Journal of Mathematical Research & Exposition
Nov., 2009, Vol. 29, No. 6, pp. 1114–1116
DOI:10.3770/j.issn:1000-341X.2009.06.022
Http://jmre.dlut.edu.cn

Minimal Hausdorff Measure of the Scattered Cantor Sets

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Abstract In this paper, we construct a scattered Cantor set having the value $\frac{1}{2}$ of $\frac{\log 2}{\log 3}$-dimensional Hausdorff measure. Combining a theorem of Lee and Baek, we can see the value $\frac{1}{2}$ is the minimal Hausdorff measure of the scattered Cantor sets, and our result solves a conjecture of Lee and Baek.

Keywords scattered Cantor set; minimal Hausdorff measure.

Document code A
MR(2000) Subject Classification 28A78
Chinese Library Classification O174.12

1. Introduction

We start with the definition of the scattered Cantor sets. Let E_0 be the interval $[0,1]$. Take two non-overlapping intervals of length 3^{-1} from E_0 and let E_1 be the union of these 2 intervals. Take two non-overlapping intervals of length 3^{-2} from each fundamental interval of E_1 and let E_2 be the union of these 2^2 intervals. We continue in this way, with E_k obtained by taking two non-overlapping intervals of length 3^{-k} from each fundamental interval of E_{k-1}. Thus E_k consists of 2^k intervals of length 3^{-k}. Let

$$F = \bigcap_{k=0}^{\infty} E_k.$$

The set F is called a scattered Cantor set.

The middle-third Cantor set is clearly a scattered Cantor set. Varying the positions of the fundamental intervals, one can easily construct many different scattered Cantor sets.

Let $s = \log 2/\log 3$. It is known that the scattered Cantor sets all have Hausdorff dimension s, with s-dimensional Hausdorff measures at most 1. As the s-dimensional Hausdorff measure of the middle-third Cantor set is 1, it is one of the biggest scattered Cantor sets in the sense of Hausdorff measure. Lee and Baek[2] proved $\mathcal{H}^s(F) \geq \frac{1}{2}$ for all scattered Cantor sets F. They conjectured that there are scattered Cantor sets of Hausdorff measure

$$\mathcal{H}^s(F) = \frac{1}{2}.$$

Received date: 2007-08-22; **Accepted date**: 2008-05-21
Foundation item: the National Natural Science Foundation of China (No. 10771164); the Education Committee of Fujian Province (No. JA08155).

In this paper, we shall construct a scattered Cantor set E and prove that its s-dimensional Hausdorff measure is $\frac{1}{2}$.

We recall that[1] the s-dimensional Hausdorff measure of F is defined by

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F),$$

where

$$\mathcal{H}^s_\delta(F) = \inf\{\sum_{n=1}^{\infty} |U_n|^s : \{U_n\}_{n=1}^{\infty} \text{is a } \delta\text{-cover of } F\}$$

and the Hausdorff dimension of F is defined by

$$\dim_H(F) = \sup\{s > 0 : \mathcal{H}^s(F) = \infty\} = \inf\{s > 0 : \mathcal{H}^s(F) = 0\}.$$

2. Construction of E

Let I=[0,1].

Step 1. Deleting the middle third open interval of I, we get two subintervals I_0 and I_1 of length 3^{-1}. I_0 and I_1 are arranged from the left to the right. Move I_0 6^{-1} to the right and I_1 6^{-1} to the left. For the sake of the convenience, we still denote the moved intervals by I_0 and I_1 and we will all use this way to label the moved intervals in the sequel. Then I_0 joins I_1 non-overlapped and form a interval, denoted by E_1^1, with length $\frac{2}{3}$.

Step 2. Deleting the middle third open intervals of I_0 and I_1, respectively, we get four subintervals I_{00}, I_{01}, I_{10} and I_{11}. They are arranged from the left to the right and their lengths are same, namely, 3^{-2}. Move I_{00} 3^{-2} to the right and I_{11} 3^{-2} to the left. After these processes, the four subintervals join together non-overlapped and form an interval E_1^2 with length $4 \cdot 3^{-2}$.

Step 3. For every subinterval of the above interval, namely, I_{00} , I_{01}, I_{10} and I_{11}, we remove the middle third open intervals of length 3^{-3}, and get eight subintervals with length 3^{-3}. They are arranged from the left to the right and denoted by I_{000}, I_{001}, I_{010}, I_{011}, I_{100}, I_{101}, I_{110}, and I_{111}. Move I_{000} 3^{-3} to the right and I_{011} 3^{-3} to the left. After these processes, the four subintervals, I_{000}, I_{001},I_{010} and I_{011}, join together non-overlapped and form an interval with length $4 \cdot 3^{-3}$; In the same way, move I_{100} 3^{-3} to the right and I_{111} 3^{-3} to the left and we get another interval with length $4 \cdot 3^{-3}$. It is not difficult to see that the gap between the two intervals is $2 \cdot 3^{-3}$.

We call I_{ε_1}, $I_{\varepsilon_1\varepsilon_2}$ and $I_{\varepsilon_1\varepsilon_2\varepsilon_3}$ the basic interval of level-1, the basic interval of level-2 and the basic interval of level-3, respectively, where $\varepsilon_i \in \{0,1\}$. Generally, we call $I_{\varepsilon_1\varepsilon_2\varepsilon_3\cdots\varepsilon_k}$ basic interval of level-k, where $\varepsilon_i \in \{0,1\}$. Therefore, after Step 3, we get two intervals E_1^3, E_2^3, and every interval is made up of four basic interval of level-3.

Continuing the above process, after step k ($k \geq 3$), we get 2^{k-2} intervals and every interval is made up of four basic intervals of level-k with length 3^{-k}. The 2^{k-2} intervals are arranged from the left to the right and denoted by E_i^k ($i = 1, 2, 3, \ldots, 2^{k-2}$). For the sake of convenience, we only show how to get E_1^{k+1} and E_2^{k+1} from E_1^k. The other intervals, E_i^k ($i = 2, 3, \ldots, 2^{k-2}$), are treated in the same way. For E_1^k, recall that it contains four basic intervals of level-k. For each one of the four basic intervals of level-k, we delete the middle third open interval and get eight subintervals with length $3^{-(k+1)}$. Then we treat the eight subintervals as in Step 3, and get two new intervals, namely, E_1^{k+1} and E_2^{k+1}.

Let

$$E = \bigcap_{k=3}^{\infty} \bigcup_{i=1}^{2^{k-2}} E_i^k.$$

Then E is exactly the scattered Cantor set which we need. By a simple calculation, we can see that the diameter of E is $\frac{1}{3}$.

Theorem 1 *Let E be defined as the above. Then $\mathcal{H}^s(E) \leq \frac{1}{2}$, where $s = \frac{\log 2}{\log 3}$.*

3. Proof of Theorem 1

Proof For any $\delta > 0$, $\varepsilon > 0$. We can choose $k \in \mathbb{N}$ ($k \geq 3$) such that $4 \cdot 3^{-k} < \delta$, then choose $n \in \mathbb{N}$ such that $2^{-n-2} < \varepsilon$. First, we note that every interval E_i^k ($i = 1, 2, 3, \ldots, 2^{k-2}$) is decomposed into 2^n intervals $E^{(k+n)}$ with length $4 \cdot 3^{-(k+n)}$ at step $k+n$. They are arranged from the left to the right and denoted by $E_{i,1}^{k+n}, E_{i,2}^{k+n}, \ldots, E_{i,2^n}^{k+n}$ (So, at step $k+n$, we get 2^{k+n-2} intervals with length $4 \cdot 3^{-(k+n)}$ in all). Let $a(E_{i,1}^{k+n})$ be the left endpoint of $E_{i,1}^{k+n}$, $b(E_{i,2^n}^{k+n})$ be the right endpoint of $E_{i,2^n}^{k+n}$ ($i = 1, 2, 3, \ldots, 2^{k-2}$). By a simple calculation, we have

$$\begin{aligned} b(E_{i,2^n}^{k+n}) - a(E_{i,1}^{k+n}) &= 4 \cdot 3^{-k} - 2(3^{-(k+1)} + \cdots + 3^{-(k+n)}) \\ &= 3^{-k+1} + 3^{-(n+k)} < \delta. \end{aligned}$$

Obviously,

$$E \subset \bigcup_{i=1}^{2^{k-2}} [a(E_{i,1}^{k+n}), b(E_{i,2^n}^{k+n})].$$

So we can regard $\{[a(E_{i,1}^{k+n}), b(E_{i,2^n}^{k+n})]\}_{i=1}^{2^{k-2}}$ as a δ-cover of E.

By the definition of $\mathcal{H}^s$ and note that $(a+b)^s \leq a^s + b^s$ when $a \geq 0, b \geq 0$ and $0 \leq s \leq 1$, we have

$$\begin{aligned} \mathcal{H}_\delta^s(E) &\leq 2^{(k-2)}(3^{-k+1} + 3^{-(n+k)})^s \leq 2^{(k-2)}((3^{-k+1})^s + (3^{-n-k})^s) \\ &\leq 2^{(k-2)}(2^{-k+1} + 2^{-k-n}) = 2^{-1} + 2^{-n-2} < 2^{-1} + \varepsilon. \end{aligned}$$

Letting $\delta \to 0$, we have $\mathcal{H}^s(E) \leq \frac{1}{2} + \epsilon$. By the arbitrariness of ε, we get $\mathcal{H}^s(E) \leq \frac{1}{2}$.

Remark By Theorem 1 and the result of Lee and Baek, we affirm the conjecture of Lee and Baek.

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