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α-Resolvable Cycle Systems for Cycle Length 4

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Abstract An m-cycle system of order v and index λ, denoted by m-CS(v,λ), is a collection of cycles of length m whose edges partition the edges of λK_v. An m-CS(v,λ) is α-resolvable if its cycles can be partitioned into classes such that each point of the design occurs in precisely α cycles in each class. The necessary conditions for the existence of such a design are $m|\frac{\lambda v(v-1)}{2}, 2|\lambda(v-1), m|\alpha v, \alpha|\frac{\lambda(v-1)}{2}$. It is shown in this paper that these conditions are also sufficient when $m=4$.

Keywords cycle; cycle system; α-resolvable.

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1. Introduction

Let m,v,λ be positive integers, X a v-set. An edge of X is an unordered pair $\{x,y\}$ where x,y are distinct vertices of X. A complete multigraph of order v and index λ, denoted by λK_v, is a graph on X in which each pair of vertices x,y is joined by exactly λ edges $\{x,y\}$. A cycle of length m is a sequence of m distinct vertices $u_1,u_2,\ldots,u_m$, denoted by $(u_1,u_2,\ldots,u_m)$, and its edge set is $\{\{u_i,u_{i+1}\}: i=1,2,\ldots,m-1\}\cup\{\{u_1,u_m\}\}$. If the edges of a λK_v can be decomposed into cycles of length m, then these cycles are called an m-cycle system, and denoted by m-CS(v,λ). An m-CS(v,λ) is said to be α-resolvable if its cycles can be partitioned into classes (called α-resolution classes) such that each point of the design occurs in precisely α cycles in each class. A 1-resolvable m-CS(v,λ) is simply called resolvable m-CS(v,λ). The existence of a resolvable m-CS(v,λ) had been solved completely.

Lemma 1.1[1] *Let λ,d,m be positive integers with $m\geq 3$. Then λK_{dm} has a resolvable m-CS(dm,λ) if and only if $\lambda(dm-1)$ is even and except the following cases:*

(1) $\lambda\equiv 2 \pmod 4$, $d=2$, $m=3$;
(2) λ *odd*, $d=2$, $m=3$;
(3) $\lambda=1$, $d=4$, $m=3$.

When $m=3$, the existence of an α-resolvable 3-CS(v,λ) had also been solved.

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Lemma 1.2[2] *An α-resolvable* 3-CS(v,λ) *exists if and only if*

$$\lambda(v-1)\equiv 0 \pmod 2,\quad \lambda v(v-1)\equiv 0 \pmod 6,\quad 3|\alpha v, \alpha|\frac{\lambda(v-1)}{2},$$

and $(v,\alpha,\lambda)\notin\{(6,1,4i+2): i\geq 0\}$.

The purpose of this paper is to investigate the existence of α-resolvable 4-CS$(v,\lambda)s$. The necessary conditions for the existence of such a design are:

$$4|\frac{\lambda v(v-1)}{2}, 2|\lambda(v-1),\quad 4|\alpha v, \alpha|\frac{\lambda(v-1)}{2}. \qquad (*)$$

From condition $(*)$, we can derive minimum values for α and λ, and call them α_0 and λ_0. Similarly to the Lammas 2.1–2.3 in [3], we have the following lammas.

Lemma 1.3 *If an α-resolvable* 4-CS(v,λ) *exists, then* $\alpha_0|\alpha$, $\lambda_0|\lambda$.

Lemma 1.4 *If an α-resolvable* 4-CS(v,λ) *exists, then a $t\alpha$-resolvable* 4-CS$(v,n\lambda)$ *exists for any positive integers n,t with* $t|\frac{\lambda(v-1)}{2\alpha}$.

Lemma 1.5 *If an α_0-resolvable* 4-CS(v,λ_0) *exists, and α,λ satisfy condition $(*)$, then an α-resolvable* 4-CS(v,λ) *exists.*

Thus, in order to show the necessary condition $(*)$ for the existence of α-resolvable 4-CS$(v,\lambda)s$ is also sufficient, we only need to prove the existence of α_0-resolvable 4-CS$(v,\lambda_0)s$.

2. Direct constructions

In order to get the existence of α-resolvable 4-CS$(v,\lambda)s$, we need some definitions and marks. Let m,v be positive integers and ∞ an infinite point. Let Z_v be the residue ring of integers modulo v. Denote $Z_v^*=Z_v\backslash\{0\}$. Let $\mathcal{C}$ be a set of cycles of length m which are constructed on Z_v or $Z_v\cup\{\infty\}$. For each cycle $C=(c_1,c_2,\ldots,c_m)$ and $j\in Z_v$, define $C+j$ to be $(c_1+j,c_2+j,\ldots,c_m+j)$ where $\infty+j=\infty$ if $\infty\in C$. Denote $\mathcal{C}+j=\{C+j: C\in\mathcal{C}\}$ for $j\in Z_v$. The differences of a cycle $C=(c_1,c_2,\ldots,c_m)$ mean $\pm(c_2-c_1),\pm(c_3-c_2),\ldots,\pm(c_m-c_{m-1}),\pm(c_1-c_m)$, where $\infty-j=j-\infty=\infty$ for any $j\in Z_v$.

In what follows, we will get α-resolvable 4-CS$(v,\lambda)s$ through direct constructions. According to condition $(*)$, α_0 and λ_0 are as follows

$$\begin{cases}\alpha_0=1,\ \lambda_0=2,\ \ v\equiv 0 \pmod 4,\\ \alpha_0=4,\ \lambda_0=1,\ \ v\equiv 1 \pmod 8,\\ \alpha_0=4,\ \lambda_0=2,\ \ v\equiv 5 \pmod 8,\\ \alpha_0=2,\ \lambda_0=4,\ \ v\equiv 2 \pmod 4,\\ \alpha_0=4,\ \lambda_0=4,\ \ v\equiv 3 \pmod 4.\end{cases}$$

Lemma 2.1 *There exists a resolvable* 4-CS$(v,2)$ *for* $v\equiv 0 \pmod 4$.

Proof Since $\alpha_0=1$, the conclusion follows from Lemma 1.1. □

Lemma 2.2 *There exists a 4-resolvable* 4-CS$(v,1)$ *for* $v\equiv 1 \pmod 8$.

Proof Let the point set $X = Z_{8k+1}$, $k > 0$. A 4-resolvable 4-CS$(v,1)$ contains $\frac{\lambda v(v-1)}{2m} = (8k+1) \times k$ cycles and $\frac{\lambda(v-1)}{2\alpha} = k$ 4-resolution classes. Let $\mathcal{C}$ consist of the following k cycles:

$$\begin{array}{llll}
(\ 1, & 0, & 2k, & 4k+1), \\
(\ 2, & 0, & 2k-1, & 4k+1), \\
(\ 3, & 0, & 2k-2, & 4k+1), \\
\ \vdots & & \ \vdots & \ \vdots \\
(k-1, & 0, & k+2, & 4k+1), \\
(\ k, & 0, & k+1, & 4k+1).
\end{array}$$

It is easy to check that the differences of all cycles of $\mathcal{C}$ give every value of Z^*_{8k+1} exactly once, which implies that $\{\mathcal{C} + i : i \in Z_{8k+1}\}$ forms a 4-CS$(v,1)$. In addition, for every $C \in \mathcal{C}$, $\{C + i : i \in Z_{8k+1}\}$ is a 4-resolution class of the 4-CS$(v,1)$. So, we derive a 4-resolvable 4-CS$(v,1)$. □

Lemma 2.3 *There exists a 4-resolvable* 4-CS$(v,2)$ *for* $v \equiv 5 \pmod 8$.

Proof Let the point set $X = Z_{8k+5}$, $k \geq 0$. A 4-resolvable 4-CS$(v,2)$ contains $(8k+5)\times(2k+1)$ cycles and $2k+1$ 4-resolution classes. Let $\mathcal{C}$ consist of the following $2k+1$ cycles:

Part 1: Construct k cycles and repeat them twice:

$$\begin{array}{llll}
(\ 1, & 0, & 2, & 4k+3\ ), \\
(\ 3, & 0, & 4, & 4k+3\ ), \\
(\ 5, & 0, & 6, & 4k+3\ ), \\
\ \vdots & & \ \vdots & \ \vdots \\
(2k-3, & 0, & 2k-2, & 4k+3\ ), \\
(2k-1, & 0, & 2k, & 4k+3\ ).
\end{array}$$

Part 2: Construct 1 cycle:

$$(2k+1, \ 0, \ 2k+2, \ 4k+3\ ).$$

Since the differences of all cycles of $\mathcal{C}$ give every value of Z^*_{8k+5} exactly twice, $\{\mathcal{C}+i : i \in Z_{8k+5}\}$ forms a 4-CS$(v,2)$. On the other hand, for every $C \in \mathcal{C}$, $\{C + i : i \in Z_{8k+5}\}$ is a 4-resolution class of the 4-CS$(v,2)$. Hence, we get a 4-resolvable 4-CS$(v,2)$. □

Lemma 2.4 *There exists a 2-resolvable* 4-CS$(v,4)$ *for* $v \equiv 2 \pmod 4$.

Proof Let the point set $X = Z_{4k+1} \cup \{\infty\}, k > 0$. A 2-resolvable 4-CS$(v,4)$ contains $(4k+1)(2k+1)$ cycles and $4k+1$ 2-resolution classes. Let $\mathcal{C}$ consist of the following $2k+1$ cycles:

Part 1: Construct k cycles:

$$
\begin{array}{llll}
(\; k, & 3k-1, & 3k, & k-2 \;), \\
(k+1, & 3k-2, & 3k+1, & k-3 \;), \\
(k+2, & 3k-3, & 3k+2, & k-4 \;), \\
\;\vdots & & \vdots & \;\vdots \\
(2k-2, & 2k+1, & 4k-2, & \;0 \quad), \\
(2k-1, & 2k, & 4k-1, & 4k \quad).
\end{array}
$$

Part 2: Construct $k-1$ cycles:

$$
\begin{array}{llll}
(\; 1, & 2k-1, & 2k+1, & 4k-1), \\
(\; 2, & 2k-2, & 2k+2, & 4k-2), \\
(\; 3, & 2k-3, & 2k+3, & 4k-3), \\
\;\vdots & & \vdots & \;\vdots \\
(k-2, & k+2, & 3k-2, & 3k+2), \\
(k-1, & k+1, & 3k-1, & 3k+1).
\end{array}
$$

Part 3: Construct 2 cycles:

$$(\infty, \ 0, \ 2k, \ 4k), \ (\infty, \ 3k, \ k, \ k-1).$$

The differences of all cycles of $\mathcal{C}$ give every value of $Z_{4k+1}^{*} \cup \{\infty\}$ exactly biquadratic, so $\{\mathcal{C}+i : i \in Z_{4k+1}\}$ forms a 4-CS$(v,4)$. Furthermore, $\mathcal{C}$ is a 2-resolution class of the 4-CS$(v,4)$, and $\mathcal{C}, \mathcal{C}+1, \ldots, \mathcal{C}+4k$ are all 2-resolution classes. So, a 2-resolvable 4-CS$(v,4)$ is given. □

Lemma 2.5 *There exists a 4-resolvable* 4-CS$(v,4)$ *for* $v \equiv 3 \pmod 4$.

Proof (1) $v \equiv 3 \pmod 8$. Let the point set $X = Z_{8k+3}$, $k > 0$. A 4-resolvable 4-CS$(v,4)$ contains $(8k+3)(4k+1)$ cycles and $4k+1$ 4-resolution classes. Let $\mathcal{C}$ consist of the following $4k+1$ cycles:

Part 1: Construct $2k$ cycles:

$$
\begin{array}{llll}
(\quad 1, & 0, & 2, & \;\; 5 \quad), \\
(\quad 5, & 0, & 6, & \;\; 13 \quad), \\
(\quad 9, & 0, & 10, & \;\; 21 \quad), \\
\;\;\vdots & & \vdots & \;\;\vdots \\
(\, 8k-7, & 0, & 8k-6, & 8k-14 \;), \\
(\, 8k-3, & 0, & 8k-2, & 8k-6 \quad).
\end{array}
$$

Part 2: Construct $2k$ cycles:

$$
\begin{array}{llll}
(\quad 3, & 0, & 4, & \;\; 9 \quad), \\
(\quad 7, & 0, & 8, & \;\; 17 \quad),
\end{array}
$$

$$\begin{array}{llll}
(\ 11, & 0, & 12, & 25 \ ), \\
\vdots & & \vdots & \vdots \\
(8k-5, & 0, & 8k-4, & 8k-10 \ ), \\
(8k-1, & 0, & 8k, & 8k-2 \ ).
\end{array}$$

Part 3: Construct 1 cycle:

$$(1, \ 0, \ 2, \ 3).$$

It is easy to check that $\{\mathcal{C}+i : i \in Z_{8k+3}\}$ forms a 4-CS$(v, 4)$. In addition, for every $C \in \mathcal{C}$, $\{C+i : i \in Z_{8k+3}\}$ is a 4-resolution class of the 4-CS$(v, 4)$. So we derive a 4-resolvable 4-CS$(v, 4)$.

(2) $v \equiv 7 \pmod 8$. Let the point set $X = Z_{8k+7}$, $k \geq 0$. A 4-resolvable 4-CS$(v, 4)$ contains $(8k+7)(4k+3)$ cycles and $4k+3$ 4-resolution classes. Let $\mathcal{C}$ consist of the following $4k+3$ cycles:

Part 1: Construct k cycles and repeat them biquadratic:

$$\begin{array}{llll}
(\ 1, & 0, & 2, & 4k+4 \ ), \\
(\ 3, & 0, & 4, & 4k+4 \ ), \\
(\ 5, & 0, & 6, & 4k+4 \ ), \\
\vdots & & \vdots & \vdots \\
(2k-3, & 0, & 2k-2, & 4k+4 \ ), \\
(2k-1, & 0, & 2k, & 4k+4 \ ).
\end{array}$$

Part 2: Construct 1 cycle and repeat them twice:

$$(2k+1, \ 0, \ 2k+2, \ 4k+4 \ ).$$

Part 3: Construct 1 cycle:

$$(2k+1, \ 0, \ 2k+3, \ 4k+4 \ ).$$

It is easy to check that $\{\mathcal{C}+i : i \in Z_{8k+7}\}$ forms a 4-CS$(v, 4)$. In addition, for every $C \in \mathcal{C}$, $\{C+i : i \in Z_{8k+7}\}$ is a 4-resolution class of the 4-CS$(v, 4)$. Therefore, we get a 4-resolvable 4-CS$(v, 4)$. □

3. Main result

Combining Lemmas 2.1–2.5, we obtain the main result:

Theorem 3.1 *There exists an α-resolvable* 4-CS(v, λ) *if and only if*

$$4 | \frac{\lambda v(v-1)}{2}, 2 | \lambda(v-1), 4 | \alpha v, \alpha | \frac{\lambda(v-1)}{2}.$$

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