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A Class of Random Approximation Theorems for Random Sums of Arbitrary Stochastic Sequence on Nonhomogeneous Markov Chains

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Abstract In this paper, the notion of limit random logarithmic likelihood ratio of stochastic sequence, as a measure of dissimilarity between the joint distribution on measure P and the Markov distribution on measure Q, is introduced. A class of random approximation theorems for arbitrary stochastic dominated sequence are obtained by using the tools of generating functions and the tailed-probability generating functions.

Keywords stochastic sequence; limit logarithmic likelihood ratio; random approximation theorem; Markov distribution; stochastic dominated sequence.

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1. Introduction

Let $\{X_n, n \geq 0\}$ be an arbitrary stochastic sequence defined on the probability space $(\Omega, \mathcal{F}, P)$ which takes values in the alphabet set $S = \{0, 1, 2, \ldots\}$ with the joint distribution:

$$P(X_0 = x_0, \ldots, X_n = x_n) = p(x_0, \ldots, x_n) > 0, \quad x_i \in S,\ 0 \leq i \leq n. \tag{1}$$

We have by virtue of the definition of the conditional probability

$$p(x_0, \ldots, x_n) = p(x_0) \prod_{k=1}^{n} p(x_k | x_0, \ldots, x_{k-1}). \tag{2}$$

Let Q be another probability measure of the measurable space $(\Omega, \mathcal{F})$, $\{X_n, n \geq 0\}$ be a nonhomogeneous Markov chain on the measure Q with the initial distribution and transition matrices as follows:

$$(q(0), q(1), q(2), \ldots), \quad q(i) > 0,\ i \in S, \tag{3}$$

$$Q_n = (q_n(i,j)), q_n(i,j) > 0, \quad i, j \in S,\ n \geq 1, \tag{4}$$

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where $q_n(i,j) = Q(X_n = j|X_{n-1} = i)$ $(n \geq 1)$. The joint distribution of $\{X_n, n \geq 0\}$ with respect to the measure Q is as follows

$$Q(X_0 = x_0, \dots, X_n = x_n) = q(x_0, \dots, x_n) = q(x_0) \prod_{k=1}^{n} q_k(x_{k-1}, x_k), \quad x_i \in S,\ 0 \leq i \leq n. \tag{5}$$

Definition 1 *Let p and q be defined as (2) and (5), set*

$$h(P|Q) = \limsup_{n\to\infty} (1/n) \log[p(X_0, \dots, X_n)/q(X_0, \dots, X_n)]. \tag{6}$$

$h(P|Q)$ is called the sample divergence of the measure P relative to the measure Q.

In fact, $h(P|Q)$ is also called the limit relative logarithmic likelihood ratio of $\{X_n, n \geq 0\}$ on the measure P relative to Q, where log is the natural logarithm. We think of it as a measure of "dissimilarity" between two joint distributions $p(x_0, \dots, x_n)$ and $q(x_0, \dots, x_n)$.

Liu[1] discussed a class of strong deviation theorems for arbitrary stochastic sequence with respect to the marginal distribution by using generating function method, and also studied the above problem by means of Laplace transform[4]. Yang[2] investigated the strong deviation theorems for arbitrary information source relative to Markov information source. Wang[6] discussed the strong deviation theorems for the random sum of arbitrary stochastic sequence. Afterwards, Wang Kangkang studied some strong limit theorems for stochastic sequence[8−10]. The paper focuses on the study of a class of random deviation theorems for arbitrary dominated stochastic sequence with respect to Markov measure by using the tools of the generating function and the tailed-probability generating function.

Lemma[3] *Let p and q be two arbitrary probability measures. Then*

$$\limsup_{n\to\infty} (1/n) \log[q(X_0, \dots, X_n)/p(X_0, \dots, X_n)] \leq 0. \quad P\text{-a.s.} \tag{7}$$

We have by (6) and (7)

$$h(P|Q) \geq \liminf_{n\to\infty} (1/n) \log[p(X_0, \dots, X_n)/q(X_0, \dots, X_n)] \geq 0 \quad P\text{-a.s.} \tag{8}$$

By (5) and (6), we arrive at

$$h(P|Q) = \limsup_{n\to\infty} (1/n) \log\Big[p(X_0, \dots, X_n) \Big/ q(X_0) \prod_{k=1}^{n} q_k(X_{k-1}, X_k)\Big]. \tag{9}$$

We express the conditional generating function and the tail-probability conditional generating function, respectively, as follows:

$$P_k(s) = \sum_{j=0}^{\infty} q_k(i,j) s^j, \quad \forall i \in S, \tag{10}$$

$$Q_k(s) = \sum_{l=0}^{\infty} \sum\nolimits_{j=l+1}^{\infty} q_k(i,j) s^l, \quad \forall i \in S. \tag{11}$$

Definition 2 *Let $\{X_n, n \geq 0\}$ be a nonhomogeneous Markov chain on the measure Q. $\{X_n, n \geq 0\}$ is said to be dominated by a random variable X if there exists a constant $D > 0$ such that*

$\forall x > 0$, $n \geq 0$,

$$Q(X_n > x | X_{n-1} = x_{n-1}) \leq D \cdot Q(X > x), \quad \forall x_{n-1} \in S \tag{12}$$

and denoted by $\{X_n, n \geq 0\} \prec X$.

2. Main results and proof

Theorem 1 *Let $\{X_n, n \geq 0\}$ be an arbitrary stochastic sequence with the joint distribution (1), and $\{X_n, n \geq 0\} \prec X$. Let $h(P|Q)$, $P_k(s)$ and $Q_k(s)$ be defined as before, $\{\sigma_n, n \geq 0\}$ be a nondecreasing nonnegative stochastic sequence, $E_Q(X) < \infty$. Denote*

$$D(\omega) = \{\omega : \lim_n \sigma_n = \infty, \quad h(P|Q) < \infty\}, \tag{13}$$

then

$$\liminf_n \frac{1}{\sigma_n(\omega)} \sum_{k=1}^{[\sigma_n(\omega)]} \{X_k - E_Q[X_k | X_{k-1}]\} \geq \alpha(h(P|Q)), \quad P\text{-a.s. } \omega \in D(\omega), \tag{14}$$

where

$$\alpha(x) = \sup\{\varphi(s, x), 0 < s < 1\}, \quad 0 \leq x < +\infty, \tag{15}$$

$$\varphi(s, x) = \liminf_{n \to \infty} \frac{1}{\sigma_n(\omega)} \sum_{k=1}^{[\sigma_n(\omega)]} [s Q_k(s) - Q_k(1)] + \frac{x}{\log s}, \quad 0 < s < 1, \ 0 \leq x < +\infty. \tag{16}$$

$$\alpha(x) \leq 0, \quad \lim_{x \to 0+} \alpha(x) = \alpha(0) = 0, \tag{17}$$

where $[c]$ represents the integral part of c, E_Q the expectation with respect to the measure Q.

Proof Let s be a nonnegative constant. Set

$$W_k(s) = E_Q[s^{X_k} | X_{k-1} = x_{k-1}] = \sum_{x_k \in S} s^{x_k} q_k(x_{k-1}, x_k), \tag{18}$$

$$m_n(s; x_0, \ldots, x_n) = q(x_0) \prod_{k=1}^{n} \frac{s^{x_k} q_k(x_{k-1}, x_k)}{W_k(s)}. \tag{19}$$

It is easy to see $m_n(s; x_0, \ldots, x_n)$, $n \geq 1$, are a family of consistent distribution functions on S^{n+1}. Let

$$T_n(s, \omega) = \frac{m_n(s; X_0, \ldots, X_n)}{p(X_0, \ldots, X_n)}. \tag{20}$$

Obviously, $\{T_n(s, \omega), n \geq 1\}$ are nonnegative super-martingales which converge almost surely[3]. Therefore, we have in virtue of Doob's martingale convergence theorem[7]

$$\lim_{n \to \infty} T_n(s, \omega) = T_\infty(s, \omega) < \infty. \quad P\text{-a.s.} \tag{21}$$

Noticing that $m_n(s; X_0, \ldots, X_n)$ and $p(X_0, \ldots, X_n)$ are two probability distributions in $(\Omega, \mathcal{F}, P)$, so we have by (13), (20) and the Lemma

$$\limsup_{n \to \infty} (1/\sigma_n) \log T_{[\sigma_n]}(s, \omega) \leq 0, \quad P\text{-a.s. } \omega \in D(\omega). \tag{22}$$

By (18)–(20), we have

$$(1/\sigma_n)\log T_{[\sigma_n]}(s,\omega) = (1/\sigma_n)\{\sum_{k=1}^{[\sigma_n]}[\log s\cdot X_k - \log W_k(s)] + \log(q(x_0)\prod_{k=1}^{[\sigma_n]} q_k(X_{k-1},X_k)\Big/ p(X_0,\dots,X_{[\sigma_n]}))\}. \tag{23}$$

By (9), (13) and (23) we have

$$\limsup_{n\to\infty}\frac{1}{\sigma_n}\sum_{k=0}^{[\sigma_n]}[\log s\cdot X_k - \log W_k(s)] \le h(P|Q), \quad P\text{-a.s. } \omega\in D(\omega). \tag{24}$$

From (24), we have

$$\limsup_{n\to\infty}\frac{1}{\sigma_n}\sum_{k=0}^{[\sigma_n]}\log s\cdot X_k \le \limsup_{n\to\infty}\frac{1}{\sigma_n}\sum_{k=0}^{[\sigma_n]}\log W_k(s) + h(P|Q), \quad P\text{-a.s. } \omega\in D(\omega). \tag{25}$$

By the property of the generating function[5], we have

$$Q_k(s) = \frac{1-W_k(s)}{1-s}, \quad |s|<1, \quad Q_k(1) = E_Q[X_k|X_{k-1}]. \tag{26}$$

Recall the inequality $1-1/x \le \log x \le x-1 (x>0)$ and let $0<s<1$. We have by (25)

$$\begin{aligned}\liminf_{n\to\infty}\frac{1}{\sigma_n}\sum_{k=1}^{[\sigma_n]}[X_k - E_Q(X_k|X_{k-1})] &\ge \liminf_{n\to\infty}\frac{1}{\sigma_n}\sum_{k=1}^{[\sigma_n]}[\frac{\log W_k(s)}{\log s} - E_Q(X_k|X_{k-1})] + \frac{h(P|Q)}{\log s}\\ &\ge \liminf_{n\to\infty}\frac{1}{\sigma_n}\sum_{k=1}^{[\sigma_n]}[\frac{W_k(s)-1}{1-1/s} - E_Q(X_k|X_{k-1})] + \frac{h(P|Q)}{\log s}\\ &= \liminf_{n\to\infty}\frac{1}{\sigma_n}\sum_{k=1}^{[\sigma_n]}[sQ_k(s) - Q_k(1)] + \frac{h(P|Q)}{\log s}, \quad P\text{-a.s. } \omega\in D(\omega).\end{aligned} \tag{27}$$

Setting $s=1$ in (22), by (9) and (23) we obtain

$$h(P|Q) \ge \liminf_{n}(1/\sigma_n)\log[p(X_0,\dots,X_{[\sigma_n]})\Big/ q(x_0)\prod_{k=1}^{[\sigma_n]} q_k(X_k|X_{k-1})] \ge 0, \quad P\text{-a.s. } \omega\in D(\omega). \tag{28}$$

Denote

$$g(s) = \liminf_{n\to\infty}\frac{1}{\sigma_n}\sum_{k=0}^{[\sigma_n]}[sQ_k(s) - Q_k(1)], \ \ 0<s<1. \tag{29}$$

Then

$$\varphi(s,x) = g(s) + x/\log s, \quad 0<s<1, \ x\ge 0, \tag{30}$$

$$\alpha(x) = \sup\{g(s) + x/\log s, 0<s<1\}, \quad x\ge 0. \tag{31}$$

Obviously, $g(s) \le 0$, $\varphi(s,x) \le 0$, hence $\alpha(x) \le 0$. Denote $q(j) = Q(X = j)$, so we have its tailed-probability generating function:

$$Q(s) = \sum_{l=0}^{\infty} \sum_{j=l+1}^{\infty} q(j) \cdot s^l. \tag{32}$$

It is obvious to see

$$Q(s) \le Q(1) = E_Q(X), \quad 0 < s \le 1. \tag{33}$$

Noticing that $\sigma_n - 1 < [\sigma_n] \le \sigma_n$, we have $\lim_n [\sigma_n]/\sigma_n = 1$ by (13). If $0 < s < s+t \le 1$, by (12), (29), (32) and (33) we have

$$\begin{aligned}
0 < g(s+t) - g(s) &= \liminf_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=0}^{[\sigma_n]} [(s+t)Q_k(s+t) - Q_k(1)] - \liminf_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=0}^{[\sigma_n]} [sQ_k(s) - Q_k(1)] \\
&= \liminf_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=0}^{[\sigma_n]} [(s+t)Q_k(s+t) - Q_k(1)] + \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=0}^{[\sigma_n]} [Q_k(1) - sQ_k(s)] \\
&\le \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=0}^{[\sigma_n]} [(s+t)Q_k(s+t) - sQ_k(s)] \\
&\le \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=0}^{[\sigma_n]} s[Q_k(s+t) - Q_k(s)] + \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=0}^{[\sigma_n]} t \cdot Q_k(s+t) \\
&\le \limsup_{n\to\infty} \frac{D}{\sigma_n} \sum_{k=0}^{[\sigma_n]} \sum_{l=0}^{\infty} \sum_{j=l+1}^{\infty} s \cdot q(j)[(s+t)^l - s^l] + \limsup_{n\to\infty} \frac{Dt}{\sigma_n} \sum_{k=0}^{[\sigma_n]} Q(s+t) \\
&\le Ds[Q(s+t) - Q(s)] + Dt \cdot E_Q(X).
\end{aligned} \tag{34}$$

By (34) we know $g(s)$ is continuous on $(0,1)$, hence $\varphi(s,x)$ is continuous on $(0,1)$ with respect to s. By (28), (30) and (31) we have for every $\omega \in D(\omega)$, $\exists \lambda_n \in Q_*$ $(n = 1, 2, \ldots)$ such that

$$\lim_n \varphi(\lambda_n, h(P|Q)) = \alpha(h(P|Q)). \tag{35}$$

By (27), (29), (30), (31) and (35) we obtain

$$\liminf_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=0}^{[\sigma_n]} [X_k - E_Q(X_k|X_{k-1})] \ge \varphi(\lambda_n, h(P|Q)), \quad n = 1, 2, \ldots, \ \omega \in D(\omega). \tag{36}$$

By (35) and (36) we have

$$\liminf_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=0}^{[\sigma_n]} [X_k - E_Q(X_k|X_{k-1})] \ge \alpha(h(P|Q)). \quad P\text{-a.s. } \omega \in D(\omega). \tag{37}$$

Hence (14) follows from (37).

Let $0 < s \le 1$. By (11), (12) and (32) we have

$$\liminf_n \frac{1}{\sigma_n} \sum_{k=0}^{[\sigma_n]} [sQ_k(s) - Q_k(1)] = \liminf_n \frac{1}{\sigma_n} \sum_{k=0}^{[\sigma_n]} \sum_{l=0}^{\infty} \sum_{j=l+1}^{\infty} [q_k(i,j)s^{1+l} - q_k(i,j)]$$

$$= \liminf_n \frac{1}{\sigma_n} \sum_{k=0}^{[\sigma_n]} \sum_{l=0}^{\infty} \sum_{j=l+1}^{\infty} q_k(i,j)(s^{1+l}-1) \geq \liminf_n \frac{D}{\sigma_n} \sum_{k=0}^{[\sigma_n]} \sum_{l=0}^{\infty} \sum_{j=l+1}^{\infty} q(j)(s^{1+l}-1)$$

$$= \liminf_n [\frac{D}{\sigma_n} \sum_{k=0}^{[\sigma_n]} \sum_{l=0}^{\infty} \sum_{j=l+1}^{\infty} q(j)s^{1+l} - \frac{D}{\sigma_n} \sum_{k=0}^{[\sigma_n]} \sum_{l=0}^{\infty} \sum_{j=l+1}^{\infty} q(j)] = D[sQ(s)-Q(1)]. \quad (38)$$

If $0<x<1$, we have by (38)

$$\alpha(x) \geq \varphi(1-\sqrt{x}, x) = g(1-\sqrt{x}) + \frac{x}{\log(1-\sqrt{x})}$$
$$\geq D[(1-\sqrt{x})Q(1-\sqrt{x}) - Q(1)] + \frac{x}{\log(1-\sqrt{x})}. \quad (39)$$

If $x=0$, we have

$$\alpha(0) \geq g(1-1/n) \geq D[(1-1/n)Q(1-1/n) - Q(1)], \quad n \geq 1. \quad (40)$$

Since $\alpha(x) \leq 0 (x \geq 0)$, (17) follows from (39) and (40).

Definition 3 *Let $\{X_n, n \geq 0\}$ be a nonhomogeneous Markov chain on the measure Q. $\{X_n, n \geq 0\}$ is said to be dominated in Cesaro sense by a random variable X if there exists a constant $D>0$ such that $\forall x>0$, $n \geq 0$,*

$$\sum_{k=1}^{n} Q(X_k > x | X_{k-1} = x_{k-1}) \leq nD \cdot Q(X > x), \quad \forall x_{k-1} \in S, \quad (41)$$

and denoted by $\{X_n, n \geq 0\} \prec X(c)$.

Theorem 2 *Under the assumption of Theorem 1, if $R>1$, then*

$$\limsup_n \frac{1}{\sigma_n} \sum_{k=0}^{[\sigma_n]} \{X_k - E_Q[X_k|X_{k-1}]\} \leq \beta(h(P|Q)), \quad P\text{-a.s. } \omega \in D(\omega), \quad (42)$$

where

$$\beta(x) = \inf\{\varphi(s,x), 1<s<R\}, \quad 0 \leq x < +\infty, \quad (43)$$

$$\varphi(s,x) = \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=0}^{[\sigma_n]} [sQ_k(s) - Q_k(1)] + \frac{x}{\log s}, \quad 1<s<R, \ 0 \leq x < +\infty, \quad (44)$$

$$\beta(x) \geq 0; \quad \lim_{x\to 0+} \beta(x) = \beta(0) = 0. \quad (45)$$

Theorem 3 *Under the assumption of Theorem 1, if $\{X_n, n \geq 0\} \prec X(c)$, $Q_k(s) < \infty$, $Q(s) < \infty$, $1<s<R$, then*

$$\limsup_n \frac{1}{\sigma_n} \sum_{k=0}^{[\sigma_n]} \{X_k - E_Q[X_k|X_{k-1}]\} \leq \beta(h(P|Q)), \quad P\text{-a.s. } \omega \in D(\omega), \quad (46)$$

$$\liminf_n \frac{1}{\sigma_n} \sum_{k=0}^{[\sigma_n]} \{X_k - E_Q[X_k|X_{k-1}]\} \geq \alpha(h(P|Q)), \quad P\text{-a.s. } \omega \in D(\omega), \quad (47)$$

where

$$\alpha(x) = \sup\{D[sQ(s) - Q(1)] + \frac{x}{\log s}, 0<s<1\}, \quad 0 \leq x < +\infty, \quad (48)$$

$$\alpha(x) \le 0, \quad \lim_{x\to 0+} \alpha(x) = \alpha(0) = 0. \tag{49}$$

$$\beta(x) = \inf\{D[sQ(s) - Q(1)] + \frac{x}{\log s}, 1 < s < R\}, \quad 0 \le x < +\infty. \tag{50}$$

$$\beta(x) \ge 0, \quad \lim_{x\to 0+} \beta(x) = \beta(0) = 0. \tag{51}$$

Remark The proofs of Theorems 2 and 3 are similar to that of Theorem 1.

Corollary 1 *Under the assumption of Theorem 3, we have*

$$\liminf_n \frac{1}{\sigma_n(\omega)} \sum_{k=1}^{[\sigma_n(\omega)]} \{X_k - E_Q[X_k|X_{k-1}]\} \ge D[e^{-1}Q(e^{-1}) - Q(1)] - h(P|Q), \quad P\text{-a.s. } \omega \in D(\omega), \tag{52}$$

$$\limsup_n \frac{1}{\sigma_n} \sum_{k=0}^{[\sigma_n]} \{X_k - E_Q[X_k|X_{k-1}]\} \le D[eQ(e) - Q(1)] + h(P|Q). \quad P\text{-a.s. } \omega \in D(\omega). \tag{53}$$

Proof Let $s = e^{-1}$ and $x = h(P|Q)$ in Theorem 3. We have by (48)

$$\alpha(x) \ge D[e^{-1}Q(e^{-1}) - Q(1)] + \frac{h(P|Q)}{\log e^{-1}} = D[e^{-1}Q(e^{-1}) - Q(1)] - h(P|Q).$$

Therefore, (52) follows from (47) and (48). Let $s = e$ and $x = h(P|Q)$ in Theorem 3. Similarly (53) follows from (46) and (50).

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