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Composition Operators from α-Bloch Spaces into Q_K Type Spaces

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Abstract Suppose ϕ is an analytic map of the unit disk D into itself, X is a Banach space of analytic functions on D. Define the composition operator C_ϕ: $C_\phi f = f \circ \phi$, for all $f \in X$. In this paper, the boundedness and compactness of the composition operators from α-Bloch spaces into $Q_K(p,q)$ and $Q_{K,0}(p,q)$ spaces are discussed, where $0 < \alpha < \infty$.

Keywords Composition operator; analytic function; $\mathcal{B}^\alpha$ space; K-Carleson measure; compact K-Carleson measure.

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1. Introduction

First, we introduce some basic notations, which are used in this paper. The unit disk in the finite complex plane C will be denoted by D. $H(D)$ will denote the space of all analytic functions on D, $B(D)$ will denote the subset of $H(D)$ consisting of these $f \in H(D)$ for which $|f(z)| < 1$, $\mathrm{d}A$ will denote the Lebesgue measure on D, normalized so that $A(D) = 1$. For $a \in D$, $\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$ is the Möbius transformation of D to itself, $g(z,a) = \log|\frac{1-\overline{a}z}{a-z}|$ is the Green function of D with singularity at a. Every analytic self-map ϕ of the unit disk D induces through composition a linear composition operator C_ϕ: $C_\phi f = f \circ \phi$ from X to itself. N^+ is the natural numbers set. We say the function $f \in \mathcal{B}^\alpha$, if $f \in H(D)$ and

$$\|f\|_\alpha = |f(0)| + \|f\|_{b^\alpha} < \infty,$$

where

$$\|f\|_{b^\alpha} = \sup_{z \in D}(1-|z|^2)^\alpha |f'(z)|.$$

We say the function $f \in \mathcal{B}_0^\alpha$, if $f \in H(D)$ and

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f'(z)| = 0.$$

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The space $\mathcal{B}^\alpha$ is a Banach space under the norm $\|\cdot\|_\alpha$, $\mathcal{B}^\alpha_0$ is the closed subset of $\mathcal{B}^\alpha$. When $\alpha = 1$, we get the Bloch space and the little Bloch space. In recent years a special class of Möbius invariant function spaces, the so-called $Q_K(p,q)$ spaces, has attracted a lot of attention. One important property of $Q_K(p,q)$ spaces is the inclusionship with α-Bloch spaces $\mathcal{B}^\alpha$. It was shown in [1] that $Q_K(p,q) \subset \mathcal{B}^{\frac{q+2}{p}}$. Furthermore, $Q_K(p,q) = \mathcal{B}^{\frac{q+2}{p}}$ if and only if

$$\int_0^1 (1-r^2)^{-2}(\log\frac{1}{r})r\,\mathrm{d}r < \infty.$$

We recall some facts about $Q_K(p,q)$ spaces. We let $K : [0,\infty) \to [0,\infty)$ be a right-continuous and nondecreasing function. For $0 < p < \infty$, $-2 < q < \infty$, we say that a function f analytic in D belongs to the space $Q_K(p,q)$[1], if

$$\|f\|^p_{K,p,q} = \sup_{a\in D}\int_D |f'(z)|^p(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z) < \infty.$$

Let $\|f\| = |f(0)| + \|f\|_{K,p,q}$. $Q_K(p,q)$ is a Banach space under the norm $\|\cdot\|$ when $p \geq 1$. If $f \in H(D)$ and

$$\lim_{|a|\to 1}\int_D |f'(z)|^p(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z) = 0,$$

then we say $f \in Q_{K,0}(p,q)$. If $p = q+2$, the space $Q_K(p,q)$ is a Möbius-invariant, i.e., $\|f\circ\varphi_a\|_{K,p,q} = \|f\|_{K,p,q}$ for all $a \in D$. $Q_{K,0}(p,q)$ is a closed subset of $Q_K(p,q)$. Three special cases are worth mentioning. When $p = 2$, $q = 0$, $Q_K(p,q) = Q_K$[2]–[4]; When $K(t) = t^s$, $Q_K(p,q) = F(p,q,s)$, $Q_{K,0}(p,q) = F_0(p,q,s)$[5]; When $p = 2$, $q = 0$, $K(t) = t^\alpha$, $Q_K(p,q) = Q_\alpha$, $Q_{K,0}(p,q) = Q_{\alpha,0}$[6]. The space $Q_K(p,q)$ is trivial, if $Q_K(p,q)$ contains only constant functions. If the integral

$$\int_0^1 (1-r^2)^q K(\log\frac{1}{r})r\,\mathrm{d}r$$

is divergent, then $Q_K(p,q)$ is trivial[1]. It is clear that the function-theoretic properties of $Q_K(p,q)$ depend on the structure of K. So, as in [7], from now on we take it for granted that the above weight function K always satisfies the following conditions:

(a) $K : [0,\infty) \to [0,\infty)$ is nondecreasing;

(b) K is two times differentiable on $(0,1)$;

(c) The above integral is convergent;

(d) $K(t) = K(1) > 0$, $t \geq 1$;

(e) $K(2t) \simeq K(t)$;

(f) $\int_0^1 \varphi_K(s)\frac{\mathrm{d}s}{s} < \infty$, where $\varphi_K(s) = \sup_{0\leq t\leq 1}\frac{K(st)}{K(t)}$ $(0 < s < \infty)$;

(g) $I = \sup_{a\in D}\int_D \frac{(1-|z|^2)^{p-2}}{|1-\bar{a}z|^p}K(\log\frac{1}{|z|})\,\mathrm{d}A(z) < \infty$.

It is a well-known consequence of Littlewood's subordination principle[8] that the formula $C_\phi f = f\circ\phi$ defines a bounded linear operator on the classical Hardy and Bergman spaces. That is, $C_\phi : H^p \to H^p$ and $C_\phi : A^p \to A^p$ are bounded operators. There has been done much research on the relations between the function theoretic properties of ϕ and the topological properties of the operator C_ϕ in different circumstances. Lou[9] discussed the boundedness and the compactness of the composition operators from $\mathcal{B}^\alpha$ to $\mathcal{B}^\beta$ when $0 < \alpha < \infty, 0 < \beta < \infty$;

Zhang[10] discussed the boundedness and the compactness of the composition operators and the weighted composition operators from $\mathcal{B}^p$ to $\mathcal{B}^q$ when $p, q \geq 0$; Zhang and Xiao[11] gave the boundedness and the compactness of the composition operators from μ-Bloch to ν-Bloch on the unit ball when μ and ν are the normal functions on $[0, 1)$;

Recently, Wulan and Zhou[1] gave many results on $Q_K(p, q)$ spaces; Wulan and Wu[4,12] discussed the boundedness and compactness of the composition operators from the Bloch space to Q_K; Li and Wulan[13] characterized compact composition operators from Q_{K_1} into Q_{K_2} spaces; Kotilainen[14] discussed the boundedness and compactness of the composition operators from $\mathcal{B}^\alpha$ to $Q_K(p, q)$; Yu and Liu[15] discussed the boundedness of the composition operators from $\mathcal{B}^\alpha$ to $Q_K(p, q)$ and composition operators from hyperbolic α-Bloch spaces into hyperbolic Q_K type spaces. We want to characterize here by means of K-Carleson measures and compact K-Carleson measures the boundedness and compactness of C_ϕ from $\mathcal{B}^\alpha$ spaces into $Q_K(p, q)$ and $Q_{K,0}(p, q)$ spaces. Throughout this paper, given a subarc $I \subset \partial D$, the boundary of D, we denote by $S(I)$ the Carleson box based on I

$$S(I) = \{r\zeta \in D : 1 - |I| < r < 1, \zeta \in I\}.$$

If $|I| \geq 1$, then we set $S(I) = D$. For $0 < p < \infty$, we say that a positive measure μ on D is a p-Carleson measure if

$$\|\mu\|_p = \sup_{I \subset \partial D} \frac{\mu(S(I))}{|I|^p} < \infty,$$

where the supremum is taken over all subarcs I of ∂D. If the right hand fractions tend to zero as $|I| \to 0$, then μ is said to be a compact p-Carleson measure. Note that the 1-Carleson measures are the classical Carleson measures. In a similar way, a positive measure μ on D is said to be a K-Carleson measure if

$$\|\mu\|_K = \sup_{I \subset \partial D} \mu_K(S(I)) < \infty,$$

where the supremum is taken over all subarcs I of ∂D, and

$$\mu_K(S(I)) = \int_{S(I)} K\Big(\frac{1 - |z|}{|I|}\Big) \, \mathrm{d}\mu(z).$$

Also, μ is said to be a compact K-Carleson measure if $\|\mu\|_K < \infty$ and

$$\lim_{|I| \to 0} \mu_K(S(I)) = 0.$$

Clearly, if $K(t) = t^p$, then μ is a K-Carleson measure if and only if the measure $(1 - |z|^2)^p \, \mathrm{d}\mu(z)$ is a p-Carleson measure.

We use the notation $a \simeq b$ to denote the comparability of the quantities a and b; i.e., the existence of two positive constants C_1 and C_2 satisfying $C_1 a \leq b \leq C_2 a$. For convenience, we will always use the letter C to denote a positive constant, which may change from one equation to the next. The constants usually depend on a and other fixed parameters.

2. Preliminaries

The following result (part (i) proved in [7]) characterizes K-Carleson measures in conformally invariant terms.

Lemma 1[7,16,17] *Suppose K satisfies (f). Then*

(i) μ is a K-Carleson measure if and only if

$$\sup_{a\in D}\int_D K(1-|\varphi_a(z)|^2)\,\mathrm{d}\mu(z)<\infty; \tag{2.1}$$

(ii) μ is a compact K-Carleson measure if and only if (2.1) holds and

$$\lim_{|a|\to 1}\int_D K(1-|\varphi_a(z)|^2)\,\mathrm{d}\mu(z)=0.$$

Lemma 2[18] *Let K satisfy (f), (g), $0<p<\infty$ and $-1<q<\infty$. Suppose n is a positive integer. Then $f\in Q_K(p,q)$ if and only if*

$$|f^{(n)}(z)|^p(1-|z|^2)^{np-p+q}\,\mathrm{d}A(z)$$

is a K-Carleson measure.

Remark When $n=1$, the result holds for $-2<q<\infty$[1].

By Lemmas 1 and 2, we have the following

Lemma 3 *Suppose $p>0$, $-2<q<\infty$. Then the following statements are equivalent:*

(1) $f\in Q_K(p,q)$;

(2) $\sup_{a\in D}\int_D|f'(z)|^p(1-|z|^2)^qK(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z)<\infty$;

(3) $|f'(z)|^p(1-|z|^2)^q\,\mathrm{d}A(z)$ is a K-Carleson measure.

The following lemma is a generalization of the result in [5].

Lemma 4 *Suppose $p>0$, $-2<q<\infty$ and μ is a K-Carleson measure. Then the following statements are equivalent:*

(1) $f\in Q_{K,0}(p,q)$;

(2) $\lim_{|a|\to 1}\int_D|f'(z)|^p(1-|z|^2)^qK(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z)=0$;

(3) $|f'(z)|^p(1-|z|^2)^q\,\mathrm{d}A(z)$ is a compact K-Carleson measure.

Proof The (ii) of Lemma 1 implies that the equivalence of (2) and (3). (1) $\Rightarrow$ (2) follows from the following inequalities

$$\int_D|f'(z)|^p(1-|z|^2)^qK(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z)\le C\int_D|f'(z)|^p(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z).$$

To prove (2) $\Rightarrow$ (1), we set $\Delta(a,\frac14)=\{z\in D:|\varphi_a(z)|\le\frac14\}$ and $D_{\frac14}=D-\Delta(a,\frac14)$, respectively. Since

$$g(z,a)\le 8(1-|\varphi_a(z)|^2),\quad |\varphi_a(z)|\ge\frac14,$$

we obtain that

$$\int_{D-\Delta(a,\frac14)}|f'(z)|^p(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)$$

$$\le C\int_{D-\Delta(a,\frac{1}{4})}|f'(z)|^p(1-|z|^2)^qK(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z)$$
$$\le C\int_{D}|f'(z)|^p(1-|z|^2)^qK(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z). \tag{2.2}$$

The Möbius-invariance of measure $(1-|z|^2)^{-2}\,\mathrm{d}A(z)$ implies that

$$\begin{aligned}&\int_{\Delta(a,\frac{1}{4})}|f'(z)|^p(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)\\ &\le \sup_{z\in\Delta(a,\frac{1}{4})}\{|f'(z)|^p(1-|z|^2)^{q+2}\}\int_{\Delta(a,\frac{1}{4})}(1-|z|^2)^{-2}K(g(z,a))\,\mathrm{d}A(z)\\ &\le \sup_{z\in\Delta(a,\frac{1}{4})}\{|f'(z)|^p(1-|z|^2)^{q+2}\}\int_{\Delta(0,\frac{1}{4})}(1-|z|^2)^{-2}K(g(z,0))\,\mathrm{d}A(z)\\ &\le \sup_{z\in\Delta(a,\frac{1}{4})}\{|f'(z)|^p(1-|z|^2)^{q+2}\}\int_0^{\frac{1}{4}}r(1-r^2)^{-2}K(\log\frac{1}{r})\,\mathrm{d}r\\ &\le C\int_D|f'(z)|^p(1-|z|^2)^qK(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z).\end{aligned}$$

So

$$\int_D|f'(z)|^p(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)\le C\int_D|f'(z)|^p(1-|z|^2)^qK(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z). \tag{2.3}$$

We get the desired condition, which completes the proof of Lemma 4.

Lemma 5[9,19] *Suppose $0<\alpha<\infty$. Then there exist $f,g\in\mathcal{B}^\alpha$, such that*

$$|f'(z)|+|g'(z)|\ge\frac{1}{(1-|z|)^\alpha}\ge\frac{1}{(1-|z|^2)^\alpha}$$

for all $z\in D$.

Lemma 6[14,15] *Suppose $0<p,\ \alpha<\infty$, $-2<q<\infty$, $\phi\in B(D)$. Then the following statements are equivalent:*

(1) $C_\phi:\mathcal{B}^\alpha\to Q_K(p,q)$ *is bounded;*
(2) $C_\phi:\mathcal{B}_0^\alpha\to Q_K(p,q)$ *is bounded;*
(3) $\sup_{a\in D}\int_D\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)<\infty$.

Lemma 7[14] *Suppose $0<\alpha<\infty$, $p\ge1$, $-2<q<\infty$, $\phi\in B(D)$. Then the following statements are equivalent:*

(1) $C_\phi:\mathcal{B}^\alpha\to Q_K(p,q)$ *is compact;*
(2) $C_\phi:\mathcal{B}_0^\alpha\to Q_K(p,q)$ *is compact;*
(3) $\phi\in Q_K(p,q)$ *and*

$$\lim_{r\to1}\sup_{a\in D}\int_{|\phi(z)|>r}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)=0.$$

3. Composition operators from the α-Bloch spaces into $Q_K(p,q)$ and $Q_{K,0}(p,q)$

In this section we are ready to prove the following results.

Theorem 1 *Suppose* $0 < p,\ \alpha < \infty$, $-2 < q < \infty$, $\phi \in B(D)$. *Then* $C_\phi : \mathcal{B}^\alpha \to Q_K(p,q)$ *is bounded if and only if* $\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q\,\mathrm{d}A(z)$ *is a* K*-Carlesom measure.*

Proof Necessity. Using Lemma 6, we have

$$\sup_{a\in D}\int_D \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z) < \infty.$$

By Lemma 1, it suffices to prove that

$$\sup_{a\in D}\int_D \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z) < \infty.$$

Since K is nondecreasing and $(1-t^2) \leq 2\log\frac{1}{t}$, for $0 < t \leq 1$, we have $(1-|\varphi_a(z)|^2) \leq 2\log\frac{1}{|\varphi_a(z)|} = 2g(z,a)$, for $z, a \in D$. Therefore,

$$\begin{aligned}
&\sup_{a\in D}\int_D \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z)\\
&\quad\leq \sup_{a\in D}\int_D \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(2g(z,a))\,\mathrm{d}A(z)\\
&\quad\leq C\sup_{a\in D}\int_D \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z)\\
&\quad< \infty.
\end{aligned}$$

Sufficiency. Assume that $\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q\,\mathrm{d}A(z)$ is a K-Carlesom measure. Then

$$\sup_{a\in D}\int_D \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z) < \infty.$$

We obtain that for all $f \in \mathcal{B}^\alpha$,

$$\begin{aligned}
&\sup_{a\in D}\int_D |(C_\phi f)'(z)|^p(1-|z|^2)^q K(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z)\\
&\quad= \sup_{a\in D}\int_D |f'(\phi(z))|^p|\phi'(z)|^p(1-|z|^2)^q K(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z)\\
&\quad\leq \|f\|_\alpha^p \sup_{a\in D}\int_D \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z)\\
&\quad< \infty.
\end{aligned}$$

By Lemma 3, $C_\phi f \in Q_K(p,q)$. Thus $C_\phi : \mathcal{B}^\alpha \to Q_K(p,q)$ is bounded. The proof is completed.□

Theorem 2 *Suppose* $0 < p, \alpha < \infty$, $-2 < q < \infty$, $\phi \in B(D)$. *Then the following statements are equivalent:*

(1) $C_\phi : \mathcal{B}_0^\alpha \to Q_{K,0}(p,q)$ *is bounded;*

(2) $\phi \in Q_{K,0}(p,q)$ *and for all* $r \in (0,1)$

$$\sup_{a\in D}\int_{|\phi(z)|>r}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z)<\infty.$$

Proof (1) $\Rightarrow$ (2). Suppose $C_\phi:\mathcal{B}_0^\alpha\to Q_{K,0}(p,q)$ is bounded. We get $C_\phi:\mathcal{B}_0^\alpha\to Q_K(p,q)$ is bounded. By Lemma 6, we get for all $r\in(0,1)$,

$$\begin{aligned}&\sup_{a\in D}\int_{|\phi(z)|>r}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z)\\ &\le \sup_{a\in D}\int_{D}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{\alpha}}(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z)\\ &<\infty.\end{aligned}$$

Take $f(z)=z\in\mathcal{B}_0^\alpha$. Since $C_\phi:\mathcal{B}_0^\alpha\to Q_{K,0}(p,q)$ is bounded, $C_\phi f=\varphi\in Q_{K,0}(p,q)$.

(2) $\Rightarrow$ (1). To prove that $C_\phi:\mathcal{B}_0^\alpha\to Q_{K,0}(p,q)$ is bounded, by the closed graph theorem we only need to show that for all $f\in\mathcal{B}_0^\alpha$, then $C_\phi f\in Q_{K,0}(p,q)$. For all $\varepsilon>0$, since $f\in\mathcal{B}_0^\alpha$, there exists $r\in(0,1)$, such that $|f'(w)|^p(1-|w|^2)^{p\alpha}<\varepsilon$, for all $|w|>r$. For $z\in\{z:|\phi(z)|>r\}$, then $|f'(\phi(z))|^p(1-|\phi(z)|^2)^{p\alpha}<\varepsilon$. For the above r, by the condition, there exists $C>0$, such that for all $a\in D$

$$\begin{aligned}&\int_{|\phi(z)|>r}|(f\circ\phi)'(z)|^p(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z)\\ &=\int_{|\phi(z)|>r}|f'(\phi(z))|^p(1-|\phi(z)|^2)^{p\alpha}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z)\\ &\le\varepsilon\sup_{a\in D}\int_{|\phi(z)|>r}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z)\\ &\le C\varepsilon.\end{aligned}$$

Also, $\phi\in Q_{K,0}(p,q)$ implies that

$$\begin{aligned}&\lim_{|a|\to1}\int_{|\phi(z)|\le r}|(f\circ\phi)'(z)|^p(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z)\\ &=\lim_{|a|\to1}\int_{|\phi(z)|\le r}|f'(\phi(z))|^p(1-|\phi(z)|^2)^{p\alpha}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z)\\ &\le\frac{\|f\|_{b^\alpha}^p}{(1-r^2)^{p\alpha}}\lim_{|a|\to1}\int_{|\phi(z)|\le r}|\phi'(z)|^p(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z)\\ &\le\frac{\|f\|_{\alpha}^p}{(1-r^2)^{p\alpha}}\lim_{|a|\to1}\int_{D}|\phi'(z)|^p(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z)\\ &=0.\end{aligned}$$

Combining all the above, we get

$$\begin{aligned}&\lim_{|a|\to1}\int_{D}|(f\circ\phi)'(z)|^p(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z)\\ &=\lim_{|a|\to1}\Big(\int_{|\phi(z)|>r}+\int_{|\phi(z)|\le r}|(f\circ\phi)'(z)|^p(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z)\Big)\\ &=0.\end{aligned}$$

Therefore $C_\phi f \in Q_{K,0}(p,q)$, i.e., $C_\phi : \mathcal{B}_0^\alpha \to Q_{K,0}(p,q)$ is bounded. The proof is completed. $\square$

Theorem 3 *Suppose* $0 < \alpha < \infty$, $p \geq 1$, $-2 < q < \infty$, $\phi \in B(D)$. *Then the following statements are equivalent:*

(1) $C_\phi : \mathcal{B}^\alpha \to Q_{K,0}(p,q)$ *is bounded;*

(2) $C_\phi : \mathcal{B}^\alpha \to Q_{K,0}(p,q)$ *is compact;*

(3) $\lim_{|a|\to 1} \int_D \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z) = 0$;

(4) $\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q\,\mathrm{d}A(z)$ *is a compact* K*-Carlesom measure;*

(5) $C_\phi : \mathcal{B}_0^\alpha \to Q_{K,0}(p,q)$ *is compact;*

(6) $\phi \in Q_{K,0}(p,q)$ *and*

$$\lim_{r\to 1} \sup_{a\in D} \int_{|\phi(z)|>r} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z) = 0. \tag{3.1}$$

Proof (2) $\Rightarrow$ (1). This implication is obvious.

(1) $\Rightarrow$ (3). Suppose (1) is satisfied. By Lemma 5, there exist $f_1, f_2 \in \mathcal{B}^\alpha$, such that

$$|f_1'(z)| + |f_2'(z)| \geq \frac{1}{(1-|z|)^\alpha} \geq \frac{1}{(1-|z|^2)^\alpha}$$

for all $z \in D$. Since $C_\phi : \mathcal{B}^\alpha \to Q_{K,0}(p,q)$ is bounded, we get $C_\phi f_1 \in Q_{K,0}(p,q)$ and $C_\phi f_2 \in Q_{K,0}(p,q)$. Thus

$$\begin{aligned}
&\lim_{|a|\to 1} \int_D \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z) \\
&\leq C \lim_{|a|\to 1} \int_D (|(f_1\circ\phi)'(z)|^p + |(f_2\circ\phi)'(z)|^p)(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z) \\
&= 0.
\end{aligned}$$

(3) $\Rightarrow$ (4). Assume (3) holds. Then we have that

$$\begin{aligned}
&\int_D \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z) \\
&\leq C \int_D \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z), \\
&\to 0 \ \text{ as } \ |a| \to 1.
\end{aligned}$$

By Lemma 1, $\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q\,\mathrm{d}A(z)$ is a compact K-Carlesom measure.

(4) $\Rightarrow$ (1). (4) gives that

$$\lim_{|a|\to 1} \int_D \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z) = 0. \tag{3.2}$$

For all $f \in \mathcal{B}^\alpha$, we have that

$$\begin{aligned}
&\int_D |(C_\phi f)'(z)|^p (1-|z|^2)^q K(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z) \\
&\leq C\|f\|_\alpha^p \int_D \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q K(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z), \\
&\to 0 \ \text{ as } \ |a| \to 1.
\end{aligned}$$

By Lemma 4, $C_\phi f \in Q_{K,0}(p,q)$, so $C_\phi : \mathcal{B}^\alpha \to Q_{K,0}(p,q)$ is bounded.

(4) $\Rightarrow$ (2). Let $\{f_n\} \subset \mathcal{B}^\alpha$, $\|f_n\|_\alpha \le 1$ and $\{f_n\}$ converge on compact subsets of D to 0 uniformly. Next, we will prove

$$\|C_\phi f_n\| \to 0 \ \text{ as } \ n \to \infty.$$

Firstly , $f_n(\phi(0)) \to 0$ as $n \to \infty$. Also, by (2.3) and (3.2), for all $\varepsilon > 0$, there exists $\delta : 0 < \delta < 1$ such that if $|a| > \delta$, then for all $n \in N^+$

$$\begin{aligned}
&\sup_{|a|>\delta} \int_D |f_n'(\phi(z))|^p |\phi'(z)|^p (1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z)\\
&\le C \sup_{|a|>\delta} \int_D |f_n'(\phi(z))|^p |\phi'(z)|^p (1-|z|^2)^q K(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z)\\
&\le C\|f_n\|_\alpha^p \sup_{|a|>\delta} \int_D \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}} (1-|z|^2)^q K(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z)\\
&< \|f_n\|_\alpha^p \varepsilon \le \varepsilon.
\end{aligned}$$

Since $\{f_n'\}$ converges on a compact subset $rD = \{z : |z| \le r\}$ of D to 0 uniformly, for $0 < r < 1$, there exists $N > 0$, such that if $n \ge N$, then $|f_n'(\phi(z))|^p < \varepsilon$, for all $z \in rD$. We then get

$$\begin{aligned}
&\int_{rD} |f_n'(\phi(z))|^p |\varphi'(z)|^p (1-|z|^2)^q K(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z)\\
&\quad < \varepsilon \int_{rD} |\phi'(z)|^p (1-|z|^2)^q K(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z).
\end{aligned}$$

Thus

$$\begin{aligned}
&\sup_{|a|\le\delta} \int_{rD} |f_n'(\phi(z))|^p |\phi'(z)|^p (1-|z|^2)^q K(1-|\varphi_a(z)|^2)\,\mathrm{d}A(z)\\
&< \varepsilon \|\phi\|_{Q_{K,p,q}}^p, \quad \forall r \in (0,1),\ n \ge N.
\end{aligned}$$

Since (4)$\Rightarrow$(1), we have $f_n \circ \phi \in Q_{K,0}(p,q) \subset Q_K(p,q)$. For all a, $|a| \le \delta$, we have

$$\int_D |(f_n \circ \phi)'(z)|^p (1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z) \le C < \infty.$$

For $0 < t < 1$, set $I_t(a) = \int_{D-tD} (f_n \circ \phi)'(z)|^p (1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z)$. We can choose $t(a) \in (0,1)$, such that

$$I_{t(a)}(a) = \int_{D-t(a)D} |(f_n \circ \phi)'(z)|^p (1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z) < \varepsilon.$$

Since $I_t(a)$ is a continuous function of a[13], there is a neighborhood $U(a)$ of a, for all $b \in U(a)$, such that

$$I_{t(a)}(b) = \int_{D-t(a)D} |(f_n \circ \phi)'(z)|^p (1-|z|^2)^q K(g(z,b))\,\mathrm{d}A(z) < \varepsilon.$$

Thus, using the compactness of $\{a : |a| \le \delta\}$, there exist $a_i \in D$, $i = 1, 2, \ldots, N_1$, such that $\{a : |a| \le \delta\} \subset \cup_{i=1}^{N_1} U(a_i)$. We take $t_0 = \max_{1\le i\le N_1} t(a_i)$, then for all a, $|a| \le \delta$,

$$\int_{D-t_0D} |(f_n \circ \phi)'(z)|^p (1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z) < \varepsilon.$$

Thus, using (3.3), we have

$$\begin{aligned}
&\sup_{|a|\le\delta}\int_D |f_n'(\phi(z))|^p|\phi'(z)|^p(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)\\
&= \sup_{|a|\le\delta}[\int_{D-t_0D}|f_n'(\phi(z))|^p|\phi'(z)|^p(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)+\\
&\quad \int_{t_0D}|f_n'(\phi(z))|^p|\phi'(z)|^p(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)]\\
&\le \sup_{|a|\le\delta}\int_{D-t_0D}|(f_n\circ\phi)'(z)|^p(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)+\\
&\quad \sup_{|a|\le\delta}\int_{t_0D}|f_n'(\phi(z))|^p|\phi'(z)|^p(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)\\
&< (1+\|\phi\|^p_{Q_{K,p,q}})\varepsilon.
\end{aligned}$$

Combining all the above, we get that for all $\varepsilon>0$, there exists $N>0$, for $n>N$,

$$\begin{aligned}
&\sup_{a\in D}\int_D |f_n'(\phi(z))|^p|\phi'(z)|^p(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)\\
&\le (2+\|\phi\|^p_{Q_{K,p,q}})\varepsilon < C\varepsilon.
\end{aligned}$$

Thus $\|C_\phi f_n\| = |f_n(\phi(0))| + \|C_\phi f_n\|_{K,p,q}\to 0$ as $n\to\infty$, i.e., $C_\phi:\mathcal{B}^\alpha\to Q_{K,0}(p,q)$ is compact.

(6)$\Rightarrow$(1). Suppose (6) is satisfied. For all $\varepsilon>0$, there exists $\delta: 0<\delta<1$, such that if $\delta<r<1$, then for $a\in D$

$$\begin{aligned}
&\int_{|\phi(z)|>r}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)\\
&\le \sup_{a\in D}\int_{|\phi(z)|>r}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)\\
&< \varepsilon.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{|\phi(z)|>r}|(f\circ\phi)'(z)|^p(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)\\
&= \int_{|\phi(z)|>r}|f'(\phi(z))|^p(1-|\phi(z)|^2)^{p\alpha}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)\\
&\le \|f\|^p_\alpha\sup_{a\in D}\int_{|\phi(z)|>r}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)\\
&\le C\varepsilon.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_{|\phi(z)|\le r}|(f\circ\phi)'(z)|^p(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)\\
&= \int_{|\phi(z)|\le r}|f'(\phi(z))|^p(1-|\phi(z)|^2)^{p\alpha}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)\\
&\le \frac{\|f\|^p_\alpha}{(1-r^2)^{p\alpha}}\int_{|\phi(z)|\le r}|\phi'(z)|^p(1-|z|^2)^qK(g(z,a))\,\mathrm{d}A(z)
\end{aligned}$$

$$\leq \frac{\|f\|_\alpha^p}{(1-r^2)^{p\alpha}} \int_D |\phi'(z)|^p (1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z)$$
$$\to 0 \quad \text{as} \quad |a| \to 1.$$

Combining all the above, we get

$$\begin{aligned}
&\lim_{|a|\to 1} \int_D |(f\circ\phi)'(z)|^p (1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z) \\
&= \lim_{|a|\to 1} \Big(\int_{|\phi(z)|>r} + \int_{|\phi(z)|\le r} |(f\circ\phi)'(z)|^p (1-|z|^2)^q K(g(z,a))\,\mathrm{d}A(z) \Big) \\
&= 0.
\end{aligned}$$

Therefore $C_\phi f \in Q_{K,0}(p,q)$, i.e., $C_\phi : \mathcal{B}^\alpha \to Q_{K,0}(p,q)$ is bounded.

(2)⇒(5). It is easy.

(5)⇒(6). Since the identical mapping belongs to $\mathcal{B}_0^\alpha$, $\phi \in Q_{K,0}(p,q)$. Since $C_\phi : \mathcal{B}_0^\alpha \to Q_{K,0}(p,q)$ is compact, $C_\phi : \mathcal{B}_0^\alpha \to Q_K(p,q)$ is compact. By Lemma 7, it is easy to see that (3.1) holds, and the proof is completed.

Corollary 4[20] *Let $0 < p < \infty$, and ϕ be an analytic self-map of D. Then the following statements are equivalent:*

(1) $C_\phi : \mathcal{B} \to Q_p$ *is bounded;*

(2) $\frac{|\phi'(z)|^2}{(1-|\phi(z)|^2)^2}(1-|z|^2)^p\,\mathrm{d}A(z)$ *is a bounded p-Carlesom measure.*

Corollary 5[20] *Let $0 < p < \infty$, and ϕ be an analytic self-map of D. Then the following statements are equivalent:*

(1) $C_\phi : \mathcal{B} \to Q_{p,0}$ *is bounded;*

(2) $C_\phi : \mathcal{B} \to Q_{p,0}$ *is compact;*

(3) $\lim_{|a|\to 1} \int_D \frac{|\phi'(z)|^2}{(1-|\phi(z)|^2)^2} g^p(z,a)\,\mathrm{d}A(z) = 0$*;*

(4) $\frac{|\phi'(z)|^2}{(1-|\phi(z)|^2)^2}(1-|z|^2)^p\,\mathrm{d}A(z)$ *is a compact p-Carlesom measure.*

Corollary 6[21] *Let $0 < p, s < \infty$, $-2 < q < \infty$, and ϕ be an analytic self-map of D. Then the following statements are equivalent:*

(1) $C_\phi : \mathcal{B}^\alpha \to F(p,q,s)$ *is bounded;*

(2) $\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^{q+s}\,\mathrm{d}A(z)$ *is a bounded s-Carlesom measure.*

Corollary 7[21] *Let $0 < s < \infty$, $p \ge 1$, $-2 < q < \infty$, and ϕ be an analytic self-map of D. Then the following statements are equivalent:*

(1) $C_\phi : \mathcal{B}^\alpha \to F_0(p,q,s)$ *is bounded;*

(2) $C_\phi : \mathcal{B}^\alpha \to F_0(p,q,s)$ *is compact;*

(3) $\lim_{|a|\to 1} \int_D \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^q g^s(z,a)\,\mathrm{d}A(z) = 0$*;*

(4) $\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}(1-|z|^2)^{q+s}\,\mathrm{d}A(z)$ *is a compact s-Carlesom measure.*

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References

[1] WULAN Hasi, ZHOU Jizhen. *Q_K type spaces of analytic functions* [J]. J. Funct. Spaces Appl., 2006, **4**(1): 73–84.

[2] ESSÉN M, WULAN Hasi. *On analytic and meromorphic functions and spaces of Q_K-type* [J]. Illinois J. Math., 2002, **46**(4): 1233–1258.

[3] WULAN Hasi. *Möbius invariant Q_p spaces: results, techniques and questions* [J]. Adv. Math. (China), 2005, **34**(4): 385–404.

[4] WULAN Hasi. *Compactness of composition operators from the Bloch space $\mathcal{B}$ to Q_K spaces* [J]. Acta Math. Sin. (Engl. Ser.), 2005, **21**(6): 1415–1424.

[5] ZHAO Ruhan. *On a general family of function spaces* [D]. Ann Acad Sci Fenn Diss 1996, 105: 56.

[6] AULASKARI R, XIAO Jie, ZHAO Ruhan. *On subspaces and subsets of BMOA and UBC* [J]. Analysis, 1995, **15**(2): 101–121.

[7] ESSÉN M, WULAN Hasi, XIAO Jie. *Several function-theoretic characterizations of Möbius invariant $\mathcal{Q}_K$ spaces* [J]. J. Funct. Anal., 2006, **230**(1): 78–115.

[8] MACCLUER B, SHAPIRO J. *Angular derivatives and compact composition operators on the Hardy and Bergman spaces* [J]. Canada. J. Math., 1986, **38**(4): 878–906.

[9] LOU Zengjian. *Composition operators on Bloch type space* [J]. Analysis, 2003, **23**(1): 81–95.

[10] ZHANG Xuejun. *Composition operators and weighted composition operators on p-Bloch spaces* [J]. Chinese Ann. Math. Ser. A, 2003, **24**(6): 711–720. (in Chinese)

[11] ZHANG Xuejun, XIAO Jianbin. *Weighted composition operators between μ-Bloch spaces on the unit ball* [J]. Sci. China Ser. A, 2005, **48**(10): 1349–1368.

[12] WU Pengcheng, WULAN Hasi. *Composition operators from the Bloch space into the spaces Q_T* [J]. Int. J. Math. Math. Sci., 2003, **31**: 1973–1979.

[13] LI Songxiao, WULAN Hasi. *Composition operators on Q_K spaces* [J]. J. Math. Anal. Appl., 2007, **327**(2): 948–958.

[14] KOTILAINEN M. *On composition operators in Q_K type spaces* [J]. J. Funct. Spaces Appl., 2007, **5**(2): 103–122.

[15] YU Yanyan, LIU Yongmin. *Composition operators from hyperbolic α-Bloch spaces into hyperbolic Q_K type spaces* [J]. Acta Math. Sci. Ser. A Chin. Ed., 2009, **29**(5): 1311–1320. (in Chinese)

[16] PAU J. *Bounded Möbius Invariant Q_K spaces* [J]. J. Math. Anal. Appl., 2008, **338**(2): 1029–1042.

[17] WULAN Hasi, ZHU Kehe. *Derivative-free characterizations of Q_K spaces* [J]. J. Aust. Math. Soc., 2007, **82**(2): 283–295.

[18] WULAN Hasi, ZHOU Jizhen. *The higher order derivatives of Q_K type spaces* [J]. J. Math. Anal. Appl., 2007, **332**(2): 1216–1228.

[19] XIAO Jie. *Holomorphic Q Classes* [M]. Springer-Verlag, Berlin, 2001.

[20] SMITH W, ZHAO Ruhan. *Composition operators mapping into the Q_p spaces* [J]. Analysis, 1997, **17**(2-3): 239–263.

[21] JIANG Lijian, HE Yuzan. *Composition operators from $\mathcal{B}^\alpha$ to $F(p, q, s)$* [J]. Acta Math. Sci. Ser. B Engl. Ed., 2003, **23**(2): 252–260.