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On CAP-Embedded Subgroups in Finite Groups

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Abstract A subgroup H of a finite group G is said to be CAP-embedded subgroup of G if, for each prime p dividing the order of H, there exists a CAP-subgroup K of G such that a Sylow p-subgroup of H is also a Sylow p-subgroup of K. In this paper some new results are obtained based on the assumption that some subgroups of prime power order have the CAP-embedded property in the group.

Keywords CAP-embedded subgroups; maximal subgroups; p-nilpotent groups; supersolvable groups.

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1. Introduction

All groups considered in this paper are finite. Our notation is standard and taken mainly from [1]. Let L be a subgroup of a group G and M/N a chief factor of G. We say that L covers M/N if $LM = LN$, while we say that L avoids M/N if $L \cap M = L \cap N$. L is said to have the cover-avoidance property in G (in short, L is a CAP-subgroup of G), if L either covers or avoids every chief factor of G. In 1962, Gaschütz[2] introduced a kind of important subgroup in studying formations which is called pre-Frattini subgroups. These subgroups have the cover-avoidance property justly. Thereafter, many authors studied this property. A natural question is: What is the influence of some CAP-subgroups on the structure of the group G? In 1993, Ezquerro[3] gave some characterization for a group G to be p-supersolvable and supersolvable based on the assumption that all maximal subgroups of some Sylow subgroups of G have the cover-avoidance property firstly. Later on, the research on the cover-avoidance property is much more developed in [4]. As a generalization of CAP-subgroups, Fan, Guo and Shum in [5] introduced the semi cover-avoidance property (i.e., semi CAP-subgroups) which generalized not only the cover-avoidance property but also c-normality and obtained some new results. The further results can be found in [6], [7]. In this paper, we generalize CAP-subgroups in another way and call it CAP-embedded subgroups (see Def. 2.1). Obviously, CAP-subgroups must be CAP-embedded subgroups, but

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the converse does not hold in general. Moreover, CAP-embedded subgroups are not necessarily semi CAP-subgroups. For example, all Sylow subgroups of the alternative group A_5 of degree 5 are CAP-embedded subgroups of A_5, but every Sylow subgroup is neither a CAP-subgroup nor a semi CAP-subgroup of A_5. We will give some necessary and sufficient conditions and some sufficient conditions for a group G to be p-nilpotent, p-supersolvable and supersolvable by means of some subgroups that have the CAP-embedded property in G.

2. Definitions and Preliminaries

For the sake of convenience, we begin by listing some definitions and lemmas which will be needed in the sequel.

Definition 2.1 *A subgroup H of a group G is said to have the CAP-embedded property in G or is called a CAP-embedded subgroup of G if, for each prime p dividing the order of H, there exists a CAP-subgroup K of G such that a Sylow p-subgroup of H is also a Sylow p-subgroup of K.*

Lemma 2.1 *Let H be a CAP-embedded subgroup of a group G and N a normal subgroup of G. Then HN/N is CAP-embedded in G/N.*

Proof Let $P \in \mathrm{Syl}_p(H)$. Then $PN/N \in \mathrm{Syl}_p(HN/N)$. By hypothesis, there exists a CAP-subgroup M of G such that $P \in \mathrm{Syl}_p(M)$. By [4, Lemma 2.3], MN is a CAP-subgroup of G. Clearly MN/N is a CAP-subgroup of G/N and $PN/N \in \mathrm{Syl}_p(MN/N)$. Hence HN/N is CAP-embedded in G/N. □

Lemma 2.2 *Let H be a subgroup of a group G, $P \in \mathrm{Syl}_p(H)$ and $N \trianglelefteq G$.*

(1) If the maximal subgroups of P are CAP-embedded in G, then the maximal subgroups of PN/N are CAP-embedded in G/N.

(2) If the 2-maximal subgroups of P are CAP-embedded in G, then the 2-maximal subgroups of PN/N are CAP-embedded in G/N.

Proof (1) Let $M/N \lessdot PN/N$. Then $M = M \cap PN = N(P \cap M)$. So we can pick a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Consequently $M = N(P \cap M) \leq NP_1$. Since $P \cap N \leq P \cap M \leq P_1$, it follows that $P \cap N = P_1 \cap N$. By $|PN|/|P_1N| = p$, $M = P_1N \lessdot PN$. By hypothesis, P_1 is a CAP-embedded subgroup of G. Hence M/N is CAP-embedded in G/N by Lemma 2.1.

(2) Let M_1/N be a 2-maximal subgroup of PN/N. Now, arguing as in the proof of (1), we can choose a 2-maximal subgroup P_2 of P such that $M_1 = P_2N$. Hence M_1/N is CAP-embedded in G/N by Lemma 2.1. □

Lemma 2.3[8, Lemma 2.6] *Let N $(N \neq 1)$ be a solvable normal subgroup of a group G. If every minimal normal subgroup of G which is contained in N is not contained in $\Phi(G)$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G contained in N.*

3. CAP-embedded subgroups and the p-nilpotency

We first characterize the p-nilpotency of G by its CAP-embedded subgroups. We have the following theorems.

Theorem 3.1 *Let p be a prime dividing the order of the group G with $(|G|, p-1) = 1$ and let H be a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that P is cyclic or every maximal subgroup of P is CAP-embedded in G, then G is p-nilpotent.*

Proof Assume that the result is false and take G a counterexample of minimal order. Then the following statements about G are true.

(1) P is not cyclic.

Let P be a cyclic group and $P = H$. By hypothesis, G/P is p-nilpotent. Let K/P be a normal p-complement of G/P. Clearly, $K \trianglelefteq G$ and $P \in \mathrm{Syl}_p(K)$. Since $N_K(P)/C_K(P) \lesssim \mathrm{Aut}(P)$ and $(|G|, p-1) = 1$, it follows that $N_K(P) = C_K(P)$. Applying Burnside's Theorem[1, II, Theorem 5.4], K is p-nilpotent. It is obvious that a normal p-complement of K is a normal p-complement of G. Thus G is p-nilpotent, a contradiction. So we can suppose that P is a cyclic group and $P < H$. By $N_H(P)/C_H(P) \lesssim \mathrm{Aut}(P)$ and $(|G|, p-1) = 1$, we have that $N_H(P) = C_H(P)$. Applying Burnside's Theorem [1, II, Theorem 5.4] again, H is p-nilpotent. Let T be a normal p-complement of H. We can consider quotient groups $\overline{G} = G/T$ and $\overline{H} = H/T$. It is clear that $\overline{G}/\overline{H} \cong G/H$ is p-nilpotent, $\overline{P} = PT/T \in \mathrm{Syl}_p(\overline{H})$ and $\overline{P}$ is cyclic. So $\overline{G}$ satisfies the hypotheses of the theorem. By the minimality of G, G/T is p-nilpotent. Therefore G is p-nilpotent, which is a contradiction.

(2) G has a unique minimal normal subgroup N contained in H, G/N is p-nilpotent and $N \nleq \Phi(G)$.

Let N be a minimal normal subgroup of G and $N \leq H$. By Lemma 2.2(1), it is easy to see that the hypotheses of the theorem hold in G/N. By the minimal choice of G, we have that G/N is p-nilpotent. Since the class of p-nilpotent groups is a saturated formation, it follows that N is the unique minimal normal subgroup contained in H and $N \nleq \Phi(G)$.

(3) N is either a p'-group or a group of order p.

Assume N is neither a p'-group nor a p-group. If $P \leq N$, then $P \in \mathrm{Syl}_p(N)$. By (1), $|P| \geqslant p^2$. For every maximal subgroup P_2 of P, P_2 is CAP-embedded in G. So there exists a CAP-subgroup A of G such that $P_2 \in \mathrm{Syl}_p(A)$. Clearly $AN \neq A$ and $N \cap A \geq P_2 \neq 1$, a contradiction. Therefore $P \nleq N$. If $P \cap N \leq \Phi(P)$, then by [10, IV, Theorem 4.7], N is p-nilpotent, which is a contradiction. Hence there exists a maximal subgroup P_3 of P such that $P = (P \cap N)P_3$. Since P_3 is a CAP-embedded subgroup of G, it follows that G has a CAP-subgroup B such that $P_3 \in \mathrm{Syl}_p(B)$. If $BN = B$, then $P_3 \cap N \in \mathrm{Syl}_p(N)$. By $P \cap N \in \mathrm{Syl}_p(N)$, $P_3 \cap N = P \cap N$. Thus $P = (P \cap N)P_3 = (P_3 \cap N)P_3 = P_3$, a contradiction. Hence $B \cap N = 1$. Since $|P||N|_{p'} = |P||N|/|P \cap N| = |PN| \geqslant |P_3N| = |P_3||N|_p|N|_{p'}$ and N is neither a p'-group nor a p-group, it follows that $|N|_p = p$. Now, by $(|G|, p-1) = 1$ and Burnside's Theorem[1, II, Theorem 5.4] again, N is p-nilpotent, a contradiction.

Let N be a p-group. If $N \leq \Phi(P)$, then $N \leq \Phi(G)$ by [9, Theorem 5.2.13], a contradiction. Hence there exists a maximal subgroup P_4 of P such that $P = P_4N$. Since P_4 is CAP-embedded in G, it follows that G has a CAP-subgroup C such that $P_4 \in \mathrm{Syl}_p(C)$. If $NC = C$, then $P = P_4N \leq C$, which is a contradiction. So $C \cap N = 1$. Noting that $C \cap N \geq P_4 \cap N = 1$, we have that $|N| = p$.

(4) Final contradiction.

Suppose N is a p'-group. Since G/N is p-nilpotent, it follows that G is p-nilpotent, a contradiction. If N is a group of order p and let K/N be a normal p-complement of G/N, then N is a Sylow p-subgroup of K. Thus, applying Burnside's Theorem[1, II, Theorem 5.4] again, K is p-nilpotent. Therefore, G is p-nilpotent, final contradiction. □

Corollary 3.2 *Let p be a prime dividing the order of the group G with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if every Sylow p-subgroup P is cyclic or every maximal subgroup of P is CAP-embedded in G.*

Proof In view of Theorem 3.1, we only need to prove the necessity part. Let G be a p-nilpotent group. So we may assume $G = PK$, where $P \in \mathrm{Syl}_p(G)$, K is a normal p-complement of G. If $p \geqslant 3$, by $(|G|, p-1) = 1$ and Feit-Thompson's Theorem[1, II, Theorem 3.8], G is solvable. If $p = 2$, by the same theorem as the above again, G is solvable. Suppose that P is not cyclic. For every maximal subgroup P_1 of P, we have that $P_1K \leq G$ and $|G : P_1K| = p$. Obviously $P_1K \lessdot G$. By [5, Theorem 2.2], P_1K is a CAP-subgroup of G. Therefore, P_1 is CAP-embedded in G. □

Theorem 3.3 *Let p be a prime dividing the order of the group G with $(|G|, p-1) = 1$ and let H be a normal subgroup of G such that G/H is p-nilpotent. If G is A_4-free, and there exists a Sylow p-subgroup P of H such that every 2-maximal subgroup of P is CAP-embedded in G, then G is p-nilpotent.*

Proof Assume that the theorem is false and let G be a counterexample of minimal order. Then, by the same arguments used in the proof of Theorem 3.1, the following statements (1) and (2) about G are true.

(1) G has a unique minimal normal subgroup N contained in H, G/N is p-nilpotent and $N \nleq \Phi(G)$.

(2) $O_{p'}(G) = 1$.

(3) G is solvable.

If $p \geqslant 3$, by Feit-Thompson's Theorem[1, II, Theorem 3.8] and $(|G|, p-1) = 1$, G is solvable. So we assume $p = 2$. If $|N|_2 \leqslant 4$, by [4, Lemma 3.12], N is 2-nilpotent. This implies that N is solvable. Since G/N is 2-nilpotent, it follows that G is solvable. Thus we assume that $|N|_2 \geqslant 8$. Using the same arguments as in the proof of Theorem 3.1(3), we have easily that $P \nleq N$ and $P \cap N \nleq \Phi(P)$. Let P_1 be a maximal subgroup of P containing $P \cap N$. Clearly, $P \cap N \nleq \Phi(P_1)$. Hence there exists a maximal subgroup P_2 of P_1 such that $P_1 = (P \cap N)P_2$. Since P_2 is CAP-embedded in G, it follows that G has a CAP-subgroup A such that $P_2 \in \mathrm{Syl}_2(A)$. Obviously, $AN \neq A$, then we have $A \cap N = 1$. By $|AN|_2 = |A|_2|N|_2 = |P_2||N|_2$, $|N|_2 \leqslant 4$, a contradiction.

Hence N is solvable. Furthermore G is solvable.

(4) Final contradiction.

By (3), N is a p-group and $N \leq P$. Because G/N is p-nilpotent, we can suppose T/N is a normal p-complement of G/N. Clearly N is a Sylow p-subgroup of T. If $|N| \geqslant p^3$, for every 2-maximal subgroup P_3 of P, there exists a CAP-subgroup B of G such that $P_3 \in \mathrm{Syl}_p(B)$ by hypothesis. If $B \cap N = 1$, then $|NP_3| > |P|$, which is a contradiction. So we have $BN = B$. Thus $N \leq P_3$. By the random choice of P_3, $N \leq \Phi(P)$. By [9, Theorem 5.2.13], we have that $N \leq \Phi(G)$, a contradiction. Consequently $|N| \leqslant p^2$. Then, by [4, Lemma 3.12] again, T is p-nilpotent. Hence G is p-nilpotent, final contradiction. □

Corollary 3.4 *Let p be a prime dividing the order of the group G with $(|G|, p^2-1) = 1$ and let H be a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every 2-maximal subgroup of P is CAP-embedded in G, then G is p-nilpotent.*

Proof If $p = 2$, then $3 \nmid |G|$ by $(|G|, p^2-1) = 1$. If $p \geqslant 3$, $2 \nmid |G|$ by $(|G|, p^2-1) = 1$ again. Hence G is A_4-free. Applying Theorem 3.3, G is p-nilpotent. □

4. CAP-embedded subgroups and the supersolvability

In this section, we characterize the supersolvability of G by its CAP-embedded subgroups. We have the following results.

Theorem 4.1 *Let p be a prime dividing the order of the group G and let H be a p-solvable normal subgroup of G such that G/H is p-supersolvable. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is CAP-embedded in G, then G is p-supersolvable.*

Proof Assume the result is false and take G a counterexample of minimal order. Now, arguing as in the proof of Theorem 3.1, the following statements (1) and (2) about G are true.

(1) G has a unique minimal normal subgroup N contained in H, G/N is p-supersolvable and $N \nleq \Phi(G)$.

(2) $O_{p'}(G) = 1$.

Since G is p-solvable and $O_{p'}(G) = 1$, N is a p-group and $N \leq P$. If $N \leq \Phi(P)$, by [9, Theorem 5.2.13], $N \leq \Phi(G)$, a contradiction. Consequently there exists a maximal subgroup P_1 of P such that $P_1N = P$. Since P_1 is a CAP-embedded subgroup of G, it follows that G has a CAP-subgroup B such that $P_1 \in \mathrm{Syl}_p(B)$. Clearly $BN \neq B$. Hence $B \cap N = 1$. Noting that $P_1 \cap N = 1$, N is a group of order p by $|P_1||N| = |P|$. The p-supersolvability of G/N implies that G is p-supersolvable, final contradiction. □

Remark 4.1 The hypothesis that H is p-solvable in Theorem 4.1 is essential. For example, if we let G be the alternating group A_5 of degree 5, $H = G$ and $p = 3$, then it is clear that the statement of Theorem 4.1 does not hold.

Corollary 4.2 *Let G be a group. Then G is supersolvable if and only if there exists a normal* *subgroup H such that G/H is supersolvable and all maximal subgroups of any Sylow subgroup of H have the CAP-embedded property in G.*

Proof The necessity part can be obtained if we let $H = G$ and apply a result due to Ezquerro[3]. So we need to prove the sufficiency part.

Let p be the smallest prime divisor of $|G|$. The supersolvability of G/H implies that G/H is p-nilpotent. By Theorem 3.1, G is p-nilpotent. Furthermore G is solvable. Applying Theorem 4.1, it is easy to see that G is supersolvable. □

Theorem 4.3 *Let G be a group. Then G is supersolvable if and only if there exists a solvable normal subgroup H such that G/H is supersolvable and all maximal subgroups of any Sylow subgroup of $F(H)$ have the CAP-embedded property in G.*

Proof Assume G is supersolvable. Let p be the largest prime divisor of $|G|$. The supersolvability of G implies that there exists a normal subgroup H of G, where the order of H is p. Clearly $H = F(H)$. Hence G/H is supersolvable and all maximal subgroups of any Sylow subgroup of $F(H)$ have the CAP-embedded property in G.

Conversely, assume the result is false and let G be a counterexample of minimal order. Then the following statements about G are true.

(1) $\Phi(G) = 1$.

Assume that $\Phi(G) \neq 1$ and take a prime p dividing $|\Phi(G)|$. Denote $P = O_p(\Phi(G)) \neq 1$. Clearly $P \trianglelefteq G$. Let $F(HP/P) = L/P$. By L/P char $HP/P \trianglelefteq G/P$, $L/P \trianglelefteq G/P$. Hence $L \trianglelefteq G$. Since L/P is a normal nilpotent subgroup of G/P and $P \leq \Phi(G)$, applying a result due to Gaschütz[10, III, Theorem 3.5], we have that L is a normal nilpotent subgroup of HP. Thus $L \leq F(HP)$. Consequently $F(HP/P) = F(HP)/P = L/P$. By [11, Lemma 3.1], $F(HP/P) = F(H)P/P$. It is clear that $(G/P)/(HP/P) \cong G/HP \cong (G/H)/(HP/H)$ is supersolvable. Now, by Lemma 2.2(1), the hypotheses of the theorem hold in G/P. By the minimality of G, G/P is supersolvable. Furthermore G is supersolvable, a contradiction.

(2) Let $T \in \mathrm{Syl}_p(F(H))$. Then $T = N_1 \times N_2 \times \cdots \times N_r$, where N_i $(i = 1, 2, \ldots, r)$ are minimal normal subgroups of G and $|N_i| = p$.

Since H is solvable, $\Phi(H) \leq \Phi(G) = 1$ and by Lemma 2.3, $T = N_1 \times N_2 \times \cdots \times N_r$, where N_i $(i = 1, 2, \ldots, r)$ are both elementary abelian and minimal normal subgroups of G. We will show that every N_i is a group of order p. If $|N_1| \geqslant p^2$, for every maximal subgroup R_1 of N_1, then it is clear that $T_1 = R_1 \times N_2 \times \cdots \times N_r \lessdot T$. By hypothesis, there exists a CAP-subgroup A of G such that $T_1 \in \mathrm{Syl}_p(A)$. Since $N_1 \nleq A$, it follows that $R_1 = N_1 \cap A = 1$, a contradiction. Therefore $|N_1| = p$. By the same arguments as the above we have that $|N_2| = p, \ldots, |N_r| = p$.

(3) Final contradiction.

By (2), we can assume that $F(H) = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_n \rangle$, where $\langle a_i \rangle$ $(i = 1, \ldots, n)$ are normal subgroups of G with prime order. By $G/C_G(\langle a_i \rangle) \lesssim \mathrm{Aut}(\langle a_i \rangle)$, $G/C_G(\langle a_i \rangle)$ is cyclic. So we have that $G/C_G(\langle a_i \rangle)$ is supersolvable. Since $C_G(F(H)) = \bigcap_{i=1}^{n} C_G(\langle a_i \rangle)$, it follows that $G/C_G(F(H))$ is supersolvable. And by the hypotheses, $G/C_H(F(H)) = G/(H \cap C_G(F(H)))$ is supersolvable. Noting that $F(H)$ is abelian, we have that $F(H) \leq C_H(F(H))$. On the other hand, $C_H(F(H)) \leq F(H)$ for H is solvable. Hence $C_H(F(H)) = F(H)$. Thus $G/F(H)$ is supersolvable. Applying Corollary 4.2, G is supersolvable, final contradiction. □

Remark 4.2 The hypothesis that H is solvable in Theorem 4.3 cannot be removed. For example, if we let $G = SL(2,5)$ and $H = G$, then $F(H)$ is a group of order 2. Thus all maximal subgroups of any Sylow subgroup of $F(H)$ have the CAP-embedded property in G, but G is not supersolvable.

Corollary 4.4 *Let N be a subgroup of a solvable group G and $G' \leq N$. If all maximal subgroups of any Sylow subgroup of $F(N)$ have the CAP-embedded property in G, then G is supersolvable.*

Proof By $G/N \cong (G/G')/(N/G')$, we have that G/N is abelian. By Theorem 4.3, G is supersolvable. □

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