$, *when* $n = 2$;

2) $\phi = \mathrm{Dig}_H + \eta_Y$, *when* $n = 3$;

3) $\phi = \mathrm{ad}_X + \mathrm{Dig}_H + \lambda_f + \eta_Y$, *when* $n \geq 4$,

where ad_X, Dig_H, λ_f, η_Y *are the inner derivation, diagonal derivation, extensible SZ-derivation and central SZ-derivation of* $N_n(R)$*, respectively.*

Theorem 2.3 *ϕ is an S-derivation of $N_n(R)$ if and only if*

1) When $n = 2$, $\phi = \mathrm{Dig}_H$;

2) When $n = 3$, $\phi = \mathrm{Dig}_H + \eta_Y$;

3) When $n \geq 4$, $\phi = \chi^c_{e_1} + \chi^r_{e_2} + \mathrm{ad}_X + \mathrm{Dig}_H + \eta_Y$,

where ad_X, Dig_H, $\chi^c_{e_1}$, $\chi^r_{e_2}$, *and* η_Y *are the inner derivation, diagonal derivation, extremal SZ-derivation and central SZ-derivation of* $N_n(R)$*, respectively,* $Y \in \sum_{i=1}^{n-1} RE_{i,i+1}$ and $2e_1 = 2e_2 = 0$.

Corollary 2.4 *ϕ is a derivation of $N_n(R)$ if and only if* $\phi = \mathrm{ad}_X + \mathrm{Dig}_H + \eta_Y$, *where* ad_X, Dig_H *and* η_Y *are the inner derivation, diagonal derivation and central SZ-derivation of* $N_n(R)$*, respectively, and* $Y \in \sum_{i=1}^{n-1} RE_{i,i+1}$.

3. Lemmas and proof of the main theorem

For $X, Y \in N_n(R)$ we denote $XY + YX$ by $X \circ Y$ for brevity. Let ϕ be a given SZ-derivation of $N_n(R)$, it is now necessary to study the invariant ideals of $N_n(R)$ under ϕ.

Lemma 3.1 *Let ϕ be an SZ-derivation of $N_n(R)$. If X, Y and $X+Y$ all are square-zero elements in $N_n(R)$, then $\phi(X) \circ Y + X \circ \phi(Y) = 0$.*

Proof An easy verification leads to the result. □

Let S be a subalgebra of $N_n(R)$, and denote by $C(S)$ the centralizer of S in $N_n(R)$:

$$C(S) = \{A \in N_n(R) \mid AX = XA = 0, \forall X \in S\}.$$

Lemma 3.2 *Let S be a subalgebra of $N_n(R)$, $\phi \in \mathrm{Dsz}\, N_n(R)$. If S and $C(S)$ both are spanned by standard matrix units and $\phi(S) \subseteq S$, then $\phi(C(S)) \subseteq C(S)$.*

Proof If $\phi(C(S)) \nsubseteq C(S)$, choose a square-zero element $X \in C(S)$ such that $\phi(X) \notin C(S)$ (recall that $C(S)$ is spanned by square-zero elements). Then there exists a matrix unit $E_{ij} \in S$ with $i < j$ such that $\phi(X)E_{ij} \neq 0$ or there exists a matrix unit $E_{kl} \in S$ with $k < l$ such that $E_{kl}\phi(X) \neq 0$. When the first case happens, it is shown that

$$\phi(X) \circ E_{ij} \neq 0.$$

Otherwise, if $\phi(X) \circ E_{ij} = 0$, assume that $\phi(X) = \sum_{1 \le p,q \le n} x_{pq}E_{pq} \in N_n(R)$, where $x_{pq} = 0$ when $p \ge q$. Then by

$$E_{ij}\phi(X) = -\phi(X)E_{ij} \neq 0$$

we obtain

$$x_{jj}E_{ij} = -x_{ii}E_{ij} \neq 0,$$

absurd. So

$$\phi(X) \circ E_{ij} \neq 0.$$

By assumption we know

$$\phi(E_{ij}) \circ X = 0.$$

Thus we have

$$\phi(X) \circ E_{ij} + \phi(E_{ij}) \circ X \neq 0.$$

This is in contradiction with Lemma 3.1 (note that E_{ij}, X and $X + E_{ij}$ all are square zero). Similarly, the later case does not happen. So $\phi(C(S)) \subseteq C(S)$. □

The center of $N_n(R)$, denoted by M_n, is RE_{1n}. The center of $N_n(R)/M_n$ obviously is M_{n-1}/M_n, where $M_{n-1} = \sum_{j-i \ge n-2} RE_{ij}$. Go on considering the center of $N_n(R)/M_{n-1}$, it is M_{n-2}/M_{n-1}, where $M_{n-2} = \sum_{j-i \ge n-3} RE_{ij}$. Generally, for $3 \le k \le n$, the center of $N_n(R)/M_k$ is M_{k-1}/M_k, where $M_{k-1} = \sum_{j-i \ge k-2} RE_{ij}$. Thus we get the upper central series of $N_n(R)$:

$$0 \subset M_n = RE_{1n} \subset M_{n-1} \subset \cdots M_k \subset \cdots \subset M_3 \subset M_2 = N_n(R), \text{ where } M_k = \sum_{j-i \ge k-1} RE_{ij}.$$

Lemma 3.3 *Let ϕ be an SZ-derivation of $N_n(R)$. Then $\phi(M_n) \subseteq M_n$.*

Proof M_n, as the centralizer of $N_n(R)$ in $N_n(R)$, naturally is invariant under ϕ (by Lemma 3.2). □

Set $I_k = \sum_{i=k}^{n} RE_{1i}$, $k = 2, 3, \ldots, n$ and set $\alpha_k = \sum_{i=k+1}^{n} RE_{ki}$, $k = 1, 3, \ldots, n-1$. Obviously, α_1 exactly is I_2.

Lemma 3.4 *If ϕ is an SZ-derivation of $N_n(R)$, then $\phi(E_{1j}) \in \alpha_1 + RE_{2n} + RE_{j+1,n}$, $j = 2, 3, \ldots, n-1$.*

Proof Fix j $(2 \le j \le n-1)$. If $1 < i < n$ and $i \neq j$, then $E_{1j}, E_{i,i+1}$ and $E_{1j} + E_{i,i+1}$ all are square-zero matrices. By Lemma 3.1, we have

$$\phi(E_{1j}) \circ E_{i,i+1} + E_{1j} \circ \phi(E_{i,i+1}) = 0. \tag{3.1}$$

By multiplying E_{1i} from the left side to the above equation we have that

$$E_{1i}\phi(E_{1j})E_{i,i+1} + E_{1,i+1}\phi(E_{1j}) = 0.$$

It is obvious that $E_{1i}\phi(E_{1j})E_{i,i+1} = 0$. So $E_{1,i+1}\phi(E_{1j}) = 0$. This shows that the $(i+1)$-th row of $\phi(E_{1j})$ is zero for every i satisfying $1 < i < n$ and $i \neq j$. So $\phi(E_{1j}) \in \alpha_1 + \alpha_2 + \alpha_{j+1}$. We have known, for any fixed i satisfying $1 < i < n$, $i \neq j$, that $E_{i,i+1}\phi(E_{1j}) = 0$. So by Equation 3.1 we get

$$\phi(E_{1j})E_{i,i+1} + E_{1j}\phi(E_{i,i+1}) = 0.$$

This shows that all positions of the i-th column of $\phi(E_{1j})$ are zero, except for the $(1,i)$-position. Then we see that

$$\phi(E_{1j}) \in \alpha_1 + RE_{2j} + RE_{2n} + RE_{j+1,n}.$$

It follows from $(E_{12} + E_{1j})^2 = 0$ that

$$E_{1j}\phi(E_{12}) + E_{12}\phi(E_{1j}) = 0,$$

which follows that the $(2,j)$-entry of $\phi(E_{1j})$ is zero. So $\phi(E_{1j}) \in \alpha_1 + RE_{2n} + RE_{j+1,n}$. □

Proof of Theorem 2.1

Case 1 $n = 2$.

When $n = 2$, there is nothing to prove.

Case 2 $n = 3$.

Suppose that

$$\phi(E_{12}) \equiv tE_{12} + sE_{23} \pmod{RE_{13}}; \quad \phi(E_{23}) \equiv uE_{23} + vE_{12} \pmod{RE_{13}}.$$

It follows from $(E_{12})^2 = 0$ that $\phi(E_{12})E_{12} + E_{12}\phi(E_{12}) = 0$, which shows that $s = 0$. Similarly, $v = 0$. Let $H = \mathrm{diag}\{0, t, t+u\} \in D_3(R)$. Then

$$(\mathrm{Dig}_H + \phi)(E_{12}) = xE_{13}; \quad (\mathrm{Dig}_H + \phi)(E_{23}) = yE_{13}$$

for certain $x, y \in R$. Assume that $(\mathrm{Dig}_H + \phi)(E_{13}) = zE_{13}$. Let $Y = -xE_{12} - yE_{23} - zE_{13}$. Then $\eta_Y + \mathrm{Dig}_H + \phi$ sends E_{12}, E_{23} and E_{13} to zero, respectively. So $\eta_Y + \mathrm{Dig}_H + \phi = 0$. By this one can get the desired expression of ϕ.

Case 3 $n \geq 4$

We give the proof of this case by steps.

Step 1. There exists an extremal SZ-derivation $\chi^c_{e_1}$ such that $(\chi^c_{e_1} + \phi)(E_{12}) \in \alpha_1$.

By Lemma 3.4, we may assume that $\phi(E_{12}) \equiv xE_{2n} + yE_{3n} \pmod{\alpha_1}$. It follows from $(E_{12})^2 = 0$ that $\phi(E_{12})E_{12} + E_{12}\phi(E_{12}) = 0$. It follows that $x = 0$. Choose $e_1 = y$, then one may verify that $(\chi^c_{e_1} + \phi)(E_{12}) \in \alpha_1$, as desired. Now we replace $\chi^c_{e_1} + \phi$ with ϕ.

Step 2. $\phi(E_{1j}) \in I_j$ for $j = 2, 3, \ldots, n$.

The case when $j = 2$ has been proved in Step 1. The case when $j = n$ is obvious by Lemma 3.3. Now we consider the case when $3 \le j \le n-1$. Since E_{12}, E_{1j} and $E_{12} + E_{1j}$ all are square zero, we see, by Lemma 3.1, that $E_{12}\phi(E_{1j}) + E_{1j}\phi(E_{12}) = 0$. This shows that the $(2, n)$-entry of $\phi(E_{1j})$ is zero. So

$$\phi(E_{1j}) \in RE_{j+1,n} + \alpha_1, \quad j = 3, \ldots, n-1.$$

Now by square zero of $E_{1j}+E_{1,j+1}$, we have that $E_{1j}\phi(E_{1,j+1})+E_{1,j+1}\phi(E_{1j}) = 0$, which implies that the $(j+1, n)$-entry of $\phi(E_{1j})$ is zero. So $\phi(E_{1j}) \in \alpha_1$ for $j = 3, 4, \ldots, n-1$. For any fixed k satisfying $2 \le k \le j-1$, it follows from $(E_{1j} + E_{kj})^2 = 0$ that

$$E_{1j}\phi(E_{kj}) + \phi(E_{1j})E_{kj} + E_{kj}\phi(E_{1j}) = 0.$$

By considering the $(1, j)$-entry of the left side we see that the $(1, k)-$entry of $\phi(E_{1j})$ is zero. So $\phi(E_{1j}) \in I_j$ for all j satisfying $3 \le j \le n-1$. Combining this with $\phi(E_{12}) \in I_2$ and $\phi(E_{1n}) \in I_n$, we finally get $\phi(E_{1j}) \in I_j$ for $j = 2, 3, \ldots, n$.

Step 3. Let $K_i = \sum_{k=1}^{i} RE_{kn}$, $i = 1, 2, \ldots, n-1$. There exists an extremal SZ-derivation $\chi^r_{e_2}$ such that $(\chi^r_{e_2} + \phi)(E_{in}) \in K_i$, $i = 1, 2, \ldots, n-1$.

The proof being analogous to that of Steps 1 and 2, is omitted. Replace $\chi^r_{e_2} + \phi$ with ϕ.

Step 4. There exists $X_1 \in N_n(R)$ such that $(\mathrm{ad}_{X_1} + \phi)(E_{1j}) \in RE_{1j} + RE_{1n}$, $j = 2, 3, \ldots, n$.

By Step 2 we may assume that

$$\phi(E_{1j}) = \sum_{k=j}^{n} c_{jk}E_{1,k}, \quad j = 2, 3, \ldots, n.$$

Let

$$X_1 = \sum_{l=2}^{n-1}\sum_{k=l+1}^{n} c_{lk}E_{lk}.$$

Then

$$(\mathrm{ad}_{X_1} + \phi)(E_{1j}) \equiv c_{jj}E_{1j} \ (\mathrm{mod}\, RE_{1n}), \quad j = 2, 3, \ldots, n.$$

As required. Now replace $\mathrm{ad}_{X_1} + \phi$ again with ϕ.

Step 5. There exists $X_2 \in N_n(R)$ such that $(\mathrm{ad}_{X_2}+\phi)(E_{ij}) \in RE_{ij}+RE_{1n}$ for all $1 \le i < j \le n$.

Firstly, we prove that $\phi(\alpha_i) \subseteq \alpha_i + \alpha_1$ for $i = 1, 2, \ldots, n-1$. By Step 4, we have known that $\phi(\alpha_1) \subseteq \alpha_1$. When $2 \le i \le n-1$, for any $x \in \alpha_i$, suppose that $\phi(x) = \sum_{1\le k<l\le n} x_{kl}E_{kl}$. When $p \ne 1$ and $p \ne i$, by square zero of $E_{1p} + x$, we have

$$\phi(E_{1p}) \circ x + \phi(x) \circ E_{1p} = 0.$$

Obviously, $\phi(x)E_{1p} = x\phi(E_{1p}) = \phi(E_{1p})x = 0$. So $E_{1p}\phi(x) = 0$. This shows that the p-th row of $\phi(x)$ is zero, which implies that $\phi(x) \in \alpha_i + \alpha_1$. Furthermore,

$$\phi(\alpha_i) \subseteq \alpha_i + \alpha_1, \quad i = 1, 2, \ldots, n-1.$$

Combining this with Step 3, we know that

$$\phi(E_{in}) \in RE_{in} + RE_{1n}, \quad i = 1, 2, \ldots, n-1.$$

Let $\beta_j = \sum_{i=1}^{j-1} RE_{ij}$, $j = 2, 3, \ldots, n$. Obviously, $\beta_n = K_{n-1}$. Next we intend to prove that

$$\phi(\beta_j) \subseteq \beta_j + \beta_n, \quad j = 2, 3, \ldots, n.$$

When $j = n$ there is nothing to prove. For $2 \leq j \leq n-1$, let $y \in \beta_j$, and suppose that $\phi(y) = \sum_{1\leq k<l\leq n} y_{kl}E_{kl}$. When $q \neq n$ and $q \neq j$, by square zero of $E_{qn} + y$, we have

$$\phi(E_{qn}) \circ y + \phi(y) \circ E_{qn} = 0.$$

This shows that the q-th column of $\phi(y)$ is zero (recall that $\phi(E_{in}) \in RE_{in} + RE_{1n}$, $i = 1, 2, \ldots, n-1$), which implies that $\phi(y) \in \beta_j + \beta_n$. Furthermore,

$$\phi(\beta_j) \subseteq \beta_j + \beta_n, \quad j = 2, 3, \ldots, n.$$

Since $E_{ij} \in \alpha_i \cap \beta_j$, we see that

$$\phi(E_{ij}) \in RE_{ij} + RE_{1j} + RE_{in} + RE_{1n}, \text{ for all pair } i, j \text{ satisfying } 1 \leq i \leq j \leq n.$$

Now assume that

$$\phi(E_{i,i+1}) \equiv s_{i,i+1}E_{i,i+1} + s_{1,i+1}E_{1,i+1} + s_{in}E_{in} \pmod{RE_{1n}}, \quad i = 2, 3, \ldots, n-2.$$

Let

$$X_2 = \sum_{l=2}^{n-2} s_{ln}E_{l+1,n} - \sum_{k=2}^{n-2} s_{1,k+1}E_{1k}.$$

Then

$$(\mathrm{ad}_{X_2} + \phi)(E_{i,i+1}) \equiv s_{i,i+1}E_{i,i+1} \pmod{RE_{1n}}, \quad i = 2, 3, \ldots, n-2.$$

Simultaneously,

$$(\mathrm{ad}_{X_2} + \phi)(E_{1j}) \in RE_{1j} + RE_{1n}, \quad j = 2, 3, \ldots, n;$$

$$(\mathrm{ad}_{X_2} + \phi)(E_{in}) \in RE_{in} + RE_{1n}, \quad i = 1, 2, \ldots, n-1.$$

Replace $\mathrm{ad}_{X_2} + \phi$ again with ϕ. Now consider $\phi(E_{ij})$ for any pair i, j satisfying $2 \leq i \leq n-3, i+2 \leq j \leq n-1$. By square zero of $E_{i,i+1} - E_{ij} + E_{i+1,n} + E_{jn}$ we know that

$$(E_{i,i+1} - E_{ij} + E_{i+1,n} + E_{jn}) \circ \phi(E_{i,i+1} - E_{i,j} + E_{i+1,n} + E_{jn}) = 0.$$

By this we see that the $(1, j)$-entry of $\phi(E_{ij})$ is zero. Similarly, by square zero of $E_{ij} - E_{j-1,j} + E_{1i} + E_{1,j-1}$ we know that the (i, n)-entry of $\phi(E_{ij})$ is zero. Thus we have

$$\phi(E_{ij}) \in RE_{ij} + RE_{1n}$$

for all pair i, j satisfying $2 \leq i \leq n-3$ and $i+2 \leq j \leq n-1$. Combining this with those we have obtained, we finally get $\phi(E_{kl}) \in RE_{kl} + RE_{1n}$ for all k, l satisfying $1 \leq k < l \leq n$.

Step 6. There exists $H \in D_n(R)$, $t \in R$, $Y \in N_n(R)$ such that $\eta_Y + \lambda_t + \mathrm{Dig}_H + \phi = 0$.

By Step 5, we may assume that

$$\phi(E_{i,i+1}) \equiv s_{i,i+1}E_{i,i+1} \pmod{RE_{1n}}, \quad i = 1, 2, \ldots, n-1.$$

Let

$$H = \mathrm{diag}\{1, s_{12}, \sum_{i=1}^{2} s_{i,i+1}, \ldots, \sum_{i=1}^{n-1} s_{i,i+1}\}.$$

Then we see that $\mathrm{Dig}_H + \phi$ sends each one of $\{E_{12}, E_{23}, \ldots, E_{n-1,n}\}$ to zero. Now we may assume that

$$(\mathrm{Dig}_H + \phi)(E_{ij}) \equiv t_{ij}E_{ij} \pmod{RE_{1n}}$$

for all i, j satisfying $1 \le i < j \le n$, where $t_{i,i+1} = 0$ for $i = 1, 2, \ldots, n-1$. Now by square zero of $E_{i,i+2} + E_{i+2,i+3} + E_{i,i+1} - E_{i+1,i+3}$, we know that

$$(E_{i,i+2} + E_{i+2,i+3} + E_{i,i+1} - E_{i+1,i+3}) \circ \phi(E_{i,i+2} + E_{i+2,i+3} + E_{i,i+1} - E_{i+1,i+3}) = 0.$$

This shows that $t_{i,i+2} = t_{i+1,i+3}$ for $i = 1, 2, \ldots, n-3$. Denote t_{13} by t. Similarly, by square zero of $E_{i,i+3} + E_{i,i+1} + E_{i+3,i+4} - E_{i+1,i+4}$, we have that

$$(E_{i,i+3} + E_{i,i+1} + E_{i+3,i+4} - E_{i+1,i+4}) \circ \phi(E_{i,i+3} + E_{i,i+1} + E_{i+3,i+4} - E_{i+1,i+4}) = 0,$$

which yields that $t_{i,i+3} = t_{i+1,i+4}$ for $i = 1, 2, \ldots, n-4$. Then by

$$(E_{i,i+3} + E_{i+3,i+4} + E_{i,i+2} - E_{i+2,i+4}) \circ \phi(E_{i,i+3} + E_{i+3,i+4} + E_{i,i+2} - E_{i+2,i+4}) = 0,$$

we have that

$$t_{i,i+3} = 2t, \quad i = 1, 2, \ldots, n-3.$$

By similar discussions, we further obtain that

$$t_{i,i+4} = 3t, \quad i = 1, 2, \ldots, n-4;$$

$$t_{i,i+5} = 4t, \quad i = 1, 2, \ldots, n-5;$$

$$\cdots \quad \cdots \quad \cdots$$

$$t_{1,n-1} = t_{2n} = (n-3)t.$$

Using $t \in R$, we construct the extensible SZ-derivation λ_{-t}. Then we have that

$$(\lambda_{-t} + \mathrm{Dig}_H + \phi)(E_{ij}) \in RE_{1n}, \text{ for all } i, j \text{ satisfying } 1 \le i < j \le n.$$

Assume that

$$(\lambda_{-t} + \mathrm{Dig}_H + \phi)(E_{ij}) = y_{ij}E_{1n} \text{ for all } i, j \text{ satisfying } 1 \le i < j \le n.$$

Let $Y = -(\sum_{1 \le i < j \le n} y_{ij}E_{ij})$. Then we see that $\eta_Y + \lambda_{-t} + \mathrm{Dig}_H + \phi$ sends all E_{ij} to zero. So

$$\eta_Y + \lambda_{-t} + \mathrm{Dig}_H + \phi = 0.$$

Now we have that

$$\eta_Y + \lambda_{-t} + \mathrm{Dig}_H + \mathrm{ad}_{X_2} + \mathrm{ad}_{X_1} + \chi^r_{e_2} + \chi^c_{e_1} + \phi = 0.$$

By this we can easily get the desired expression of ϕ. $\square$

Proof of Theorem 2.2 Firstly, one can easily see that $\mathrm{Dpz}\, N_n(R)$ forms an additive subgroup of $\mathrm{Dsz}\, N_n(R)$. It is trivial to verify that ad_X, Dig_H, λ_f and η_Y all belong to $\mathrm{Dpz}\, N_n(R)$, so the sufficient condition obviously holds. Now consider another direction. When $n = 2$ or $n = 3$, there is nothing to say.

When $n \geq 4$, if $\phi \in \mathrm{Dpz}\, N_n(R)$, naturally $\phi \in \mathrm{Dsz}\, N_n(R)$. Then by Theorem 2.1, ϕ takes the form

$$\phi = \chi^c_{e_1} + \chi^r_{e_2} + \mathrm{ad}_X + \mathrm{Dig}_H + \lambda_f + \eta_Y.$$

Since

$$\mathrm{ad}_X + \mathrm{Dig}_H + \lambda_f + \eta_Y \in \mathrm{Dpz}\, N_n(R),$$

we know that

$$\chi^c_{e_1} + \chi^r_{e_2} \in \mathrm{Dpz}\, N_n(R).$$

To achieve the aim, it suffices to show that $e_1 = e_2 = 0$. By $E_{23}E_{12} = 0$, we have that

$$[(\chi^c_{e_1} + \chi^r_{e_2})(E_{23})] \cdot E_{12} + E_{23} \cdot [(\chi^c_{e_1} + \chi^r_{e_2})(E_{12})] = -e_1 E_{2,n} = 0,$$

which leads to $e_1 = 0$. Then $\chi^r_{e_2} \in \mathrm{Dpz}\, N_n(R)$. Similarly, by $E_{n-1,n}E_{n-2,n-1} = 0$, we have that

$$[\chi^r_{e_2}(E_{n-1,n})] \cdot E_{n-2,n-1} + E_{n-1,n} \cdot \chi^r_{e_2}(E_{n-2,n-1}) = -e_2 E_{1,n-1} = 0,$$

which leads to $e_2 = 0$. So $\phi = \mathrm{ad}_X + \mathrm{Dig}_H + \lambda_f + \eta_Y$, as desired. $\square$

Proof of Theorem 2.3 We only prove the case when $n \geq 4$. Firstly, one can easily see that $\mathrm{Ds}\, N_n(R)$ forms an additive subgroup of $\mathrm{Dsz}\, N_n(R)$. ad_X and Dig_H, being derivations of $N_n(R)$, naturally are S-derivations of $N_n(R)$. It is easy to verify that $\eta_Y \in \mathrm{Ds}\, N_n(R)$ if $Y \in M_3$. It is not difficult to verify that $\chi^c_{e_1} \in \mathrm{Ds}\, N_n(R)$ if $2e_1 = 0$ and $\chi^r_{e_2} \in \mathrm{Ds}\, N_n(R)$ if $2e_2 = 0$. So the sufficient condition holds. Now consider the contrary direction. If $\phi \in \mathrm{Ds}\, N_n(R)$, naturally $\phi \in \mathrm{Dsz}\, N_n(R)$. Then by Theorem 2.1, ϕ takes the form

$$\phi = \chi^c_{e_1} + \chi^r_{e_2} + \mathrm{ad}_X + \mathrm{Dig}_H + \lambda_f + \eta_Y.$$

Since

$$\mathrm{ad}_X + \mathrm{Dig}_H \in \mathrm{Ds}\, N_n(R),$$

we know that $\chi^c_{e_1} + \chi^r_{e_2} + \lambda_f + \eta_Y$ is an S-derivation of $N_n(R)$. We denote it by ϕ_1. By applying ϕ_1 to $(E_{12} + E_{23})^2 = E_{13}$, we get

$$(E_{12} + E_{23}) \circ \phi(E_{12} + E_{23}) = \phi(E_{13}),$$

which yields

$$-e_1 E_{3n}(E_{12} + E_{23}) - e_1(E_{12} + E_{23})E_{3n} \equiv f E_{13} + e_1 E_{2n} \pmod{R E_{1n}}.$$

This leads to $2e_1 = f = 0$. By a similar discussion, one can get $2e_2 = 0$. Note that if $2e_1 = 2e_2 = 0$, then $\chi^c_{e_1}$ and $\chi^r_{e_2}$ both are S-derivations of $N_n(R)$. Then we further get $\eta_Y \in \mathrm{Ds}\, N_n(R)$.

Write $Y=\sum_{1\le i<j\le n} y_{ij}E_{ij}\in N_n(R)$. If $j-i\ge 2$, we can choose k such that $i<k<j$. Then by applying η_Y to $(E_{ik}+E_{kj})^2=E_{ij}$, we have that

$$(E_{ik}+E_{kj})\circ\phi(E_{ik}+E_{kj})=\phi(E_{ij}),$$

which results in $y_{ij}=0$. So $Y\in\sum_{i=1}^{n-1} RE_{i,i+1}$, as desired. This completes the proof. $\square$

Proof of Corollary 2.4 If $Y\in\sum_{i=1}^{n-1} RE_{i,i+1}$, then $\eta_Y\in Der\ N_n(R)$. So the sufficient condition holds. On the contrary, if ϕ is a derivation of $N_n(R)$, then $\phi\in \mathrm{Ds}\,N_n(R)\cap\mathrm{Dpz}\,N_n(R)$. Then by using Theorems 2.2 and 2.3, one easily obtains that $\phi=\mathrm{ad}_X+\mathrm{Dig}_H+\eta_Y$, where $Y\in\sum_{i=1}^{n-1} RE_{i,i+1}$. $\square$

References

[1] JϕNDRUP S. *Automorphisms and derivations of upper triangular matrix rings* [J]. Linear Algebra Appl., 1995, **221**: 205–218.

[2] WANG Dengyin, YU Qiu. *Derivations of the parabolic subalgebras of the general linear Lie algebra over a commutative ring* [J]. Linear Algebra Appl., 2006, **418**(2-3): 763–774.

[3] WANG Dengyin, OU Shikun, YU Qiu. *Derivations of the intermediate Lie algebras between the Lie algebra of diagonal matrices and that of upper triangular matrices over a commutative ring* [J]. Linear Multilinear Algebra, 2006, **54**(5): 369–377.

[4] OU Shikun, WANG Dengyin, YAO Ruiping. *Derivations of the Lie algebra of strictly upper triangular matrices over a commutative ring* [J]. Linear Algebra Appl., 2007, **424**(2-3): 378–383.

[5] BENKOVIČ D. *Jordan derivations and antiderivations on triangular matrices* [J]. Linear Algebra Appl., 2005, **397**: 235–244.