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Randomly Weighted Sums for Negatively Associated Random Variables with Heavy Tails

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Abstract In 2003, Tang Qihe et al. obtained a simple asymptotic formula for independent identically distributed (i.i.d.) random variables with heavy tails. In this paper, under certain moment conditions, we establish a formula as the same as Tang's, when random variables are negatively associated (NA).

Keywords asymptotic; heavy tails; negatively associated; uniformity.

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1. Introduction

For two positive infinitesimals $A(\cdot)$ and $B(\cdot)$, as usual, we write $A(\cdot) \lesssim B(\cdot)$ if $\limsup \frac{A(\cdot)}{B(\cdot)} \le 1$, $A(\cdot) \gtrsim B(\cdot)$ if $\liminf \frac{A(\cdot)}{B(\cdot)} \ge 1$, and $A(\cdot) \sim B(\cdot)$ if both. All limit relations are for $x \to \infty$ unless stated otherwise.

A finite family of random variables $\{X_i; 1 \le i \le n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets A_1 and A_2 of $\{1, 2, \ldots, n\}$,

$$\mathrm{Cov}(f_1(X_{k_1}, k_1 \in A_1), f_2(X_{k_2}, k_2 \in A_2)) \le 0,$$

whenever functions f_1 and f_2 are coordinatewise non-decreasing (or non-increasing) and the covariance exists. The concept of negative association was independently introduced by Alam and Saxena[1] and Joag-Dev and Proschan[2].

A well-known notion in extremal value theory, the subexponentiality, describes an important property of the right tail of a distribution. The subexponential class of distributions, denoted by $\mathcal{S}$, is the most well-known heavy tailed distributions. Given a non-negative random variable X, its distribution function (d.f.) is defined by $F(x) = P(X \le x)$ and its tail by $\overline{F}(x) = 1 - F(x)$. We say that X (or its d.f.) is heavy tailed if it has no finite exponential moments. A distribution F with support $[0, \infty)$ belongs to $\mathcal{S}$, if for some $n \ge 2$ (or, equivalently, for any $n \ge 2$),

$$\lim_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n.$$

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There are other heavy-tailed subclasses, class $\mathcal{L}$ of long-tailed distributions, class $\mathcal{D}$ of distributions with dominated varying tails, class $\mathcal{R}$ of distributions with regularly varying tails, and class ERV of distributions with extended regularly varying tails.

F belongs to class $\mathcal{L}$ if $\lim_{x\to\infty}\frac{\overline{F}(x+L)}{\overline{F}(x)} = 1$ for any fixed $L > 0$ (or, equivalently, for some $L > 0$);

F belongs to class $\mathcal{D}$ if $\lim_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)} < \infty$ for any $0 < y < 1$ (or, equivalently, for some $0 < y < 1$);

F belongs to class $\mathcal{R}_{-\alpha}$ for some $\alpha > 1$ if $\lim_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}$ for any $y > 0$;

F belongs to $ERV(-\alpha,-\beta)$ for some α,β with $0 < \alpha \le \beta < \infty$ if

$$y^{-\beta} \le \liminf_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)} \le \limsup_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)} \le y^{-\alpha} \quad \text{for any } y > 1.$$

For $0 < \alpha \le \gamma \le \beta < \infty$, the following inclusion relationship holds:

$$\mathcal{R}_{-\gamma} \subset ERV(-\alpha,-\beta) \subset \mathcal{L}\cap\mathcal{D} \subset \mathcal{S} \subset \mathcal{L}.$$

For more details on heavy-tailed distributions and their applications to insurance and finance, the readers are referred to Embrechts et al.[3], Goldie and Klüppelberg[4] and Meerschaert and Scheffler[5].

Recently, Tang et al.[6] showed following proposition.

Proposition 1 *Let $\{X_k, 1 \le k \le n\}$ be a sequence of i.i.d. random variables with common distribution function F, and $\{\theta_k, 1 \le k \le n\}$ be another sequence of positive random variables independent of $\{X_k, 1 \le k \le n\}$ satisfying $P(0 < \theta_k \le b) = 1$, for some $0 < b < \infty$. If $F \in \mathcal{L}\cap\mathcal{D}$, then*

$$P(\sum_{k=1}^{n}\theta_k X_k > x) \sim \sum_{k=1}^{n} P(\theta_k X_k > x). \tag{1}$$

The work mentioned above is restricted to the independence case. Motivated by this consideration, the purpose of the present paper is to derive asymptotic relationships like (1) for randomly weighted sums of NA random variables with heavy tails. See Theorem 1 below.

It should be also mentioned that Wang and Tang[7] obtained the asymptotic relationships for the tail probabilities of the maximum of partial sums and random sums of NA random variables with heavy tails. Wang et al.[8] obtained results on precise large deviations for the partial sums and random sums of NA random variables with common distribution $F \in \mathcal{D}$.

Our main result is the following:

Theorem 1 *Let $\{X_k, 1 \le k \le n\}$ be a sequence of non-negative NA random variables with common distribution function F, and $\{\theta_k, 1 \le k \le n\}$ be another sequence of positive random variables independent of $\{X_k, 1 \le k \le n\}$ such that $P(0 < \theta_k \le b) = 1$, for some $0 < b < \infty$. If $F \in \mathcal{L}\cap\mathcal{D}$ and $EX_1^{\gamma} < \infty$ for some $\gamma > 1$, then the relation (1) still holds.*

2. Preliminaries

For a d.f. F and any $y > 0$, as done recently by Tang and Tsitsiashvili[11], we set

$$\overline{F_*}(y) = \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}$$

and then define

$$\gamma_F = -\lim_{y\to\infty} \frac{\log \overline{F_*}(y)}{\log y}.$$

We call γ_F as the upper Matuszewska index of the d.f. F. For details we refer to Bingham et al[9].

The lemma below was proved by Ng et al.[10, Lemma 2.1] and Tang and Tsitsiashvili[11, Lemma 3.5].

Lemma 1 *For a distribution $F \in \mathcal{D}$ with a finite expectation, $1 \le \gamma_F < \infty$ and, as $x \to \infty$,*

$$x^{-\nu} = o(\overline{F}(x)) \quad \text{for any} \quad \nu > \gamma_F.$$

Wang and Tang[7, Lemma 3.1] proved the following result:

Lemma 2 *Let $\{X_k; k \le 1\}$ be a sequence of NA random variables with common d.f. F and finite mean $\mu < 0$. If $E|X_1|^\gamma < \infty$ for some $\gamma > 1$, then for any $0 < \rho < 1$ such that $(r-1)/(8\rho) > 1$, there is some constant $C > 0$ independent of x and n such that the inequality*

$$P\Big(\max_{1\le k\le n} \sum_{k=1}^{n} X_k > x, \bigcap_{k=1}^{n} (X_k \le \rho x)\Big) \le C x^{-\frac{r-1}{8\rho}+1}$$

holds for all $n \ge 1$ and all $x > 0$.

The following lemma will play a crucial role in the proof of our main result.

Lemma 3 *Let $\{X_k, 1 \le k \le n\}$ be a sequence of non-negative NA random variables with common d.f. F. If $F \in \mathcal{L} \cap \mathcal{D}$ and $E|X_1|^r < \infty$ for some $r > 1$, then for any fixed $0 < a \le b < \infty$, the relation*

$$P(\sum_{k=1}^{n} c_k X_k > x) \sim \sum_{k=1}^{n} P(c_k X_k > x) \tag{2}$$

holds uniformly for $\underline{c_n} =: (c_1, c_2, \dots, c_n) \in [a, b]^n$, where the uniformity is understood as

$$\lim_{x\to\infty} \sup_{\underline{c_n}\in[a,b]^n} \left| \frac{P(\sum\limits_{k=1}^{n} c_k X_k > x)}{\sum\limits_{k=1}^{n} P(c_k X_k > x)} - 1 \right| = 0.$$

Proof By the NA property, for $x > 0$, we have

$$\begin{aligned} P\Big(\sum_{k=1}^{n} c_k X_k > x\Big) &\ge \sum_{k=1}^{n} P(c_k X_k > x) - \sum_{1\le i<j\le n} P(c_i X_i > x, c_j X_j > x) \\ &\ge \sum_{k=1}^{n} P(c_k X_k > x) - \sum_{1\le i<j\le n} P(c_i X_i > x, b X_j > x) \end{aligned}$$

$$\geq \sum_{k=1}^{n} P(c_k X_k > x) - \sum_{1\leq i<j\leq n} P(c_i X_i > x) P(bX_j > x)$$
$$\sim \sum_{k=1}^{n} P(c_k X_k > x).$$

This proves that uniformly for $\underline{c_n} \in [a,b]^n$ and $x > 0$

$$P\Big(\sum_{k=1}^{n} c_k X_k > x\Big) \gtrsim \sum_{k=1}^{n} P(c_k X_k > x).$$

Next, we aim at proving that the upper asymptotic bound holds uniformly for $\underline{c_n} \in [a,b]^n$, i.e.,

$$P\Big(\sum_{k=1}^{n} c_k X_k > x\Big) \lesssim \sum_{k=1}^{n} P(c_k X_k > x), \text{ holds uniformly for } \underline{c_n} \in [a,b]^n.$$

Clearly, for arbitrary fixed numbers $0 < \theta < 1$ and $L > 0$, we have

$$\begin{aligned} P\Big(\sum_{k=1}^{n} c_k X_k > x\Big) \leq & P\Big(\bigcup_{k=1}^{n} (c_k X_k > x - L)\Big) + P\Big(\sum_{k=1}^{n} c_k X_k > x, \bigcap_{i=1}^{n} (c_i X_i \leq \theta x)\Big) + \\ & P\Big(\sum_{k=1}^{n} c_k X_k > x, \bigcap_{j=1}^{n} (c_j X_j \leq x - L), \bigcup_{i=1}^{n} (c_i X_i > \theta x)\Big) \\ \leq & \sum_{k=1}^{n} P(c_k X_k > x - L) + P\Big(\sum_{k=1}^{n} c_k X_k > x, \bigcap_{i=1}^{n} (c_i X_i \leq \theta x)\Big) + \\ & \sum_{i=1}^{n} P\Big(\sum_{k=1}^{n} c_k X_k - c_i X_i > L, c_i X_i > \theta x\Big) \\ =: & I_1 + I_2 + I_3. \end{aligned} \tag{3}$$

First, we deal with I_1. Notice that, $\frac{c_k}{b} < 1$, $k = 1, \ldots, n$, we have

$$P(c_k X_k > x - L) = P\Big(bX_k > \frac{x}{\frac{c_k}{b}} - \frac{L}{\frac{c_k}{b}}\Big) \leq P\Big(bX_k > \frac{x}{\frac{c_k}{b}} - \frac{L}{\frac{a}{b}}\Big) \tag{4}$$

and

$$P(c_k X_k > x - L) \geq P\Big(bX_k > \frac{x}{\frac{c_k}{b}} - L\Big), \quad k = 1, \ldots, n. \tag{5}$$

Hence, by the condition $F \in \mathcal{L}$, we have

$$I_1 = \sum_{k=1}^{n} P\Big(c_k X_k > x - L\Big) \sim \sum_{k=1}^{n} P(c_k X_k > x). \tag{6}$$

Now, we deal with I_3. For fixed θ, by the NA property, we have

$$\begin{aligned} I_3 &= \sum_{i=1}^{n} P\Big(\sum_{k=1}^{n} c_k X_k - c_i X_i > L, c_i X_i > \theta x\Big) \\ &\leq \sum_{i=1}^{n} P(c_i X_i > \theta x) P\Big(\sum_{k=1}^{n} c_k X_k - c_i X_i > L\Big) \\ &\leq P(bX_1 > \theta x) \sum_{i=1}^{n} P\Big(\sum_{k=1}^{n} c_k X_k - c_i X_i > L\Big). \end{aligned} \tag{7}$$

Hence, by $F \in \mathcal{D}$, we obtain

$$\lim_{L\to\infty} \limsup_{x\to\infty} \frac{P(bX_1 > \theta x)}{P(X_1 > x)} \sum_{i=1}^{n} P\Big(\sum_{k=1}^{n} c_k X_k - c_i X_i > L\Big) = 0. \tag{8}$$

Finally, we turn to I_2. For any fixed constant $\mu > EX_1$ and all large x, we have

$$\begin{aligned} I_2 &= P\Big(\sum_{k=1}^{n} c_k X_k > x, \bigcap_{i=1}^{n} (c_i X_i \le \theta x)\Big) \\ &\le P\Big(\sum_{k=1}^{n} b X_k > x, \bigcap_{i=1}^{n} (a X_i \le \theta x)\Big) \\ &\le P\Big(\sum_{k=1}^{n} (X_k - \mu) > \frac{x}{b} - n\mu, \bigcap_{i=1}^{n} (X_i - \mu) \le \frac{\theta x}{a}\Big) \\ &\le P\Big(\sum_{k=1}^{n} (X_k - \mu) > \frac{x}{b} - n\mu, \bigcap_{i=1}^{n} (X_i - \mu) \le \frac{2b\theta}{a}\Big(\frac{x}{b} - n\mu\Big)\Big). \end{aligned}$$

We choose some $0 < \theta < 1$, such that

$$\frac{a(r-1)}{16b\theta} - 1 > \gamma_F$$

holds. Applying Lemmas 1 and 2 gives

$$I_2 \le C\Big(\frac{x}{b} - n\mu\Big)^{-\frac{a(r-1)}{16b\theta}+1} = o\Big(P\Big(X_1 > \frac{x}{b} - n\mu\Big)\Big) = o\Big(\overline{F}\Big(\frac{x}{b} - n\mu\Big)\Big) = o(\overline{F}(x)). \tag{9}$$

Hence, substituting (6)–(9) above into (13), we see that the relation holds uniformly for $\underline{c_n} \in [a, b]^n$ and large x. This completes the proof of the lemma. □

The following Lemma 4 can be considered as the continuation of Lemma 3 above.

Lemma 4 *Let $\{X_k, 1 \le k \le n\}$ be a sequence of non-negative NA random variables with common distribution function F, and $\{\theta_k, 1 \le k \le n\}$ be another sequence of positive random variables independent of $\{X_k, 1 \le k \le n\}$ satisfying $P(a \le \theta_k \le b) = 1$ for some $0 < a \le b < \infty$. If $F \in \mathcal{L} \cap \mathcal{D}$ and $EX_1^\gamma < \infty$ for some $\gamma > 1$,*

$$P(\sum_{k=1}^{n} \theta_k X_k > x) \sim \sum_{k=1}^{n} P(\theta_k X_k > x) \tag{10}$$

holds.

Proof By Lemma 3 above and the dominated convergence theorem, we obtain

$$\begin{aligned} P\Big(\sum_{k=1}^{n} \theta_k X_k > x\Big) &= E\Big[P\Big(\sum_{k=1}^{n} \theta_k X_k > x | \theta_1, \ldots, \theta_n\Big)\Big] \\ &\sim E\Big[\sum_{k=1}^{n} P(\theta_k X_k > x | \theta_1, \ldots, \theta_n)\Big] \\ &= \sum_{k=1}^{n} P(\theta_k X_k > x). \end{aligned}$$

This completes the proof of the lemma. □

3. Proof of Theorem 1

Proof Clearly,

$$P\Big(\sum_{k=1}^n \theta_k X_k > x\Big) = P\Big(\sum_{k=1}^n \theta_k X_k > x, \bigcup_{k=1}^n (0 < \theta_k < \delta)\Big) + \Big(\sum_{k=1}^n \theta_k X_k > x, \bigcap_{k=1}^n (\delta \le \theta_k \le b)\Big)$$
$$=: L_1 + L_2.$$

As for L_1, we have

$$L_1 \le P\Big(\sum_{k=1}^n bX_k > x\Big) P\Big(\bigcup_{k=1}^n (0 < \theta_k < \delta)\Big).$$

By Lemma 4 and $F \in \mathcal{D}$, we have $P(\sum_{k=1}^n bX_k > x) \sim n\overline{F}(\frac{x}{b}) = O(\overline{F}(x))$. It follows that

$$\lim_{\delta \downarrow 0} \limsup_{x \to \infty} \frac{L_1}{\overline{F}(x)} = 0.$$

For I_2, by Lemma 4, we have

$$\begin{aligned} L_2 &\sim \sum_{k=1}^n P\Big(\theta_k X_k > x, \bigcap_{k=1}^n (\delta \le \theta_k \le b)\Big) \\ &= \sum_{k=1}^n P(\theta_k X_k > x) - \sum_{k=1}^n P\Big(\theta_k X_k > x, \bigcup_{k=1}^n (0 < \theta_k < \delta)\Big) \\ &= \sum_{k=1}^n P(\theta_k X_k > x) - L_3. \end{aligned}$$

Similarly to L_1

$$\lim_{\delta \downarrow 0} \limsup_{x \to \infty} \frac{L_3}{\overline{F}(x)} = 0.$$

This completes the proof of Theorem 1. □

4. Some corollary

Let X be a random variable with a distribution $F \in \mathcal{D}$. For any $y > 0$, we set

$$f_*(y) = \liminf_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)}, \quad f^*(y) = \limsup_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)}.$$

Let θ be another random variable independent of the random variable X satisfying $P(0 < \theta < b) = 1$ for some $0 < b < \infty$. Applying Theorem 3.3 (iv) of Cline and Samorodnitsky[12], we know that

$$0 < Ef_*(\theta^{-1}) \le \liminf_{x \to \infty} \frac{P(\theta X > x)}{P(X > x)} \le \limsup_{x \to \infty} \frac{P(\theta X > x)}{P(X > x)} \le Ef^*(\theta^{-1}) < \infty.$$

Hence by Theorem 1, we easily obtain the following corollary:

Corollary 1 *Let $\{X_k, 1 \le k \le n\}$ be a sequence of non-negative NA random variables with common distribution function F and $EX_1^\gamma < \infty$ for some $\gamma > 1$, and $\{\theta_k, 1 \le k \le n\}$ be another sequence of positive random variables independent of $\{X_k, 1 \le k \le n\}$ such that $P(0 < \theta_k \le b) = 1$, for some $0 < b < \infty$.*

1) If $F \in \mathcal{L} \cap \mathcal{D}$, then

$$\overline{F}(x) \sum_{k=1}^{n} E f_*(\theta_k^{-1}) \lesssim P\Big(\sum_{k=1}^{n} \theta_k X_k > x\Big) \lesssim \overline{F}(x) \sum_{k=1}^{n} E f^*(\theta_k^{-1});$$

2) If $F \in ERV(-\alpha, -\beta)$ for some $0 < \alpha \le \beta < \infty$, then

$$\overline{F}(x) \sum_{k=1}^{n} E\big[\min \theta_k^{\alpha}, \theta_k^{\beta}\big] \lesssim P\Big(\sum_{k=1}^{n} \theta_k X_k > x\Big) \lesssim \overline{F}(x) \sum_{k=1}^{n} E\big[\max \theta_k^{\alpha}, \theta_k^{\beta}\big];$$

3) If $F \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$, then

$$P\Big(\sum_{k=1}^{n} \theta_k X_k > x\Big) \sim \overline{F}(x) \sum_{k=1}^{n} E\theta_k^{\alpha}.$$

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