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Hopf Ore Extension over Dihedral Group

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Abstract In this paper, the Hopf Ore extension and corresponding module extension of the group algebra over dihedral group are studied. It turns out that the 1-dimensional and 2-dimensional simple representations can both be extended to the simple representations over a class of Hopf Ore extension.

Keywords dihedral group; Hopf Ore extension; module extension.

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1. Introduction

The general theory of the Ore extension is firstly introduced in [1]. The method of the Ore extension commenced in ring theory to construct a class of noncommutative rings, that is, skew polynomial ring. In recent literatures, the idea of the Ore extension theory was widely applied, especially in the theory of quantum group. In [2], Panov studied some classes of the Hopf Ore extension over Hopf algebra. It turns out that one may add $(1, r)$-primitive element to Hopf algebra, and then obtain a new Hopf algebra. In addition, Panov gave the relation between Hopf Ore extension and 1-cocycle algebra over the group Hopf algebra. Based on [2], in this paper, we continue to study the related problem about the Hopf Ore extension of the group algebra over dihedral group. Firstly, we introduce the explicit relation between the Hopf Ore extension and 1-cocycle, and then classify the set of 1-cocycle over the group algebra kD_n over dihedral group D_n. Secondly, we give explicitly the Hopf algebra structure of the Hopf Ore extension of kD_n. Finally, we prove that the 1-dimensional simple representations and 2-dimensional simple representations can both be extended to some classes of Hopf Ore extension over kD_n.

Throughout this paper, we assume that Hopf algebra is over the complex field $k = \mathbb{C}$. We denote the multiplication by m, the unit by μ, and the comultiplication by Δ, the counit by ϵ, and the antipode by s. We denote the set of group-like elements by $G(H)$ for any Hopf algebra H. The basic notations and simple facts about Hopf algebra can be referred to [3] and [4].

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2. Hopf Ore extension

In this section, we shall recall some basic facts about the Hopf Ore extension. Let A be a k-algebra. Consider an endomorphism σ of and a σ-derivation δ of the algebra A. This means that $\delta(k) = 0$ and

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b,$$

for any $a, b \in A$.

The Ore extension $R = A[x, \sigma, \delta]$ of the k-algebra A is the k-algebra R generated by the variable x and the algebra A with the relation

$$xa = \sigma(a)x + \delta(a), \quad \forall a \in A.$$

It is easy to see that every element can uniquely be represented as $\sum \lambda_i x^i$, where $\lambda \in k$[5].

Definition 2.1 *Let A and $R = A[x, \sigma, \delta]$ be Hopf algebras over k. The Hopf algebra $R = A[x, \sigma, \delta]$ is called the Hopf-Ore extension if $\Delta(x) = x \otimes 1 + r \otimes x$ and A is a Hopf subalgebra in R, where r is a group-like element in $G(H)$.*

Remark If $R = A[x, \sigma, \delta]$ is a Hopf-Ore extension, it is easy to get $\epsilon(x) = 0$ and $s(x) = -r^{-1}x$.

The following theorem and proposition are the basic facts about the general theory of the Hopf Ore extension and the proof can be found in [2].

Theorem 2.2 *Let A be Hopf algebra, σ be an endomorphism of the algebra A, δ be a σ-derivation. Then the Hopf algebra $R = A[x, \sigma, \delta]$ is a Hopf-Ore extension of A if and only if the following conditions are satisfied for any $a \in A$:*

(1) There is a character $\chi : A \to k$ such that

$$\sigma(a) = \sum \chi(a_1)a_2;$$

(2) The following relation holds:

$$\sum \chi(a_1)a_2 = \sum ad_r(a_1)\chi(a_2),$$

where $ad_r(a) = rar^{-1}$;

(3) The σ-derivation δ satisfies the relation

$$\Delta(\delta(a)) = \sum \delta(a_1) \otimes a_2 + ra_1 \otimes \delta(a_2).$$

Let G be a group, $A = kG$ be the group algebra, and χ be a character over kG. Then the linear form $\alpha : kG \to k$ is determined by the values $\alpha(g)$, $g \in G$. We say that α is a 1-cocycle associated with χ, if α satisfies the following condition

$$\alpha(gh) = \alpha(g) + \chi(g)\alpha(h), \quad \forall g, h \in G.$$

We denote by $Z^1_\chi(G)$ the set of all 1-cocycle associated with χ over kG.

Proposition 2.3 *Let $A = kG$ be a group Hopf algebra. Then every Hopf-Ore extension of*

$A = kG$ is of the form $A[x,\sigma,\delta]$, where $\sigma(a) = \sum \chi(a_1)a_2$, δ is a σ-derivation of the form

$$\delta(a) = \sum \alpha(a_1)(1-r)a_2,$$

for some $\alpha \in Z^1_\chi(G)$, where χ is a group character and r is an element in the center of the group G.

3. 1-cocycle over kD_n

In this section, we describe explicitly the set of all 1-cocycle over kD_n. For $n \in \mathbb{N}$, let $D_n = \langle a, b | a^n = b^2 = (ab)^2 = 1\rangle$ be the $2n$ order dihedral group. Assume that n is an even number and $n \geq 2$. It is well-known that D_n has the following four linear representations over $k = \mathbb{C}$:

$$\chi_1(a) = \chi_1(b) = 1,\ \chi_2(a) = 1,\ \chi_2(b) = -1,$$

$$\chi_3(a) = -1,\ \chi_3(b) = 1,\ \chi_4(a) = -1,\ \chi_4(b) = -1.$$

And we also know that the center of D_n is $\{1, a^\nu\}$, where $n = 2\nu$.

Lemma 3.1 *Let $\chi_i (i = 1, 2, 3, 4)$ be the trivial character of kD_n. Then we have*

(1) $Z^1_{\chi_1}(D_n) = 0$,

(2) $Z^1_{\chi_2}(D_n) = k$,

(3) $Z^1_{\chi_3}(D_n) = k$,

(4) $Z^1_{\chi_4}(D_n) = k$.

Proof (1) Let $g \in D_n$. Notice that the order of g is finite, hence there exists an $m \in \mathbb{N}$ such that $g^m = 1$. For any $\alpha \in Z^1_{\chi_1}(D_n)$, it is easy to see that $\chi_1(g) = 1$, $\alpha(1) = 0$ and $\alpha(g^2) = 2\alpha(g)$. By induction, we have $\alpha(g^m) = m\alpha(g)$, so we get that $\alpha(g) = 0$.

(2) Let $\alpha \in Z^1_{\chi_2}(D_n)$. Note that $a^n = 1$ and $\chi_2(a) = 1$, it follows that $\alpha(a) = 0$ by using the proof of Lemma 3.1. And also we have

$$\alpha(b^2) = \alpha(b) + \chi_2(b)\alpha(b) = 0.$$

Thus, we have

$$\alpha(ab) = \alpha(b) = \alpha(ba^{n-1}),$$

so, there exists $\lambda \in k$ such that $\alpha(b) = \lambda$. Conversely, for any $\lambda \in k$, we set

$$\alpha(a^i) = 0,\ \alpha(a^i b) = \lambda, \quad 0 \leq i \leq n-1.$$

Then it can be easily proved that $\alpha \in Z^1_{\chi_2}(D_n)$. Thus, we have that $Z^1_{\chi_2}(D_n) = k$.

(3) Let $\alpha \in Z^1_{\chi_3}(D_n)$. Note that $b^2 = 1$ and $\chi_3(b) = 1$, it follows that $\alpha(b) = 0$ by using the proof of (1). And also we have

$$\alpha(a^i) = \begin{cases} 0, & i \text{ is even,} \\ \alpha(a), & i \text{ is odd,} \end{cases}$$

and

$$\alpha(a^i b) = \begin{cases} 0, & i \text{ is even,} \\ \alpha(a), & i \text{ is odd.} \end{cases}$$

Thus, there exists $\lambda \in k$ such that $\alpha(a) = \lambda$. Conversely, for any $\lambda \in k$, we set

$$\alpha(a^i) = \begin{cases} 0, & i \text{ is even}, \\ \lambda, & i \text{ is odd}. \end{cases}$$

And

$$\alpha(a^i b) = \begin{cases} 0, & i \text{ is even}, \\ \lambda, & i \text{ is odd}. \end{cases}$$

Then it can be easily proved that $\alpha \in Z^1_{\chi_3}(D_n)$. Thus, we have that $Z^1_{\chi_3}(D_n) = k$.

(4) Let $\alpha \in Z^1_{\chi_4}(D_n)$. Note that $a^n = 1$ and $\chi_4(a) = -1$, it follows that

$$\alpha(a^i) = \begin{cases} 0, & i \text{ is even}, \\ \alpha(a), & i \text{ is odd}. \end{cases}$$

From the relation $(ba)^2 = 1$, we have

$$0 = \alpha(1) = \alpha((ba)^2) = \alpha(ba) + \chi(ba)\alpha(ba) = 2\alpha(ba) = 2(\alpha(b) - \alpha(a)),$$

and hence $\alpha(a) = \alpha(b)$. Thus, there exists $\lambda \in k$ such that $\alpha(a) = \alpha(b) = \lambda$. Conversely, for any $\lambda \in k$, we set

$$\alpha(a^i) = \begin{cases} 0, & i \text{ is even}, \\ \lambda, & i \text{ is odd}, \end{cases} \quad \alpha(a^i b) = \begin{cases} \lambda, & i \text{ is even}, \\ 0, & i \text{ is odd}. \end{cases}$$

Then it can be easily proved that $\alpha \in Z^1_{\chi_4}(D_n)$. Therefore, the assertion follows.

At the end of this section, let $n = 2\nu + 1$ be odd. We know that kD_n has two linear representations:

$$\rho_1(a) = \rho_1(b) = 1, \ \rho_2(a) = 1, \ \rho_2(b) = -1. \qquad \square$$

Similar to the proof above, we have

Lemma 3.2 Let n be odd. Then we have $Z^1_{\rho_1}(D_n) = 0$ *and* $Z^1_{\rho_2}(D_n) = k$.

4. Hopf Ore extension over kD_n

In this section, we shall explicitly give the generators and relations of the Hopf Ore extension over kD_n. Throughout this section, we assume that $n = 2\nu$ is even. Set $\chi = \chi_3$, $\alpha(a) = q$,$\alpha(b) = 0$, and $r = a^\nu$. We denote the corresponding Hopf Ore extension $kD_n[x, \sigma, \delta]$ by $A(n, q)$, where

$$\sigma(a) = -a, \ \sigma(b) = b, \ \delta(a) = q(1 - a^\nu)a, \ \delta(b) = 0.$$

Thus, we have the following theorem

Theorem 4.1 *The Hopf Ore extension $A(n, q)$ is a k-algebra generated by a, b and x with the following relations:*

$$a^n = b^2 = 1, \ (ba)^2 = 1, \ xa = -ax + q(1 - a^\nu)a, \ bx = xb.$$

The Hopf algebra structure is determined by

$$\Delta(a) = a \otimes a, \ \Delta(b) = b \otimes b, \ \Delta(x) = x \otimes 1 + a^\nu \otimes x,$$

$$\epsilon(a) = \epsilon(b) = 1,\ \epsilon(x) = 0,\ s(a) = a^{-1},\ s(b) = b^{-1},\ s(x) = -a^{\nu}x.$$

5. Module extension

In this section, we shall prove that the 1-dimension and 2-dimension simple representations over kD_n can both be extended to the simple representations over $A(n,q)$. We continue to use the notations in Section 4.

Firstly, we need the following Lemma.

Lemma 5.1[6] *Let R, E be algebras over k, and $f : R \to E$ be an algebra homomorphism. And let σ be an algebra homomorphism of R, δ be α-derivation, $\xi \in E$. If (f, σ, δ, ξ) satisfies*

$$\xi f(r) = f(\sigma(r))\xi + f(\delta(r)), \quad \forall r \in R,$$

then there exists a unique algebra homomorphism $\bar{f} : R[x, \sigma, \delta] \to E$, such that $\bar{f}(x) = \xi$, $\bar{f}|_R = f$.

Let k_i be the 1-dimension kD_n representation corresponding to the character χ_i for $i = 1, 2, 3, 4$. Define an algebra homomorphism $\bar{\chi_i} : A(n,q) \to k$ determined by

$$\bar{\chi_i}(a) = \chi_i(a),\ \bar{\chi_i}(b) = \chi_i(b),\ \bar{\chi_i}(x) = \frac{q(1 - \chi_i(a^{\nu}))\chi_i(a)}{2\chi_i(a)}.$$

From Lemma 5.1, it follows that $\bar{\chi_i}$ is well-defined. Thus we have four 1-dimension representations over $A(n,q)$, which will also be denoted by k_i for $i = 1, 2, 3, 4$.

Theorem 5.2 *$A(n,q)$ has only four 1-dimension simple representations k_i for $i = 1, 2, 3, 4$.*

Proof Note that 1-dimension simple module over $A(n,q)$ must be simple module over kD_n, thus the conclusion can be derived from Lemma 5.1 and the above statement.

In the following, we shall discuss the case of 2-dimension simple modules.

For $n = 2\nu$, $\nu > 1$, $1 \le i \le \nu - 1$, we know that kD_n has 2-dimension representation corresponding to the following algebra homomorphism $\varphi_i : kD_n \to \mathbb{M}_2(k)$ determined by

$$a \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where ω is n-primitive root of unity. Denote these 2-dimension representations of kD_n by V_i for $1 \le i \le \nu - 1$. It is known from [4] that V_i are all 2-dimension simple representations over kD_n. $\square$

Theorem 5.3 *Let $T = \begin{pmatrix} \eta & \xi \\ \xi & \eta \end{pmatrix} \in \mathbb{M}_2(k)$, where $\eta = \frac{1}{2}q(1 - \omega^{iv})$, $\xi = 0$ $(1 \le i \le \nu - 1)$ or $\xi \in k$ for $i = \frac{\nu}{2}$ when ν is even. Then there exists a unique algebra homomorphism*

$$\bar{\varphi_i} : A(n,q) \to \mathbb{M}_2(k)$$

such that $\bar{\varphi_i}|_{kD_n} = \varphi_i$ and $\bar{\varphi_i}(x) = T$.

Proof Since $T = \begin{pmatrix} \eta_1 & \xi_1 \\ \xi_2 & \eta_2 \end{pmatrix} \in \mathbb{M}_2(k)$, it is known from Lemma 5.1 that there exists unique algebra homomorphism $\bar{\varphi}_i : A(n,q) \to \mathbb{M}_2(k)$ such that $\bar{\varphi}_i|_{kD_n} = \varphi_i$ and $\bar{\varphi}_i(x) = T$ with the following relations:

$$T\varphi_i(a) = -\varphi_i(a)T + q(E - \varphi_i(a^\nu))\varphi_i(a),\ T\varphi_i(b) = \varphi_i(b)T.$$

From the second equality above, it follows that T must have the form of $T = \begin{pmatrix} \eta & \xi \\ \xi & \eta \end{pmatrix}$. From the first one, we have

$$\begin{pmatrix} \eta & \xi \\ \xi & \eta \end{pmatrix}\begin{pmatrix} \omega^i & 0 \\ 0 & \omega^{-i} \end{pmatrix} = -\begin{pmatrix} \omega^i & 0 \\ 0 & \omega^{-i} \end{pmatrix}\begin{pmatrix} \eta & \xi \\ \xi & \eta \end{pmatrix} + q\begin{pmatrix} (1-\omega^{i\nu})\omega^i & 0 \\ 0 & (1-\omega^{-i\nu})\omega^{-i} \end{pmatrix},$$

that is,

$$\begin{pmatrix} \eta\omega^i & \xi\omega^{-i} \\ \xi\omega^i & \eta\omega^{-i} \end{pmatrix} = -\begin{pmatrix} \eta\omega^i & \xi\omega^i \\ \xi\omega^{-i} & \eta\omega^{-i} \end{pmatrix} + q\begin{pmatrix} (1-\omega^{i\nu})\omega^i & 0 \\ 0 & (1-\omega^{-i\nu})\omega^{-i} \end{pmatrix}.$$

Consequently, we have

$$\xi\omega^i = -\xi\omega^{-i},$$

$$\eta\omega^i = -\eta\omega^i + q(1-\omega^{i\nu})\omega^i,\quad \eta\omega^{-i} = -\eta\omega^{-i} + q(1-\omega^{-i\nu})\omega^{-i},$$

and

$$\xi(\omega^{2i}+1) = 0,\ \eta = \frac{1}{2}q(1-\omega^{i\nu}).$$

So we have that $\xi = 0$ or $\omega^{2i}+1 = 0$. If $\omega^{2i}+1=0$, then $\omega^{4i} = 1$. Note that $\omega^{2\nu} = 1$ and $1 \le i \le \nu - 1$, it follows that $2i = \nu$. Thus, the assertion holds. □

From Theorem 5.3, we get four 2-dimension simple representations over $A(n,q)$. Let $\varphi : kD_n \to \mathbb{M}_2(k)$ be an algebra homomorphism. We say that φ can be trivially extended to $A(n,q)$ if $\bar{\varphi}(x) = 0$. Using Theorem 5.3, it is easy to prove the following corollary

Corollary 5.4 *Let $q \in k^*$. Then φ_i can be trivially extended to $A(n,q)$ if and only if i is even.*

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