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Residual Cusum Test for Parameters Change in ARCH Errors Models with Deterministic Trend

JIN Hao[1], *TIAN Zheng*[1,2]
(1. Department of Applied Mathematics, Northwestern Polytechnical University, Shaanxi 710072, China;
2. State Key Laboratory of Remote Sensing Science, Chinese Academy of Science, Beijing 100101, China)
(E-mail: jinhao_2004@126.com)

Abstract This paper analyzes the problem of testing for parameters change in ARCH errors models with deterministic trend based on residual cusum test. It is shown that the asymptotically limiting distribution of the residual cusum test statistic is still the sup of a standard Brownian bridge under null hypothesis. In order to check this, we carry out a Monte Carlo simulation and examine the return of IBM data. The results from both simulation and real data analysis support our claim. We also can explain this phenomenon from a theoretical viewpoint that the variance in ARCH model in mainly determined by its parameters.

Keywords residual cusum test; invariance principle; Brownian bridge; change point.

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1. Introduction

Testing for structural breaks is an often important step in an anlysis of a stationary time series because a myriad of political and economic factors can cause the relationships among economic variables to change over time. Since the early work of Chow[1] and Quant[2], numerous studies have been undertaken with an upsurge of interest in various models with an unknown change point. With respect to the problem of testing for structural breaks, recent contributions include Krishnaiah and Miao[3], Bhattcharya[4], Andrews[5,6] as well as the monograph by Csorgo and Horvath[7]. Issues about the distributional properties of the estimates, in particular those of break date, have also been considered by Bai[8]. These test and inference issues have been addressed in the context of multiple structural breaks by Bai and Perron[9,10].

Most of the work in statistic and econometric literatures are concentrated on the case where the regressors and the errors are stationary. Debates related to structural breaks are also important in context of deterministic trend regression models following the work of Perron[11]. In this paper, we draw our attention to test for ARCH errors models with deterministic trend.

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The ARCH processes have long been popular in financial time series analysis. The literature on the subject is so vast that we have to refer the reader to the fairly comprehensive work of Engle[12], Kokoszka[13] and Hall[14]. The main feature of ARCH processes is that the sequences are uncorrelated, but the square of the sequences has a rich dependence structure.

Despite the importance of testing for structural changes, the problem of detecting parameter changes has not received as much attention as the context of ARCH models as in the setting of linear time series models. Kim, Cho and Lee[15] applied the cusum test to ARCH models taking account of the fact that the variance is functional of ARCH parameters, and their change can be detected by examining the existence of the variance change. Although this reasoning was correct, it turned out that the cusum test generates spurious rejection of the null hypothesis and produces low power. Hence, in order to overcome such drawbacks, Lee, Tokutsu and Meakawa[16] proposed to use the cusum test based on the residuals, given as the squares of observations divided by estimated conditional variances. Since the residual based test conventionally discards correlation effects and enhances the performance of the test, their models do not contain the case of the deterministic trend regressors taking place at a change point. The aim of this paper is to fill this gap by analyzing the limiting distribution of the residual cusum test in models where the deterministic trend regressors exhibit a slope change at some unknown date.

These and other issues will be addressed in this paper whose structure is as follows. Section 2 first describes the models considered, the assumptions made on the various components and how the residual cusum test is obtained. Section 3 presents the empirical results for a Monte Carlo simulation and the return of IBM data. Finally, Section 4 presents brief concluding remarks.

2. The models and residual cusum test

We consider the following model

$$\begin{aligned} y_t &= \mu + \beta t + \eta_t, \\ \eta_t &= h_t \varepsilon_t, \\ h_t^2 &= a + \sum_{j=1}^{\infty} p_j \eta_{t-j}^2, \end{aligned} \tag{1}$$

where $a \geq 0$, $p_j \geq 0$, $\sum_{j=1}^{\infty} p_j < \infty$ and ε_t are independent identical distribution random variables with zero mean and unit variance. We assume that $y_s, s < t$ are independent of ε_u, $u \geq t$ and $\{(\eta_t, h_t)\}$ is strong mixing. The objective here is to test the following hypothesis,

(H_0) $\theta = (\mu, \beta, a, p_1, p_2, \ldots)$ remains the same for the whole series,

(H_1) Not (H_0).

We state the assumptions which we need to prove asymptotic validity of our approach:

(A1) $E|\eta_t|^{4+\delta} < \infty$ and $E|\varepsilon_t|^{4+\delta} < \infty$, for some $\delta > 0$,

(A2) There exists a sequence of positive integers with $q \to \infty$, $q/\sqrt{T} \to 0$, and $\sum_{j=q+1}^{\infty} p_j \to 0$, as $T \to \infty$,

(A3) $\{(\eta_t, h_t)\}$ is strong mixing.

Now we can construct the residual cusum test based on $\hat{\varepsilon}_t$. In analysis of h_t^2 and η_t^2, we define that

$$\hat{h}_t^2 = \hat{a} + \sum_{j=1}^{q} \hat{p}_j \hat{\eta}_{t-j}^2,$$
$$\hat{\eta}_t = y_t - \hat{\mu} - \hat{\beta} t,$$
$$\hat{\varepsilon}_t = \hat{\eta}_t / \hat{h}_t, \qquad (2)$$

where $\hat{\mu} - \mu = O_p(T^{-1/2})$, $\hat{a} - a = O_p(T^{-1/2})$, $\hat{p}_j - p_j = O_p(T^{-1/2})$ and $\hat{\beta} - \beta = O_p(T^{-3/2})$, and have the following result.

Theorem 1 *Assume that (A1)–(A3) hold. Let*

$$\hat{R}_T = \frac{1}{\sqrt{T}\hat{\tau}} \max_{1\le k\le T} \Big| \sum_{t=1}^{k} \hat{\varepsilon}_t^2 - \frac{k}{T} \sum_{t=1}^{T} \hat{\varepsilon}_t^2 \Big|,$$

where $\hat{\tau}^2 = \frac{1}{T}\sum_{t=1}^{T} \hat{\varepsilon}_t^4 - (\frac{1}{T}\sum_{t=1}^{T} \hat{\varepsilon}_t^2)^2$. *Then, under* (H_0),

$$\hat{R}_T \xrightarrow{P} \sup_{0\le v\le 1} |W^0(v)|, \quad T\to\infty,$$

where $v = k/T$.

Remark Lee et al.[16] considered the regression models with GARCH(1,1) errors. However, the proof in that paper should not all be copied directly to the deterministic trend regression situation. In contract to [16], we have simplified the assumptions for (A2). In what follows we will prove the asymptotical distribution of the residual cusum test is the sup of a standard Brownian bridge.

Proof Split $\hat{\varepsilon}_T^2$ into $\varepsilon_T^2 + \sum_{i=1}^{7} V_{i,t}$, where

$$\begin{aligned}
\hat{\varepsilon}_t^2 = \frac{\hat{\eta}_t^2}{\hat{h}_t^2} &= \frac{(y_t - \hat{\mu} - \hat{\beta} t)^2}{\hat{h}_t^2} \\
&= \frac{\eta_t^2}{\hat{h}_t^2} + \frac{1}{\hat{h}_t^2}\Big[2\eta_t(\hat{\beta}-\beta)t + 2\eta_t(\hat{\mu}-\mu) + 2(\hat{\mu}-\mu)(\hat{\beta}-\beta)t + (\hat{\mu}-\mu)^2 + t^2(\hat{\beta}-\beta)^2\Big] \\
&= \frac{\eta_t^2}{\hat{h}_t^2} + \sum_{i=3}^{7} V_{i,T},
\end{aligned}$$

$$\frac{\eta_t^2}{\hat{h}_t^2} = \frac{\eta_t^2}{h_t^2} + \frac{(h_t^2 - \hat{h}_t^2)\varepsilon_t^2}{h_t^2} + \frac{(h_t^2 - \hat{h}_t^2)^2 \varepsilon_t^2}{h_t^2 \hat{h}_t^2} = \varepsilon_t^2 + \sum_{i=1}^{2} V_{i,T}.$$

We want to claim that

$$R_{i,T} := \frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{t=1}^{k} V_{i,T} - \frac{k}{T} \sum_{t=1}^{T} V_{i,T} \Big| = o_p(1), \quad i = 1, \ldots, 7. \qquad (3)$$

First, we handle with $V_{1,T}$. Note that

$$h_t^2 - \hat{h}_t^2 = (a - \hat{a}) + \sum_{j=q+1}^{\infty} p_j \eta_{t-j}^2 + \sum_{j=1}^{q} (p_j - \hat{p}_j)\eta_{t-j}^2 + \sum_{j=1}^{q} \hat{p}_j (\eta_{t-j}^2 - \hat{\eta}_{t-j}^2)$$

$$= \sum_{i=1}^{4} W_{i,T}.$$

Owing to (A3) and the invariance principle for strong mixing process (cf. Theorem 1.7 of Peligrad[17]), we have

$$\frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{t=1}^{k} \frac{\varepsilon_t^2}{h_t^2} - \frac{k}{T} \sum_{t=1}^{T} \frac{\varepsilon_t^2}{h_t^2} \Big| = O_p(1),$$

$$\frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{t=1}^{k} \frac{\varepsilon_t^2 \eta_{t-j}^2}{h_t^2} - \frac{k}{T} \sum_{t=1}^{T} \frac{\varepsilon_t^2 \eta_{t-j}^2}{h_t^2} \Big| = O_p(1)$$

which implies

$$\begin{aligned} &\frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{t=1}^{k} \frac{\varepsilon_t^2 W_{1,T}}{h_t^2} - \frac{k}{T} \sum_{t=1}^{T} \frac{\varepsilon_t^2 W_{1,T}}{h_t^2} \Big| \\ &= |(a-\hat{a})| \frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{t=1}^{k} \frac{\varepsilon_t^2}{h_t^2} - \frac{k}{T} \sum_{t=1}^{T} \frac{\varepsilon_t^2}{h_t^2} \Big| \\ &= O_p(T^{-1/2}) \cdot O_p(1) = o_p(1), \end{aligned} \tag{4}$$

and

$$\begin{aligned} &\frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{t=1}^{k} \frac{\varepsilon_t^2 W_{2,T}}{h_t^2} - \frac{k}{T} \sum_{t=1}^{T} \frac{\varepsilon_t^2 W_{2,T}}{h_t^2} \Big| \\ &= \frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{j=q+1}^{\infty} p_j \Big(\sum_{t=1}^{k} \frac{\varepsilon_t^2 \eta_{t-j}^2}{h_t^2} - \frac{k}{T} \sum_{t=1}^{T} \frac{\varepsilon_t^2 \eta_{t-j}^2}{h_t^2} \Big) \Big| \\ &= O_p(1) \cdot O_p\Big(\sum_{j=q+1}^{\infty} p_j \Big) = o_p(1). \end{aligned} \tag{5}$$

In term of $R_{1,T}$, $W_{3,T}$ and $W_{4,T}$ remain to be proved. Noting that $p_j - \hat{p}_j = O_p(T^{-1/2})$ and (A2), we can get

$$\begin{aligned} &\frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{t=1}^{k} \frac{W_{3,T} \varepsilon_t^2}{h_t^2} - \Big(\frac{k}{T}\Big) \sum_{t=1}^{T} \frac{W_{3,T} \varepsilon_t^2}{h_t^2} \Big| \\ &= \frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{j=1}^{q} (p_j - \hat{p}_j) \Big(\sum_{t=1}^{k} \frac{\varepsilon_t^2 \eta_{t-j}^2}{h_t^2} - \frac{k}{T} \sum_{t=1}^{T} \frac{\varepsilon_t^2 \eta_{t-j}^2}{h_t^2} \Big) \Big| \\ &= O_p\Big(\sum_{j=1}^{q} (p_j - \hat{p}_j) \Big) = O_p(q/\sqrt{T}) = o_p(1). \end{aligned} \tag{6}$$

Now, we verify that

$$\frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{t=1}^{k} \frac{W_{4,T} \varepsilon_t^2}{h_t^2} - \frac{k}{T} \sum_{t=1}^{T} \frac{W_{4,T} \varepsilon_t^2}{h_t^2} \Big| = o_p(1). \tag{7}$$

Note that

$$\hat{p}_j [\eta_{t-j}^2 - \hat{\eta}_{t-j}^2]$$

$$= \hat{p}_j[2(\hat{\mu}-\mu)\eta_{t-j} + 2(\hat{\beta}-\beta)t\eta_{t-j} - 2(\hat{\mu}-\mu)(\hat{\beta}-\beta)t - (\hat{\mu}-\mu)^2 - (\hat{\beta}-\beta)^2 t^2]$$
$$= \sum_{i=1}^{5} Q_{i,T}.$$

It suffices to prove that

$$\frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{j=1}^{q}\sum_{t=1}^{k} \frac{Q_{i,T}\varepsilon_t^2}{h_t^2} - \frac{k}{T}\sum_{j=1}^{q}\sum_{t=1}^{T} \frac{Q_{i,T}\varepsilon_t^2}{h_t^2} \Big| = o_p(1), \quad i = 1, \dots, 5. \tag{8}$$

Since by (A3)

$$\frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{t=1}^{k} \frac{\eta_{t-j}\varepsilon_t^2}{h_t^2} - \frac{k}{T}\sum_{t=1}^{T} \frac{\eta_{t-j}\varepsilon_t^2}{h_t^2} \Big| = O_p(1),$$

$$\frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{t=1}^{k} \frac{t\eta_{t-j}\varepsilon_t^2}{h_t^2} - \frac{k}{T}\sum_{t=1}^{T} \frac{t\eta_{t-j}\varepsilon_t^2}{h_t^2} \Big| = O_p(T),$$

and $\hat{\beta} - \beta = O_p(T^{-3/2})$ which implies

$$\frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{j=1}^{q}\sum_{t=1}^{k} \frac{Q_{1,T}\varepsilon_t^2}{h_t^2} - \frac{k}{T}\sum_{j=1}^{q}\sum_{t=1}^{T} \frac{Q_{1,T}\varepsilon_t^2}{h_t^2} \Big|$$
$$= \frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{j=1}^{q} \hat{p}_j(\hat{\mu}-\mu)\Big(\sum_{t=1}^{k} \frac{\eta_{t-j}\varepsilon_t^2}{h_t^2} - \frac{k}{T}\sum_{t=1}^{T} \frac{\eta_{t-j}\varepsilon_t^2}{h_t^2} \Big) \Big|$$
$$= O_p\Big(T^{-1/2}\sum_{j=1}^{q}\hat{p}_j\Big) = O_p\Big(T^{-1/2}\sum_{j=1}^{q}p_j\Big) + o_p(1) = o_p(1), \tag{9}$$

and

$$\frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{j=1}^{q}\sum_{t=1}^{k} \frac{Q_{2,T}\varepsilon_t^2}{h_t^2} - \frac{k}{T}\sum_{j=1}^{q}\sum_{t=1}^{T} \frac{Q_{2,T}\varepsilon_t^2}{h_t^2} \Big|$$
$$= \frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{j=1}^{q} \hat{p}_j(\hat{\beta}-\beta)\Big(\sum_{t=1}^{k} \frac{t\eta_{t-j}\varepsilon_t^2}{h_t^2} - \frac{k}{T}\sum_{t=1}^{T} \frac{t\eta_{t-j}\varepsilon_t^2}{h_t^2} \Big) \Big|$$
$$= O_p\Big(T^{-1/2}\sum_{j=1}^{q}\hat{p}_j\Big) = O_p\Big(T^{-1/2}\sum_{j=1}^{q}p_j\Big) + o_p(1) = o_p(1). \tag{10}$$

The proofs of $Q_{3,T}, Q_{4,T}, Q_{5,T}$ are essentially the same as those of $Q_{1,T}$ and $Q_{2,T}$, and are omitted for brevity. Hence, we have proved that $R_{1,T} = o_p(1)$.

Now we deal with $R_{2,T}$. Since $h_t^2 - \hat{h}_t^2 = \sum_{i=1}^{4} W_{i,T}$, $\hat{h}_t^2 \ge \hat{a}$ and $h_t^2 \ge a$, we want first to prove $W_{2,T}^2$ satisfies

$$\frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| \sum_{t=1}^{k} \frac{W_{2,T}^2\varepsilon_t^2}{h_t^2\hat{h}_t^2} - \frac{k}{T}\sum_{t=1}^{T} \frac{W_{2,T}^2\varepsilon_t^2}{h_t^2\hat{h}_t^2} \Big|$$
$$= \frac{1}{\sqrt{T}} \max_{1\le k\le T} \Big| (\sum_{j=q+1}^{T} p_j)^2 \Big(\sum_{t=1}^{k} \frac{\varepsilon_t^2\eta_{t-j}^4}{h_t^2\hat{h}_t^2} - \frac{k}{T}\sum_{t=1}^{T} \frac{\varepsilon_t^2\eta_{t-j}^4}{h_t^2\hat{h}_t^2} \Big) \Big|$$

$$\leq \frac{1}{\sqrt{T}\hat{a}} \max_{1\leq k\leq T} \Big|(\sum_{j=q+1}^{T} p_j)^2 \max_{1\leq t\leq T} \eta_t^2 \Big(\sum_{t=1}^{k} \frac{\varepsilon_t^2 \eta_{t-j}^2}{h_t^2} - \frac{k}{T}\sum_{t=1}^{T} \frac{\varepsilon_t^2 \eta_{t-j}^2}{h_t^2}\Big)\Big| = o_p(1). \tag{11}$$

It is obvious that

$$\begin{aligned} &\frac{1}{\sqrt{T}} \max_{1\leq k\leq T} \Big| \sum_{t=1}^{k} \frac{W_{3,T}^2 \varepsilon_t^2}{h_t^2 \hat{h}_t^2} - \frac{k}{T}\sum_{t=1}^{T} \frac{W_{3,T}^2 \varepsilon_t^2}{h_t^2 \hat{h}_t^2} \Big| \\ &= \frac{1}{\sqrt{T}} \max_{1\leq k\leq T} \Big|(\sum_{j=1}^{q} p_j - \hat{p}_j)^2 \Big(\sum_{t=1}^{k} \frac{\varepsilon_t^2 \eta_{t-j}^4}{h_t^2 \hat{h}_t^2} - \frac{k}{T}\sum_{t=1}^{T} \frac{\varepsilon_t^2 \eta_{t-j}^4}{h_t^2 \hat{h}_t^2}\Big)\Big| \\ &\leq \frac{1}{\sqrt{T}\hat{a}} \max_{1\leq k\leq T} \Big|(\sum_{j=1}^{q} p_j - \hat{p}_j)^2 \max_{1\leq t\leq T} \eta_t^2 \Big(\sum_{t=1}^{k} \frac{\varepsilon_t^2 \eta_{t-j}^2}{h_t^2} - \frac{k}{T}\sum_{t=1}^{T} \frac{\varepsilon_t^2 \eta_{t-j}^2}{h_t^2}\Big)\Big| = o_p(1). \end{aligned} \tag{12}$$

Since

$$\frac{1}{\sqrt{T}}\Big|\sum_{t=1}^{k} W_{i,T}^2 \varepsilon_t^2 - \frac{k}{T}\sum_{t=1}^{T} W_{i,T}^2 \varepsilon_t^2\Big| \leq \frac{2}{\sqrt{T}}\sum_{t=1}^{T} W_{i,T}^2 \varepsilon_t^2,$$

to show $R_{2,T} = o_p(1)$ is equivalent to showing

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} W_{i,T}^2 \varepsilon_t^2 = o_p(1), \quad i = 1, 4. \tag{13}$$

Owing to $a - \hat{a} = O_p(T^{-1/2})$, it is easy to see that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} W_{1,T}^2 \varepsilon_t^2 = \frac{1}{\sqrt{T}}\sum_{t=1}^{T} (a-\hat{a})^2 \varepsilon_t^2 = o_p(1). \tag{14}$$

We just only need to prove

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} W_{4,t}^2 \varepsilon_t^2 = \frac{1}{\sqrt{T}}\sum_{t=1}^{T} \Big(\sum_{j=1}^{q} \hat{p}_j(\eta_t^2 - \hat{\eta}_t^2)\Big)^2 \varepsilon_t^2 = o_p(1). \tag{15}$$

Because of $\hat{p}_j(\eta_{t-j}^2 - \hat{\eta}_{t-j}^2) = \sum_{i=1}^{5} Q_{i,T}$, it suffices to prove

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} \Big(\sum_{j=1}^{q} Q_{i,T}\Big)^2 \varepsilon_t^2 = o_p(1), \quad i = 1, \ldots, 5. \tag{16}$$

To deal with $Q_{1,T}$, we have

$$\begin{aligned} \frac{1}{\sqrt{T}}\sum_{t=1}^{T} \Big(\sum_{j=1}^{q} Q_{1,T}\Big)^2 \varepsilon_t^2 &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T} \Big(\sum_{j=1}^{q} \hat{p}_j \eta_{t-j}(\hat{\mu} - \mu)\Big)^2 \varepsilon_t^2 \\ &= O_p\Big(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} T^{-1}(\sum_{j=1}^{q} \hat{p}_j)^2\Big) \\ &= O_p\Big(\frac{1}{\sqrt{T}}(\sum_{j=1}^{q} p_j)^2\Big) + o_p(1) = o_p(1). \end{aligned} \tag{17}$$

By $\beta - \hat{\beta} = O_P(T^{-3/2})$, we can get the same result for $Q_{2,T}$ that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} \Big(\sum_{j=1}^{q} Q_{2,T}\Big)^2 \varepsilon_t^2 = \frac{1}{\sqrt{T}}\sum_{t=1}^{T} \Big(\sum_{j=1}^{q} \hat{p}_j \eta_{t-j}(\hat{\beta} - \beta)t\Big)^2 \varepsilon_t^2$$

$$= O_p\Big(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} t^2 T^{-3}(\sum_{j=1}^{q}\hat{p}_j)^2\Big)$$

$$= O_p\Big(\frac{1}{\sqrt{T}}(\sum_{j=1}^{q}p_j)^2\Big) + o_p(1) = o_p(1). \quad (18)$$

The proofs of $Q_{3,T}, Q_{4,T}, Q_{5,T}$ are essentially the same as those of $Q_{1,T}$ and $Q_{2,T}$, and are omitted for brevity again. Hence, we have proved that $R_{2,T} = o_p(1)$. By the way, we also can get the analogous results for $W_{2,T}$ and $W_{3,T}$ that

$$\frac{1}{T}\sum_{t=1}^{T} W_{i,T}^2\varepsilon_t^2 = o_p(1), \quad i = 2, 3. \quad (19)$$

Together with (13) and (19), we can obtain

$$\frac{1}{T}\sum_{t=1}^{T}(h_t^2 - \hat{h}_t^2)^2\varepsilon_t^2 = o_p(1). \quad (20)$$

Now it remains to show $R_{i,T} = o_p(1)$, $i = 3, \ldots, 7$. It is trivial to show that $R_{6,T} = o_p(1)$. Also, one can verify the negligibility of $R_{i,T} = o_p(1)$, $i = 3, 4, 5, 7$, in a similar fashion to prove that of $R_{1,T} = o_p(1)$ and $R_{2,T} = o_p(1)$, respectively. Hence (3) is established, which directly implies that

$$\frac{1}{\sqrt{T}}\max_{1\le k\le T}\Big|\sum_{t=1}^{k}\hat{\varepsilon}_t^2 - \frac{k}{T}\sum_{t=1}^{T}\hat{\varepsilon}_t^2\Big| = \frac{1}{\sqrt{T}}\max_{1\le k\le T}\Big|\sum_{t=1}^{k}\varepsilon_t^2 - \frac{k}{T}\sum_{t=1}^{T}\varepsilon_t^2\Big| + o_p(1). \quad (21)$$

Since $\frac{1}{\sqrt{T}\tau}\sum_{t=1}^{k}\Big(\varepsilon_t^2 - E\varepsilon_t^2\Big) \xrightarrow{d} W(v)$, $\frac{k}{T}\cdot\frac{1}{\sqrt{T}\tau}\sum_{t=1}^{T}\Big(\varepsilon_t^2 - E\varepsilon_t^2\Big) \xrightarrow{d} vW(1)$, by the CMT (Continuous Mapping Theorem), we can prove

$$\begin{aligned} R_T &= \frac{1}{\sqrt{T}\tau}\max_{1\le k\le T}\Big|\sum_{t=1}^{k}\varepsilon_t^2 - \frac{k}{T}\sum_{t=1}^{T}\varepsilon_t^2\Big| \\ &= \frac{1}{\sqrt{T}\tau}\max_{1\le k\le T}\Big|\sum_{t=1}^{k}\Big(\varepsilon_t^2 - E\varepsilon_t^2\Big) - \frac{k}{T}\sum_{t=1}^{T}\Big(\varepsilon_t^2 - E\varepsilon_t^2\Big)\Big| \\ &\xrightarrow{d} \sup_{0\le v\le 1}\Big|W^0(v)\Big|. \end{aligned} \quad (22)$$

Finally, we show that $\hat{\tau}^2 \xrightarrow{P} \tau^2 = Var(\varepsilon_1^2) = E\varepsilon_1^4 - (E\varepsilon_1^2)^2$. Note that

$$\hat{\varepsilon}_t^2 - \varepsilon_t^2 = \frac{(h_t^2 - \hat{h}_t^2)\varepsilon_t^2}{\hat{h}_t^2} + \frac{\hat{\eta}_t^2 - \eta_t^2}{\hat{h}_t^2}. \quad (23)$$

According to what has been proved above, we know $\frac{\hat{\eta}_t^2 - \eta_t^2}{\hat{h}_t^2}$ satisfies

$$\frac{1}{T}\sum_{t=1}^{T}\frac{\hat{\eta}_t^2 - \eta_t^2}{\hat{h}_t^2} = o_p(1) \quad \text{and} \quad \frac{1}{T}\sum_{t=1}^{T}\Big(\frac{\hat{\eta}_t^2 - \eta_t^2}{\hat{h}_t^2}\Big)^2 = o_p(1). \quad (24)$$

Thus in view of (23) and (24)

$$\Big|\frac{1}{T}\sum_{t=1}^{T}(\hat{\varepsilon}_t^2 - \varepsilon_t^2)\Big| \le \Big|\frac{1}{T\hat{a}}\sum_{t=1}^{T}(h_t^2 - \hat{h}_t^2)\varepsilon_t^2\Big| + o_p(1)$$

$$\leq \frac{1}{\hat{a}}\Big(\frac{1}{T}\sum_{t=1}^{T}(h_t^2-\hat{h}_t^2)^2\varepsilon_t^2\Big)^{1/2}\cdot\Big(\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t^2\Big)^{1/2}, \tag{25}$$

which is $o_p(1)$ due to (20). Hence

$$\frac{1}{T}\sum_{t=1}^{T}\hat{\varepsilon}_t^2 \stackrel{P}{\longrightarrow} E\varepsilon_1^2. \tag{26}$$

Now by (24)

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}(\hat{\varepsilon}_t^2-\varepsilon_t^2)^2 &= \frac{1}{T\hat{a}^2}\sum_{t=1}^{T}(\hat{h}_t^2-h_t^2)^2\varepsilon_t^4+o_p(1)\\ &\leq \frac{1}{\hat{a}^2}\Big(\max_{1\leq t\leq T}\varepsilon_t^2\Big)\cdot\Big(\frac{1}{T}\sum_{t=1}^{T}(h_t^2-\hat{h}_t^2)^2\varepsilon_t^2\Big)+o_p(1)\\ &= o_p(1),\end{aligned} \tag{27}$$

and furthermore

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}(\hat{\varepsilon}_t^2+\varepsilon_t^2)^2 &= \frac{1}{T}\sum_{t=1}^{T}(\hat{\varepsilon}_t^2-\varepsilon_t^2)^2+\frac{4}{T}\sum_{t=1}^{T}\varepsilon_t^2\hat{\varepsilon}_t^2\\ &\leq \frac{2}{T}\sum_{t=1}^{T}(\hat{\varepsilon}_t^2-\varepsilon_t^2)^2+\frac{8}{T}\sum_{t=1}^{T}\varepsilon_t^4=O_p(1).\end{aligned} \tag{28}$$

Hence

$$\begin{aligned}\Big|\frac{1}{T}\sum_{t=1}^{T}\hat{\varepsilon}_t^4-\frac{1}{T}\sum_{t=1}^{T}\varepsilon_t^4\Big| &\leq \Big(\frac{1}{T}\sum_{t=1}^{T}(\hat{\varepsilon}_t^2-\varepsilon_t^2)^2\Big)^{1/2}\Big(\frac{1}{T}\sum_{t=1}^{T}(\hat{\varepsilon}_t^2+\varepsilon_t^2)^2\Big)^{1/2}\\ &= o_p(1)\cdot O_p(1)=o_p(1).\end{aligned} \tag{29}$$

We can obtain

$$\frac{1}{T}\sum_{t=1}^{T}\hat{\varepsilon}_t^4 \stackrel{P}{\rightarrow} E\varepsilon_1^4. \tag{30}$$

This together with (26) yields $\hat{\tau}^2 \stackrel{P}{\rightarrow} \tau^2$. Therefore, in view of this, (21) and (22), we have completed the proof of the theorem. □

3. Simulation and a data example

In the section, we evaluate the performance of the test statistic $\hat{R}_T$ through a simulation study. In particular, the result is compared with Lee et al[16]. In this simulation, we use Monte Carlo methods to investigate the finite sample size and power properties of the residual cusum tests. Let ε_t be independent identically distributed standard normal random variables. The tests require knowledge of the true breakpoint, and experiments are programmed using 1000 replications. All results refer to tests run at the 0.05 nominal asymptotic level, for samples of size $T=500, 1000, 2000$.

In order to check the performance of $\hat{R}_T$, we consider the models

$$y_t=\mu+\beta t+\eta_t,$$

$$\eta_t = h_t \varepsilon_t,$$
$$h_t^2 = a + p_1 \eta_{t-1}^2 + p_2 \eta_{t-2}^2$$

and y_0 can be assumed to be 0 without loss of generality. The application of our residuals method also depends on a choice of the block size q; the problem is very similar to the choice of the bandwidth in applying smoothing or kernel methods. For sample size used in our simulation a simple practical recommendation for the test is to use the block size q approximately equal to $q = [\log T]^2$ (in bracket) and 10% of the sample size T (out bracket).

Now we consider the problem of test under the following hypothesis: the parameters change from $\theta = (\mu, \beta, a, p_1, p_2)$ to $\theta^* = (\mu^*, \beta^*, a^*, p_1^*, p_2^*)$ at $T/2$.

θ^*	$n = 500$	$n = 1000$	$n = 2000$
$(0.02, 0.28, 0.4, 0.3, 0.3)$	0.037(0.032)	0.041(0.035)	0.052(0.048)
$(0.03, 0.38, 0.4, 0.3, 0.3)$	0.381(0.334)	0.692(0.715)	0.857(0.864)
$(0.04, 0.48, 0.4, 0.3, 0.3)$	0.417(0.456)	0.733(0.704)	0.882(0.889)

Table 3.1 Experimental size and power, $\theta = (0.02, 0.28, 0.4, 0.3, 0.3)$

θ^*	$n = 500$	$n = 1000$	$n = 2000$
$(0.02, 0.28, 0.4, 0.3, 0.3)$	0.033(0.038)	0.036(0.031)	0.043(0.047)
$(0.02, 0.28, 0.5, 0.1, 0.4)$	0.481(0.517)	0.742(0.764)	0.907(0.901)
$(0.02, 0.28, 0.3, 0.5, 0.2)$	0.557(0.588)	0.843(0.829)	0.942(0.938)

Table 3.2 Experimental size and power, $\theta = (0.02, 0.28, 0.4, 0.3, 0.3)$

θ^*	$n = 500$	$n = 1000$	$n = 2000$
$(0.02, 0.28, 0.4, 0.3, 0.3)$	0.034(0.029)	0.038(0.032)	0.049(0.049)
$(0.03, 0.38, 0.4, 0.1, 0.5)$	0.494(0.534)	0.751(0.732)	0.917(0.920)
$(0.04, 0.48, 0.4, 0.5, 0.1)$	0.596(0.563)	0.859(0.874)	0.958(0.963)

Table 3.3 Experimental size and power, $\theta = (0.02, 0.28, 0.4, 0.3, 0.3)$

θ^*	$n = 500$	$n = 1000$	$n = 2000$
$(0.02, 0.28, 0.4, 0.3, 0.3)$	0.039(0.035)	0.042(0.036)	0.060(0.057)
$(0.03, 0.38, 0.3, 0.2, 0.5)$	0.687(0.656)	0.892(0.876)	0.957(0.953)
$(0.04, 0.48, 0.5, 0.4, 0.1)$	0.657(0.625)	0.903(0.881)	0.982(0.977)

Table 3.4 Experimental size and power, $\theta = (0.02, 0.28, 0.4, 0.3, 0.3)$

As would be expected, we can see that our tests have no size distortions, and the empirical size and power properties summarized in Table 3.1-3.4 are improved as sample size is increased. The sizes of all the tests are close to the asymptotic 0.05 level with impressive power properties. It is interesting to note that the more the parameters a, p_1 and p_2 are changed , the higher is the power and the greater the probability of the sample containing a change point. However, the power seems to be slightly less reliable in terms of other parameters, such as μ and β. This occurs maybe because the observation variance is mainly determined by these ARCH parameters. Finally, a choice of q may in actual practice affect the test, despite the affection would not be so serious for fairly large samples. Overall, the simulation evidence is strongly in favor of using residual cusum based tests to detect parameters change in ARCH errors models with deterministic trend.

For further study, we apply our method to the return of IBM data from 1961, May 1 to 1962, July 2, which have been studied by Shephard[18] and Giraitis[19]. The original data are in Figure (a). Figure (b) stands for first order difference of the data.

The change point estimator is computed by $\hat{k} = \min_{1\leq k\leq T}\{k : |R_k| = \max_{1\leq t\leq T}|R_t|\}$, and we can obtain $\hat{k} = 230$. The rank of the fixed ARCH processes should be calculated by the BIC criterion.

The original data prior to the estimated change point, from $t = 1$ to $t = 230$, appear to follow the models:

$$\begin{aligned} y_t &= 0.07 + 0.23t + \eta_t, \\ \eta_t &= h_t\varepsilon_t, \\ h_t^2 &= 0.78 + 0.16\eta_{t-1}^2 + 0.02\eta_{t-2}^2, \end{aligned}$$

while the orginal data posterior to the change point, from $t = 231$ to $t = 280$, follow the other models:

$$\begin{aligned} y_t &= 0.46 - 0.76t + \eta_t, \\ \eta_t &= h_t\varepsilon_t, \\ h_t^2 &= 0.65 + 0.23\eta_{t-1}^2 + 0.08\eta_{t-2}^2. \end{aligned}$$

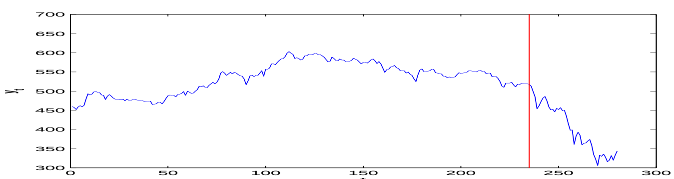

Figure (a) The original data

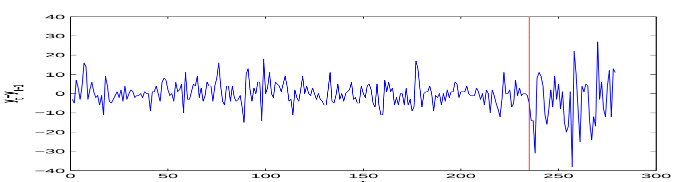

Figure (b) The first order difference of data

4. Concluding remarks

We propose a method for testing parameters change in ARCH errors models with deterministic trend. We construct a cusum test based on the residuals: the squares of observations divided by estimated conditional variance, and prove that the asymptotically limiting distribution of the residuals cusum test statistic is still the sup of a standard Brownian bridge under null hypothesis. In the proof, we use the invariance principle result for mixing processes, which was possible thanks to the results of Pelrigrad[17]. As most nonparametric methods, our procedure also depends on a choice of "bandwidth parameter" q in our case. A choice of q may in actual practice affects the test, despite the affection would not be so serious for fairly large samples. The simulation results appear to be remarkably favorable to our test: sizes and powers have been shown to perform well. This phenomenon also can be explained from a theoretical viewpoint employing an idea proposed by Lee et al.[16]. In Section 3, the test is applied to the return of IBM data and detects one change point. In a word, for testing parameters change, we can establish the asymptotic of this method and assess its performance both theoretically and numerically.

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