

Order and Type of the Generalized Dirichlet Series

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Abstract For an entire function represented by a generalized dirichlet series, we define its maximal term, maximal modulus, order and type. We use the classical methods to study the relation between order, type and coefficients, exponents, which improve and generalize some results of the dirichlet series with real exponents.

Keywords maximal term; order; type; maximal modulus.

Document code A

MR(2000) Subject Classification 30B50

Chinese Library Classification O174.5; O174.52

1. Introduction

Let $\Lambda = \{\lambda_n = |\lambda_n|e^{i\theta_n} : n = 1, 2, \dots\}$ be a sequence of complex numbers in the right half plane satisfying the following conditions:

- (a) $\liminf_{n \rightarrow \infty} (|\lambda_{n+1}| - |\lambda_n|) = \delta(\Lambda) > 0$;
- (b) $\sup\{|\arg \theta_n| : n = 1, 2, \dots\} \leq \alpha < \frac{\pi}{2}$;
- (c) $\limsup_{n \rightarrow \infty} \frac{n}{|\lambda_n|} = D < +\infty$.

Assume that F is an entire function represented by a generalized dirichlet series

$$F(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \quad s = \sigma + it : \sigma, t \in \mathbb{R}, \quad (1)$$

where $\{a_n\}$ is a sequence of complex numbers and $\{\lambda_n\}$ is expressed as above. Let

$$G_\theta = \{|\arg s - \pi| \leq \theta < \pi/2 : s = \sigma + it, \sigma, t \in \mathbb{R}\}.$$

Now we introduce some definitions which will be discussed in the sequel.

Definition 1 Similar to [1] or [4], we define the order ρ of $F(s)$ by

$$\rho = \limsup_{\sigma \rightarrow -\infty} \frac{\log \log M(\sigma)}{-\sigma}, \quad (2)$$

where the maximal modulus $M(\sigma) = \sup_{t \in \mathbb{R}} \{|F(\sigma + it)| : s = \sigma + it \in G_\theta\}$.

Received date: 2007-08-23; **Accepted date:** 2008-01-04

Foundation item: the Natural Science Youth Foundation of Jiangxi Province (No. 2007GQS0159); Research Plan Program of Education Bureau of Jiangxi Province (Nos. GJJ08161; GJJ09463).

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Definition 2 If $0 < \rho < \infty$, then define the type τ of $F(s)$ by

$$\tau = \limsup_{\sigma \rightarrow -\infty} \frac{\log M(\sigma)}{e^{-\rho\sigma}}, \quad (3)$$

where $M(\sigma)$ is defined by (2).

Definition 3 Assume that

$$m(\sigma) = \sup\{|a_n e^{-\lambda_n s}| : s = \sigma + it \in G_\theta, n \in \mathbb{N}_+\}, \quad (4)$$

which is called the maximal term of the series (1).

If $0 < \rho < \infty$, by [1], we introduce

$$\rho_r = \limsup_{n \rightarrow \infty} \frac{\operatorname{Re} \lambda_n \log \operatorname{Re} \lambda_n}{\log |1/a_n|}, \quad \tau_r = \limsup_{n \rightarrow \infty} \frac{\operatorname{Re} \lambda_n}{e\rho} |a_n|^{\frac{\rho}{\operatorname{Re} \lambda_n}}, \quad (5)$$

and

$$\rho_m = \limsup_{n \rightarrow \infty} \frac{|\lambda_n| \log |\lambda_n|}{\log |1/a_n|}, \quad \tau_{m_i} = \limsup_{n \rightarrow \infty} \frac{|\lambda_n|}{e\rho_i} |a_n|^{\frac{\rho_i}{|\lambda_n|}} \quad (0 < \rho_i < \infty), \quad i = 1, 2, \quad (6)$$

where $\rho_1 = \rho \cos \theta$, $\rho_2 = \rho \frac{1}{1 + \tan \theta \tan \alpha}$.

In this paper, firstly we discuss the relation between the maximal term and the maximal modulus, then apply it to estimate the order ρ and the type τ , respectively.

At last, we point out that we denote by A a positive constant, and by $A(\cdot)$ a positive constant only depending on (\cdot) for the whole article, not necessarily the same at each occurrence. ε is an arbitrary small positive number.

2 Preliminary results

In this section, we will give lemmas which play an important role in the proof of theorems.

Lemma 1 Assume that a , b and λ are positive constants. Then

$$\varphi(\sigma) = be^{\sigma a} - \sigma \lambda$$

reaches its minimal value at the point $\sigma = \frac{1}{a} \log \frac{\lambda}{ab}$.

Lemma 2 If the sequence $\Lambda = \{\lambda_n\}$ ($n = 1, 2, \dots$) satisfies

$$\limsup_{n \rightarrow \infty} \frac{\log n}{\operatorname{Re} \lambda_n} = E < +\infty, \quad (7)$$

then $M(\sigma) \leq A(\varepsilon)m(\sigma - E - \varepsilon)$.

Proof Since $\lim_{n \rightarrow \infty} \frac{\log n}{\operatorname{Re} \lambda_n} = E < +\infty$, $\forall \varepsilon > 0$, $\exists N > 0$, s.t., $\forall n > N$, we have $\log n < (E + \varepsilon/2)\operatorname{Re} \lambda_n$. Obviously, if $\sigma + it \in G_\theta$, then $\sigma - E - \varepsilon + it \in G_\theta$. Hence

$$\begin{aligned} M(\sigma) &\leq \sum_{n=1}^N |a_n e^{-\lambda_n s}| + \sum_{N+1}^{\infty} |a_n e^{-\lambda_n (\sigma - E - \varepsilon) + it}| \cdot e^{-(E + \varepsilon)\operatorname{Re} \lambda_n} \\ &\leq Nm(\sigma) + m(\sigma - E - \varepsilon) \sum_{N+1}^{\infty} \frac{1}{n} \frac{e^{E + \varepsilon}}{e^{E + \varepsilon/2}} \leq A(\varepsilon)m(\sigma - E - \varepsilon). \end{aligned}$$

Lemma 3 If the sequence $\Lambda = \{\lambda_n\}$ ($n = 1, 2, \dots$) satisfies the conditions (a), (b) and (c), then

$$m(\sigma) \leq A(D, \varepsilon)M(\sigma + \delta_0 - \frac{hD}{\cos \alpha}),$$

where $\delta_0 = (\pi D + \varepsilon) \cos \phi_0$, $0 \leq \phi_0 \leq 2\pi$, $h = 9 - 3 \log \delta(\Lambda)D$.

Proof Let

$$T_n(z) = \prod_{k=1, k \neq n}^{\infty} \left(1 - \frac{z^2}{\lambda_k^2}\right).$$

Since the sequence $\Lambda = \{\lambda_n\}$ ($n = 1, 2, \dots$) satisfies the conditions (a) and (b), using the similar method to [1, Lemma 3.1.3], we can show that $T_n(z)$ is an entire function of exponential type πD and because

$$|T_n(\pm \lambda_n)| \geq \prod_{k=1, k \neq n}^{\infty} \left(\frac{||\lambda_k|^2 - |\lambda_n|^2|}{|\lambda_k|^2}\right),$$

we have

$$\limsup_{n \rightarrow \infty} \frac{\log |T_n^{-1}(\pm \lambda_n)|}{|\lambda_n|} \leq hD, \quad D > 0,$$

where $h = [9 - 3 \log \delta(\Lambda)D]$. For $D = 0$ is trivial, we omit this case. Then for sufficiently large N , $\forall n > N$, we can obtain

$$\frac{1}{|T_n(\lambda_n)|} < e^{D(h+\varepsilon)|\lambda_n|}. \quad (8)$$

By (1.3.3) and (1.3.5) in [2], we have

$$|a_n| = \frac{1}{|T_n(-\lambda_n)|} \cdot |L_n[F(s)]| \cdot |e^{\lambda_n s}| \quad (9)$$

and

$$|L_n[F(s)]| \leq A(D, \varepsilon)M(\sigma + \delta_0), \quad (10)$$

where s is an arbitrary point in the complex s plane, $\delta_0 = (\pi D + \varepsilon) \cos \phi_0$, $0 \leq \phi_0 \leq 2\pi$. Using the similar method to [1], we can prove $\delta(\Lambda)D \leq 1$, so $\delta_0 - \frac{hD}{\cos \alpha} < 0$. By (8), (9) and (10), for σ sufficiently small, we obtain

$$|a_n e^{-\lambda_n s}| \leq A(D, \varepsilon)M(\sigma + \delta_0) \exp\{hD|\lambda_n|\} \quad (11)$$

$$\leq A(D, \varepsilon)M(\sigma + \delta_0) \exp\left\{\frac{hD}{\cos \alpha} \operatorname{Re} \lambda_n\right\}. \quad (12)$$

Hence,

$$m(\sigma) \leq A(D, \varepsilon)M(\sigma + \delta_0 - \frac{hD}{\cos \alpha}).$$

Lemma 4 If the sequence $\Lambda = \{\lambda_n\}$ ($n = 1, 2, \dots$) satisfies the conditions (a), (b) and (c), then

$$\rho = \limsup_{\sigma \rightarrow -\infty} \frac{\log \log m(\sigma)}{-\sigma}.$$

If $0 < \rho < \infty$, then

$$\tau = \limsup_{\sigma \rightarrow -\infty} \frac{\log m(\sigma)}{e^{-\rho \sigma}}.$$

Proof By the condition (c), we have $\limsup_{n \rightarrow \infty} \frac{\log n}{\operatorname{Re} \lambda_n} = 0$. So from Lemmas 2 and 3, we obtain the lemma.

Remark Since $F(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ ($s = \sigma + it : \sigma, t \in \mathbb{R}$) is an entire function in the complex plane, according to [5], we have

$$\lim_{n \rightarrow \infty} \frac{\log |a_n|}{|\lambda_n|} = -\infty. \quad (13)$$

So $\rho_r \leq \rho_m$.

3 Main theorems

Our main conclusions are as follows:

Theorem 1 If the sequence $\Lambda = \{\lambda_n\}$ ($n = 1, 2, \dots$) satisfies the conditions (a), (b) and (c), then

$$\rho_r \leq \rho \leq \sec \theta \rho_m, \quad (*)$$

and

$$\rho_r \leq \rho \leq (1 + \tan \theta \tan \alpha) \rho_r. \quad (**)$$

Proof 1) We prove $\rho_r \leq \rho$. By the definition of ρ , for sufficiently small σ , we have

$$\log M(\sigma) \leq e^{-(\rho+\varepsilon)\sigma}.$$

For equation (11) holds for all $s \in D_\theta$, without loss of generality, putting $s = \sigma \in D_\theta$, $\sigma \in \mathbb{R}$ sufficiently small, then

$$|a_n| \leq A(D, \varepsilon) M(\sigma + \delta_0) e^{\sigma \operatorname{Re} \lambda_n + hD|\lambda_n|}.$$

So

$$\log |a_n| \leq \log A(D, \varepsilon) + hD|\lambda_n| - \delta_0 \operatorname{Re} \lambda_n + e^{-(\rho+\varepsilon)(\sigma+\delta_0)} + (\sigma + \delta_0) \operatorname{Re} \lambda_n.$$

By Lemma 1, we have

$$e^{-(\rho+\varepsilon)(\sigma+\delta_0)} + (\sigma + \delta_0) \operatorname{Re} \lambda_n \leq \frac{\operatorname{Re} \lambda_n}{\rho + \varepsilon} (1 + \log \frac{\operatorname{Re} \lambda_n}{\rho + \varepsilon}).$$

Hence, we obtain

$$-\frac{1}{\rho_r} = \limsup_{n \rightarrow \infty} \frac{\log |a_n|}{\operatorname{Re} \lambda_n \log \operatorname{Re} \lambda_n} < -\frac{1}{\rho}.$$

2) We prove $\rho \leq \sec \theta \rho_m$. By the definition of ρ_m , for sufficiently large n , we have

$$|a_n| \leq \exp \left\{ \frac{-|\lambda_n| \log |\lambda_n|}{\rho_m + \varepsilon} \right\}.$$

Then for $-\operatorname{Re}(\lambda_n s) \leq |\lambda_n s| \leq |\lambda_n \sigma| \sec \theta$, $\sigma < 0$, we have

$$\log m(\sigma) \leq \max \left\{ \frac{-|\lambda_n| \log |\lambda_n|}{\rho_m + \varepsilon} - \operatorname{Re}(\lambda_n s) \right\} \leq \max \left\{ \frac{-|\lambda_n| \log |\lambda_n|}{\rho_m + \varepsilon} - \sigma |\lambda_n| \sec \theta \right\}.$$

Let $h_1(t) = -a_1 t \log t + b_1 t$, $t > 0$, where $t = |\lambda_n|$, $a_1 = \frac{1}{\rho_m + \varepsilon}$, $b_1 = -\sigma \sec \theta$. Then $h_1(t)$ gets its maximal value $a_1 e^{-1+b_1/a_1}$ at the point $t_1 = e^{-1+b_1/a_1}$. So by Lemma 4 we have

$$\rho = \limsup_{\sigma \rightarrow -\infty} \frac{\log \log m(\sigma)}{-\sigma} \leq \rho_m \sec \theta.$$

3) We prove $\rho \leq (1 + \tan \theta \tan \alpha) \rho_r$. By the definition of ρ_r , for sufficiently large n , we have

$$|a_n| \leq \exp\left\{\frac{-\operatorname{Re}\lambda_n \log \operatorname{Re}\lambda_n}{\rho_r + \varepsilon}\right\}.$$

Then for $-t \operatorname{Im}\lambda_n \leq -\sigma \tan \theta \tan \alpha \operatorname{Re}\lambda_n$, $\sigma < 0$, we have

$$\begin{aligned} \log m(\sigma) &\leq \max\left\{\frac{-\operatorname{Re}\lambda_n \log \operatorname{Re}\lambda_n}{\rho_m + \varepsilon} - \operatorname{Re}(\lambda_n s)\right\} \\ &\leq \max\left\{\frac{-\operatorname{Re}\lambda_n \log \operatorname{Re}\lambda_n}{\rho_r + \varepsilon} - \sigma \operatorname{Re}\lambda_n - \sigma \tan \theta \tan \alpha \operatorname{Re}\lambda_n\right\}. \end{aligned}$$

Let $h_2(t) = -a_2 t \log t + b_2 t$, $t > 0$, where $t = \operatorname{Re}\lambda_n$, $a_2 = \frac{1}{\rho_r + \varepsilon}$, $b_2 = -\sigma(1 + \tan \theta \tan \alpha)$. Then $h_2(t)$ gets its maximal value $a_2 e^{-1+b_2/a_2}$ at the point $t_2 = e^{-1+b_2/a_2}$. So from Lemma 4, we have

$$\rho = \limsup_{\sigma \rightarrow -\infty} \frac{\log \log m(\sigma)}{-\sigma} \leq \rho_r(1 + \tan \theta \tan \alpha).$$

Corollary 1 In Theorem 1, if $\alpha = 0$, i.e., $\Lambda = \{\lambda_n\}$ ($n = 1, 2, \dots$) is a sequence of real numbers satisfying the conditions (a) and (c), by equation (**), we get

$$\rho = \rho_r = \rho_m = \limsup_{n \rightarrow \infty} \frac{\lambda_n \log \lambda_n}{\log |1/a_n|},$$

which is similar to [1, p.40, Theorem 3.2.1].

Corollary 2 Let $\theta = \frac{\pi}{4}$ in Theorem 1. Then by equation (*), we have $\rho_r \leq \rho \leq \sqrt{2}\rho_m$, which is also got in [3, Theorem 1].

Theorem 2 If the sequence $\Lambda = \{\lambda_n\}$ ($n = 1, 2, \dots$) satisfies the conditions (a), (b) and (c), then

$$e^{-hD\rho \sec \alpha} \tau_r \leq \tau \leq \tau_{m_1}, \quad (*)$$

and

$$\tau \leq \tau_{m_2}. \quad (**)$$

Proof 1) We prove $e^{-hD\rho \sec \alpha} \tau_r \leq \tau$. By the definition of τ , for sufficiently small σ , we have

$$M(\sigma) \leq \exp\{(\tau + \varepsilon)e^{-\rho\sigma}\}.$$

By equation (12) and Lemma 1, for sufficiently small σ , we have

$$\begin{aligned} |a_n| &\leq A(D, \varepsilon) M(\sigma + \delta_0) e^{\sigma \operatorname{Re}\lambda_n + hD|\lambda_n|} \\ &\leq A(D, \varepsilon) e^{hD|\lambda_n|} \exp\left\{\frac{\operatorname{Re}\lambda_n}{\rho} (1 + \log \frac{\rho(\tau + \varepsilon)}{\operatorname{Re}\lambda_n})\right\}. \end{aligned}$$

Then for sufficiently large n , we have

$$\frac{\operatorname{Re}\lambda_n}{e\rho} |a_n|^{\rho/\operatorname{Re}\lambda_n} \leq e^{\rho hD|\lambda_n|/\operatorname{Re}\lambda_n} (\tau + \varepsilon).$$

So

$$\tau_r = \limsup_{n \rightarrow \infty} \frac{\operatorname{Re}\lambda_n}{e\rho} |a_n|^{\frac{\rho}{\operatorname{Re}\lambda_n}} \leq e^{hD\rho \sec \alpha} \tau.$$

2) We prove $\tau \leq \tau_{m_1}$. By the definition of τ_{m_1} , for n sufficiently large,

$$\log |a_n| \leq \frac{|\lambda_n|}{\rho_1} \log \left[(\tau_{m_1} + \varepsilon) \frac{e\rho_1}{|\lambda_n|} \right].$$

For σ sufficiently small,

$$\begin{aligned}\log m(\sigma) &\leq \max \left\{ \frac{|\lambda_n|}{\rho_1} \log \left[(\tau_{m_1} + \varepsilon) \frac{e\rho_1}{|\lambda_n|} \right] - \operatorname{Re}(\lambda_n s) \right\} \\ &\leq \max \left\{ \frac{|\lambda_n|}{\rho_1} \log \left[(\tau_{m_1} + \varepsilon) \frac{e\rho_1}{|\lambda_n|} \right] - |\lambda_n| \sigma \sec \theta \right\} \\ &= \max \{-a_3 t \log t - b_3 t\},\end{aligned}$$

where $t = \frac{|\lambda_n|}{(\tau_{m_1} + \varepsilon)e\rho_1} > 0$, $a_3 = (\tau_{m_1} + \varepsilon)e$, $b_3 = -(\tau_{m_1} + \varepsilon)e\rho_1 \sigma \sec \theta$, then $h_3(t) = -a_3 t \log t - b_3 t$ has its maximal value $a_3 e^{-1+b_3/a_3}$.

So by the estimate above, we obtain

$$\tau = \limsup_{\sigma \rightarrow -\infty} \frac{\log m(\sigma)}{e^{-\sigma\rho}} \leq \limsup_{\sigma \rightarrow -\infty} \frac{(\tau_{m_1} + \varepsilon)e^{-\sigma\rho_1 \sec \theta}}{e^{-\sigma\rho}}.$$

By the equation (6), we know $\rho_1 = \rho \cos \theta$. Therefore, $\tau \leq \tau_{m_1}$.

3) We prove $\tau \leq \tau_{m_2}$. By the definition of τ_{m_2} , for n sufficiently large, we have

$$\log |a_n| \leq \frac{\operatorname{Re} \lambda_n}{\rho_2} \log \left[(\tau_{m_2} + \varepsilon) \frac{e\rho_2}{\operatorname{Re} \lambda_n} \right].$$

Then for σ sufficiently small, we get

$$\begin{aligned}\log m(\sigma) &\leq \max \left\{ \frac{\operatorname{Re} \lambda_n}{\rho_2} \log \left[(\tau_{m_2} + \varepsilon) \frac{e\rho_2}{\operatorname{Re} \lambda_n} \right] - \operatorname{Re}(\lambda_n s) \right\} \\ &\leq \max \left\{ \frac{|\lambda_n|}{\rho_2} \log \left[(\tau_{m_2} + \varepsilon) \frac{e\rho_2}{\operatorname{Re} \lambda_n} \right] - (1 + \tan \theta \tan \alpha) \sigma \operatorname{Re} \lambda_n \right\} \\ &= \max \{-a_4 t \log t - b_4 t\},\end{aligned}$$

where $t = \frac{\operatorname{Re} \lambda_n}{(\tau_{m_2} + \varepsilon)e\rho_2} > 0$, $a_4 = (\tau_{m_2} + \varepsilon)e$, $b_4 = -(\tau_{m_2} + \varepsilon)e\rho_2 \sigma (1 + \tan \theta \tan \alpha)$, then $h_4(t) = -a_4 t \log t - b_4 t$ has its maximal value $b_4 e^{-1+b_4/a_4}$.

So by the estimate above, we obtain

$$\tau = \limsup_{\sigma \rightarrow -\infty} \frac{\log m(\sigma)}{e^{-\sigma\rho}} \leq \limsup_{\sigma \rightarrow -\infty} \frac{(\tau_{m_2} + \varepsilon)e^{-\sigma\rho_2(1+\tan \theta \tan \alpha)}}{e^{-\sigma\rho}}.$$

By the equation (6), we know $\rho_2 = \rho \frac{1}{1+\tan \theta \tan \alpha}$. Therefore, $\tau \leq \tau_{m_2}$.

Corollary 3 Let $D = 0$, $\alpha = 0$ in Theorem 2. Then by equations (\star) and $(\star\star)$, we have $\tau = \tau_r = \tau_{m_2}$.

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