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On S-Semipermutable Subgroups of Finite Groups

LU Jia Kuan[1], *LI Shi Rong*[2]
(1. Department of Mathematics, Shanghai University, Shanghai 200444, China;
2. Department of Mathematics, Guangxi University, Guangxi 530004, China)
(E-mail: ljk_607@163.com)

Abstract Let d be the smallest generator number of a finite p-group P and let $\mathcal{M}_d(P) = \{P_1, \ldots, P_d\}$ be a set of maximal subgroups of P such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$. In this paper, we study the structure of a finite group G under the assumption that every member in $\mathcal{M}_d(G_p)$ is S-semipermutable in G for each prime divisor p of $|G|$ and a Sylow p-subgroup G_p of G.

Keywords S-semipermutable subgroups; p-nilpotent groups; supersolvable groups.

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1. Introduction

All groups considered in this paper are finite.

A subgroup H of a group G is called S-permutable in G if H permutes with all Sylow subgroups of G, i.e., $HS = SH$ for any Sylow subgroup S of G. This concept was introduced by Kegel in [1] and has been studied by some authors[2−5]. A subgroup H of a group G is called S-semipermutable in G if H permutes with every Sylow p-subgroup S of G with $(p, |S|) = 1$[6]). Obviously, an S-permutable subgroup is an S-semipermutable subgroup. The converse does not hold in general. For example, a Sylow 3-subgroup of the symmetric group S_4 of degree 4 is S-semipermutable in S_4 but not S-permutable in S_4. Wang and Zhang have studied the influence of S-semipermutability of some subgroups of prime power order on the structure of finite groups[7,8].

Let G be a group and let $\mathcal{M}(G)$ be the set of all maximal subgroups of all Sylow subgroups of G. Many authors have investigated the structure of a group G under the assumption that every member in $\mathcal{M}(G)$ is well-situated in G[9−17]. In many cases, the assumption that every member in $\mathcal{M}(G)$ is well-situated in G is too strong. It seems to be natural to replace $\mathcal{M}(G)$ by a small subset of $\mathcal{M}(G)$. As a choice to such a subset, we have the following:

Definition 1.1 *Let d be the smallest generator number of a finite p-group P and let $\mathcal{M}_d(P) = \{P_1, \ldots, P_d\}$ be a set of maximal subgroups of P such that $\cap_{i=1}^{d} P_i = \Phi(G)$.*

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We know that $|\mathcal{M}(P)| = (p^d - 1)/(p-1)$, $|\mathcal{M}_d(P)| = d$ and

$$\lim_{d\to\infty} (p^d - 1/p - 1)/d = \infty,$$

thus $|\mathcal{M}(P)| \gg |\mathcal{M}_d(P)|$.

In this paper, we investigate the structure of a group G under the assumption that every member in $\mathcal{M}_d(G_p)$ is S-semipermutable in G for each prime divisor p of $|G|$ and a Sylow p-subgroup G_p of G.

2. Preliminaries

In this section we collect some lemmas which are useful to the proof of our theorems.

Lemma 2.1[8] *(1) Let G be a group. If $H \le K \le G$ and H is S-semipermutable in G, then H is S-semipermutable in K.*

(2) Let H be p-subgroup of a group G for some prime p. If H is S-semipermutable in G and $K \trianglelefteq G$, then HK/K is S-semipermutable in G/K.

Lemma 2.2[18] *Let P be a Sylow p-subgroup of a group G and $N \trianglelefteq G$. If $P \cap N \le \Phi(P)$, then N is p-nilpotent.*

Agrawal defined in [19] the generalized center of a group G, $\mathrm{genz}(G)$, as the subgroup of G generated by all elements g of G such that $\langle g\rangle$ is S-permutable in G, and the generalized hypercenter, $\mathrm{genz}_\infty(G)$, as the largest term of the chain

$$1 = \mathrm{genz}_0(G) \le \mathrm{genz}_1(G) = \mathrm{genz}(G) \le \mathrm{genz}_2(G) \le \cdots$$

where $\mathrm{genz}_{i+1}(G)/\mathrm{genz}_i(G) = \mathrm{genz}(G/\mathrm{genz}_i(G))$, for $i \ge 0$. He proved that:

Lemma 2.3 *A group G is supersolvable if and only if $G = \mathrm{genz}_\infty(G)$.*

Lemma 2.4 *Let P be an elementary abelian p-group of order p^d with $d \ge 2$ and let $\mathcal{M}_d(P) = \{M_1, \ldots, M_d\}$. Then*

(1) $X_i = \bigcap_{j\ne i} M_j$ is cyclic of order p,

(2) $P = \langle X_1, \ldots, X_d\rangle$.

Lemma 2.5[20] *Let H be a solvable normal subgroup of G with $H \ne 1$. If every minimal normal subgroup of G contained in H is not contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of H is the direct product of some minimal normal subgroups of G which are contained in H.*

3. The results

Theorem 3.1 *Let p be the smallest prime dividing the order of G and let G_p be a Sylow p-subgroup of G. If every member in $\mathcal{M}_d(G_p)$ is S-semipermutable in G, then G is p-nilpotent.*

Proof Assume that the theorem is false and let G be a counterexample of minimal order. It follows from [21, IV, 2.8] that G_p is not cyclic. Furthermore, we claim the following facts.

(i) $O_{p'}(G) = 1$.

By Lemma 2.1(2), we observe that the hypothesis is still true for $G/O_{p'}(G)$. If $O_{p'}(G) \neq 1$, then the minimality of G implies that $G/O_{p'}(G)$ is p-nilpotent. It follows that G is p-nilpotent, a contradiction. Thus we may assume that $O_{p'}(G) = 1$. Similarly, we know that if $G_p \leq H < G$, then H is p-nilpotent, by the choice of G.

(ii) Let Q be a Sylow q-subgroup of G, where $q \neq p$. Then G_pQ is a subgroup of G.

Let $\mathcal{M}_d(G_p) = \{P_1, \ldots, P_d\}$. We have $d \geq 2$ and $G_p = P_1P_2$. By the hypothesis, each P_i is S-semipermutable in G, so $P_1Q = QP_1$ and $P_2Q = QP_2$. Thus $G_pQ = P_1P_2Q = P_1QP_2 = QP_1P_2 = QG_p$, i.e., G_pQ is a subgroup of G.

Now, we make use of the above claims to prove our theorem and we treat two cases.

Case 1 $|\pi(G)| = 2$.

In this case, let Q be a Sylow q-subgroup of G, where $q \neq p$ is a prime dividing the order of G. Then assertion (ii) implies $G = G_pQ$. For any $i \in \{1, \ldots, d\}$, by the hypothesis, P_i is S-semipermutable in G and so P_iQ is a subgroup of G. Since p is the smallest prime dividing the order of G, it follows that $P_iQ \trianglelefteq G$ and G/P_iQ is a group of order p. Set

$$N = \bigcap_{i=1}^{d} P_iQ.$$

Then N is a normal subgroup of G such that G/N is a p-group. Since P_i is a maximal subgroup of G_p, we have that $G_p \cap N = \bigcap_{i=1}^{d} P_i = \Phi(G_p)$. By Lemma 2.2, N is p-nilpotent. It follows from $O_{p'}(G) = 1$ that N is a p-group, contradicting that Q is a subgroup of N.

Case 2 $|\pi(G)| \geq 3$.

In this case, let U be a subgroup of G_p with $U \neq 1$. Let Q_1 be a Sylow q-subgroup of $N_G(U)$ and Q be a Sylow q-subgroup of G which contains Q_1 where $q \neq p$ is a prime. Set $K = G_pQ$. Then assertion (ii) implies K is a group of G. It is obvious that K is a proper group of G. Applying assertion (1), K is p-nilpotent. It follows that $Q_1 = Q \cap N_K(U) \trianglelefteq N_K(U)$ and so $UQ_1 = U \times Q_1$. This implies that $N_G(U)/C_G(U)$ is a p-group. By the theorem of Frobenius[21, IV, 5.8], G is p-nilpotent, a contradiction. The proof is completed. □

Corollary 3.2 *Let p be the smallest prime dividing the order of a group G, N a normal subgroup of G such that G/N is p-nilpotent, and let P be a Sylow p-subgroup of N. If every member in $\mathcal{M}_d(P)$ is S-semipermutable in G, then G is p-nilpotent.*

Proof Let K/N be the normal Hall p'-subgroup of G/N. By Lemma 2.1(1) and Theorem 3.1, K is p-nilpotent and therefore G is p-nilpotent. □

Corollary 3.3 *Let G be a group. If, for each Sylow subgroup P of G, every member in $\mathcal{M}_d(P)$ is S-Semipermutable in G, then G is a Sylow tower group.*

Proof Let p be the smallest prime of $|G|$. Then, by Theorem 3.1, G is p-nilpotent. By the same arguments and induction, we see that G is a Sylow tower group. □

Theorem 3.4 *For a group G, the following statements are equivalent:*

(i) G is supersolvable;

(ii) There is a normal subgroup H of G such that G/H is supersolvable and for each non-cyclic Sylow subgroup P of H, every member in $\mathcal{M}_d(P)$ is S-semipermutable in G.

Proof We only need to show that (ii) implies (i). Assume that (ii) holds. By Lemma 2.1(1) and Corollary 3.3, H is a Sylow tower group. Let q be the largest prime dividing the order of H and let Q be a Sylow q-subgroup of H. Then Q is normal in G. By Lemma 2.1(2), the hypothesis is still true for G/Q. Then by induction G/Q is supersolvable.

Assume that $\Phi(Q) \neq 1$. Then, by Lemma 2.1(2), the hypothesis is still true for $G/\Phi(Q)$. Then by induction $G/\Phi(Q)$ is supersolvable and therefore G is supersolvable. Consequently, we may assume that $\Phi(Q) = 1$ and so Q is an elementary abelian group of order q^d. If Q is cyclic, then G is supersolvable. Therefore we may assume that Q is not cyclic.

Let $\mathcal{M}_d(Q) = \{Q_1, \ldots, Q_d\}$, where $d \geq 2$. For any $i \in \{1, \ldots, d\}$, by the hypothesis, Q_i is S-semipermutable in G. Let p be a prime dividing the order of G with $p \neq q$ and let G_p be a Sylow p-subgroup of G. Then Q_iG_p is a group. This implies that $Q_i = Q \cap Q_iG_p \trianglelefteq Q_iG_p$. In particular, G_p normalizes Q_i. It follows that $K = O^q(G)Q$ normalizes Q_i.

Set $X_j = \bigcap_{i \neq j} Q_i$. By Lemma 2.4, X_j is of order q. Since all Q_i are normal in K, we have that $X_j \trianglelefteq K$. Now any Sylow subgroup G_p of G with $p \neq q$ is contained in K, so G_p normalizes X_j. On the other hand, let G_q be a Sylow q-subgroup of G. Then $X_jG_q = G_q = G_qX_j$, since $X_i \leq Q \leq G_q$. Hence X_j permutes with every Sylow subgroup of G and therefore every X_j is contained in the generalized center of G, i.e., $X_j \leq genz(G)$ for all j. Again applying Lemma 2.4, we have $Q = \langle X_1, \ldots, X_d \rangle$. So $Q \leq \mathrm{genz}(G)$. It follows that $G/\mathrm{genz}(G)$ is supersolvable. Thus G is supersolvable by Lemma 2.3. □

Corollary 3.5 *Let G' be the derived subgroup of a group G. If, for each non-cyclic Sylow subgroup P of G', every member in $\mathcal{M}_d(P)$ is S-semipermutable in G, then G' is nilpotent.*

Proof Take $H = G'$. By Theorem 3.4, G is supersolvable. Since the derived subgroup of a supersolvable group is nilpotent, it follows that G' is nilpotent. □

Recall that a class $\mathcal{F}$ of groups is called a formation if $G \in \mathcal{F}$ and $N \trianglelefteq G$, then $G/N \in \mathcal{F}$, and if $G/N_i \in \mathcal{F}$, $i = 1, 2$, then $G/N_1 \cap N_2 \in \mathcal{F}$. If, in addition, $G/\Phi(G) \in \mathcal{F}$ implies $G \in \mathcal{F}$, then $\mathcal{F}$ is called saturated. The class $\mathcal{U}$ of all supersolvable groups is an interesting example of saturated formations.

Theorem 3.6 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let G be a group. Then the following two statements are equivalent:*

(i) $G \in \mathcal{F}$;

(ii) There exists a normal subgroup H of G such that $G/H \in \mathcal{F}$ and for every Sylow subgroup P of H, every member of $\mathcal{M}(P)$ is S-semipermutable in G.

Proof Only $(ii) \Rightarrow (i)$ needs to be proved. By Lemma 2.1(1) and Theorem 3.4, H is supersolvable. Let q be the largest prime dividing H and let Q be a Sylow q-subgroup of H. Then Q is normal in G. Clearly, $(G/Q)/(H/Q) \cong G/H \in \mathcal{F}$. By Lemma 2.1(2), G/Q satisfies the hypothesis. Then by induction $G/Q \in \mathcal{F}$. Let $\mathcal{M}(Q) = \{Q_1, \ldots, Q_n\}$. For any $i \in \{1, \ldots, n\}$, since Q_i is S-semipermutable in G, we may see that Q_iG_p is a subgroup of G, where p is a prime dividing the order of G with $p \neq q$ and G_p is a Sylow p-subgroup of G. Also, $Q_iG_q = G_q = G_qQ_i$ because $Q_i \leq Q \leq G_q$, where G_q is a Sylow q-subgroup of G. Therefore, each member of $\mathcal{M}(Q)$ is S-permutable in G. Thus, by [15, Theorem 3.3], $G \in \mathcal{F}$. □

The following example which is from a manuscript of the second author shows that Theorem 3.6 is false if one replaces $\mathcal{M}(P)$ by $\mathcal{M}_d(P)$ in Theorem 3.6.

Example 3.7 There exists a saturated formation $\mathcal{F}$ containing $\mathcal{U}$ and a solvable group G with a normal p-subgroup P such that $G/P \in \mathcal{F}$ and each member in $\mathcal{M}_d(P)$ is S-permutable in G (hence S-semipermutable in G). But $G \notin \mathcal{F}$.

Proof Let f be a formation function defined by $f(p) =$ the class of p'- groups for any prime p and let $\mathcal{F}$ be the formation locally defined by $\{f(p)\}$. If Y is a supersolvable group, then any p-chief factor H/N of Y is cyclic of order p, so $Y/C_Y(H/N)$ is cyclic of order dividing $p-1$ and hence $Y/C_Y(H/N) \in f(p)$. Therefore, $Y \in \mathcal{F}$ and so $\mathcal{F}$ contains $\mathcal{U}$. Clearly, $A_4 \in \mathcal{F}$.

Let $P = \langle a, b, c\rangle$ be an elementary abelian group of order 3^3 and let α and β be two automorphisms of P defined respectively by

$$\alpha = \begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}, \quad \beta = \begin{pmatrix} a & b & c \\ b & c^{-1} & a^{-1} \end{pmatrix}.$$

Then $\alpha^3 = \beta^3 = (\alpha\beta)^2 = 1$, so $H = \langle \alpha, \beta\rangle \cong A_4$. Then H acts on P by automorphism. Let $G = PH$ be the corresponding semidirect product. In fact, P is an irreducible and faithful A_4-module on $GF(p)$ and so P is a minimal normal subgroup of G with $C_H(P) = 1$. Because $A_4 \in \mathcal{F}$ and $G/P \cong H = A_4$, we have $G/P \in \mathcal{F}$. Let $K = PS$ where S is a Sylow 2- subgroup of G. We have $O^3(G) \leq K \trianglelefteq G$. Since S is elementary abelian of order 4, it follows that a minimal normal subgroup of K contained in P is of order p. By Maschke's theorem[21, I, 17.7], P is a completely reducible S-module. Hence $P = \langle a_1\rangle \times \langle a_2\rangle \times \langle a_3\rangle$, where $\langle a_i\rangle (i = 1, 2, 3)$ are S-invariant. Let $P_i = \langle a_j | j \neq i\rangle$. Then every P_i is S-quasinormal in G and $\mathcal{M}_d(P) = \{P_1, P_2, P_3\}$. On the other hand, P is a 3-chief factor of G and $G = C_G(P) = G/P \cong A_4$, which is not $3'$-group. Hence $G \notin \mathcal{F}$. □

Theorem 3.8 *Let p be a prime dividing the order of a p-solvable group G and let P be a Sylow p-subgroup of G. If every member in $\mathcal{M}_d(P)$ is S-semipermutable in G, then G is p-supersolvable.*

Proof Assume that the theorem is false and let G be a counterexample of minimal order. Then

(1) $O_p(G) > 1$.

It is obvious that $G/O_{p'}(G)$ satisfies the hypothesis. If $O_{p'}(G) > 1$, then minimality of G implies that $G/O_{p'}(G)$ is p-supersolvable and therefore G is p-supersolvable, a contradiction. So

$O_{p'}(G) = 1$. It follows that $O_p(G) > 1$.

(2) $O_p(G) = S_1 \times \cdots \times S_r$ where S_i $(i = 1, \ldots, r)$ is minimal normal subgroup of G of order p.

Let N be a minimal normal subgroup of G contained in $O_p(G)$. If $N \le \Phi(P)$, then, by Lemma 2.1(2), G/N satisfies the hypothesis. Then G/N is p-supersolvable, by the choice of G. Since N is normal in G, by [22, Theorem 5.2.13], we see that $N \le \Phi(G)$. It follows that G is p-supersolvable, a contradiction. Thus $N \nleq \Phi(P)$. We may assume that $N \nleq P_1$ with $P_1 \in \mathcal{M}_d(P)$. Let $N_1 = N \cap P_1$. Then $|N : N_1| = p$. Let $|G| = p^a q_1^{b_1} \cdots q_t^{b_t}$ be the prime factorization. For any $i \in \{1, \ldots, t\}$, let Q_{q_i} be a Sylow q_i-subgroup of G. By the hypothesis, $P_1 Q_{q_i}$ is a subgroup of G and so $N_1 = N \cap P_1 Q_{q_i} \trianglelefteq P_1 Q_{q_i}$. Hence $N_1 \trianglelefteq \langle P_1 Q_{q_1}, \ldots, P_1 Q_{q_t}, N \rangle = G$. The minimality of N implies that $N_1 = 1$. Then N is a cyclic group of order p. Now, N is an abelian subgroup and $NP_1 = P$, $N \cap P_1 = 1$. By Gaschütz's Theorem[21, I, 17.4], there exists a subgroup M of G such that $G = NM$, $N \cap M = 1$. It is obvious that M is a maximal subgroup of G, i.e., $N \nleq \Phi(G)$. Now, by Lemma 2.5, we see that $O_p(G) = S_1 \times \cdots \times S_r$ with S_i is minimal normal subgroup of G. By the same arguments as above, we see S_i has order p, as desired.

(3) $G/O_p(G)$ p-supersolvable.

Because $G/C_G(S_i)$ is cyclic, $G/C_G(S_i)$ is p-supersolvable. Since the class of p-supersolvable groups is a formation, We have that $G/\bigcap_{i=1}^{t} C_G(S_i)$ is p-supersolvable, i.e., $G/C_G(O_p(G))$ is p-supersolvable. On the other hand, since G is p-solvable, it follows from [22, Theorem 9.3.1] that $C_G(O_p(G)) \le O_p(G)$. Thus $G/O_p(G)$ p-supersolvable.

Applying our claims (2) and (3), G is p-supersolvable. □

Theorem 3.9 *Let p be an odd prime dividing the order of G and let P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every member in $\mathcal{M}_d(P)$ is S-semipermutable in G, then G is p-nilpotent.*

Proof Assume that the theorem is false and let G be a counterexample of minimal order.

(1) Every proper subgroup of G containing P is p-nilpotent and $O_{p'}(G) = 1$.

Let $H \le G$ with $P \le H < G$. Then $N_H(P) \le N_G(P)$ and $N_H(P)$ is p-nilpotent. Applying Lemma 1.1(1), we see that H satisfies the hypothesis. Thus H is p-nilpotent, by the choice of G. It is clear that the quotient group $G/O_{p'}(G)$ satisfies the hypothesis by Lemma 1.1(2). Thus, if $O_{p'}(G) \ne 1$, then the minimality of G implies that $G/O_{p'}(G)$ is p-nilpotent and therefore G is p-nilpotent, a contradiction. Hence $O_{p'}(G) = 1$.

(2) G is p-solvable.

Since G is not p-nilpotent, by a result of Thompson[23,Corollary], there exists a characteristic subgroup T of P such that $N_G(T)$ is not p-nilpotent. Since $N_G(P)$ is p-nilpotent, we may choose a characteristic subgroup T of P such that $N_G(T)$ is not p-nilpotent and $N_G(K)$ is p-nilpotent for every characteristic subgroup K of P with $T < K \le P$. Since T is a characteristic subgroup of P, we have $N_G(P) \le N_G(T)$. Moreover, $N_G(P) < N_G(T)$. By (1), we see that $N_G(T) = G$. Then $T = O_p(G)$, by the choice of T. Using the result of Thompson[23,Corollary] again, we see that $G/O_p(G)$ is p-nilpotent and therefore G is p-solvable.

(3) G is p-nilpotent.

By (2) and Theorem 3.8, G is p-supersolvable. Since a p-supersolvable group is p-solvable group of p-rank at most 1, it follows from [21, VI, 6.6] that the p-length of G is at most 1. By (1), we have that $G = O_{pp'}(G)$. In particular, $N_G(P) = G$. It follows that G is p-nilpotent, a contradiction. The proof is completed. □

Corollary 3.10 *Let p be an odd prime dividing the order of a group G and N a normal subgroup of G such that G/N is p-nilpotent. If $N_G(P)$ is p-nilpotent and every member in $\mathcal{M}_d(P)$ is S-semipermutable in G, then G is p-nilpotent, where P is a Sylow p-subgroup of N.*

Proof Let K/N be the normal Hall p'-subgroup of G/N. By Lemma 2.1(1) and Theorem 3.9, K is p-nilpotent and therefore G is p-nilpotent. □

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