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On Singularity of Spline Space Over Morgan-Scott's Type Partition

Zhong Xuan LUO[1,*], **Feng Shan LIU**[2], **Xi Quan SHI**[2]
1. *School of Mathematical Sciences, Dalian University of Technology, Liaoning* 116024*, P. R. China;*
2. *Applied Mathematics Research Center, Delaware State University, Dover* 19901*, U. S. A*

Abstract Multivariate spline function is an important research object and tool in Computational Geometry. The singularity of multivariate spline spaces is a difficult problem that is ineritable in the research of the structure of multivariate spline spaces. The aim of this paper is to reveal the geometric significance of the singularity of bivariate spline space over Morgan-Scott type triangulation by using some new concepts proposed by the first author such as characteristic ratio, characteristic mapping of lines (or ponits), and characteristic number of algebraic curve. With these concepts and the relevant results, a polished necessary and sufficient conditions for the singularity of spline space $S^{\mu}_{\mu+1}(\Delta^{\mu}_{MS})$ are geometrically given for any smoothness μ by recursion. Moreover, the famous Pascal's theorem is generalized to algebraic plane curves of degree $n \geq 3$.

Keywords singularity of spline space; Morgan-Scott's partition; planar algebraic curve; characteristic ratio; characteristic mapping; characteristic number.

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1. Introduction

Algebraic geometry is the theoretical foundation of multivariate polynomial splines and multivariate interpolation by polynomial. The definition of multivariate spline is stated as follows [28]: Let Ω be a given planar polygonal region and Δ be a triangulation or partition of Ω. The linear space

$$S^{\mu}_{k}(\Delta) := \{s \mid s|_{T_i} \in \mathbb{P}_k, s \in C^{\mu}(\Omega), \forall T_i \in \Delta\}$$

is called the spline space of degree k with smoothness μ, where T_i is a cell of the Δ and $\mathbb{P}_{\mathbf{k}}$ is the polynomial space of total degree less than or equal to k.

Previous research has shown that there is an equivalent relationship between the study of the intrinsic properties of planar algebraic curves and the singularity of bivariate spline spaces. It thus leads to the possibility of studying intrinsic properties of plane curves by the spline method.

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* Corresponding author
E-mail address: luozx9609@vip.sina.com (Z. X. LUO)

The singularity of multivariate spline spaces is an important object that is inevitable in the research of the structure of multivariate spline spaces. Morgan and Scott [20] pointed out that the dimension of the multivariate spline space depends not only on the topological property of its partition, but also heavily on the geometric property of the partition. Although the singularity of multivariate spline over any triangulation has not been completely settled to date, many achievements concerning the structure of multivariate spline space can be found in many of references published in the past 30 years [1–12, 16, 21, 22, 27, 28]. For Morgan-Scott's triangulation, Shi [23] and Diener [12] independently obtained the geometric significance of the necessary and sufficient condition of $\dim(S_2^1(\Delta_{MS})) = 7$, respectively. Du [13] obtained another equivalent geometric necessary and sufficient condition of singularity of $S_2^1(\Delta_{MS})$ from the viewpoint of the projective geometry. More precisely, if the six quasi-inner edges are regarded as six points in the projective plane, then they lie on a conic. Obviously, the equivalence of the results obtained by Shi and Du is sustained by Pascal's Theorem: If a hexagon is inscribed in a conic in the project plane, then the opposite sides of the hexagon meet in collinear points. The equivalent relationship of the results to the singularity of $S_2^1(\Delta_{MS})$ can be interpreted clearly in table 1.

$S_2^1(\Delta_{MS})$ is singular	$\Longleftrightarrow$	$\sum_{i=1}^{6} \lambda_i l_i^2(x, y, z) = 0$ has nonzero solutions
$\Updownarrow$		$\Updownarrow$
$\frac{k_3k_4}{k_1k_2} \cdot \frac{k_7k_8}{k_5k_6} \cdot \frac{k_{11}k_{12}}{k_9k_{10}} = 1$[15,24] Aa, Bb, Cc concurrent[23]	Pascal $\Longleftrightarrow$ Theorem	$l_i(i = 1, 2, \ldots, 6)$ lie on a conic

Table 1 The equivalent relation of the singularity

To make an intensive study of the singularity of multivariate spline spaces over triangulations, Luo & Chen [17] investigated the singularity of the space $S_{\mu+1}^{\mu}(\Delta_{MS}^{\mu})$ $(\mu \geq 2)$ and gave out an algebraic necessary and sufficient condition to the singularity. Taking $\mu = 2$ for instance, it follows from Luo & Wang's results [16] that an equivalent geometric condition to the singularity of $S_3^2(\Delta_{MS}^2)$ is: the 9 quasi-inner edges of the triangulation Δ_{MS}^2 (regarded as points on the projective plane) lie on a "cubic". Hence, it is reasonable to believe that there must be some profound mathematical relationship between the algebraic and geometric conditions as Pascal's theorem in the case of $\mu = 1$.

It is necessary to observe Pascal's theorem in a different way. Pascal's theorem implies that any given six points on a conic determine three points through simple linear mapping (from intersection points of some lines) and the three points lie on an algebraic curve of degree of 1 (instead of "collinear"). The following extended problem is spontaneously proposed: is there a homothetic result in the case of plane curves of degree 3? or are there 6 points inscribed in a conic determined by any 9 or 10 points on a cubic through simple linear mapping (intersections of some lines or other simple mapping)? How about the cases of plane curves of higher degrees? Of cause, it is not difficult to prove that the above problems are invalid for general cases. Whereas what kind of characters does a cubic has if the cubic preserves similar to Pascal's theorem? We believe that the conclusion to this problem results a direct generalization of Pascal's theorem.

The necessary and sufficient condition in algebraic form to the space $S^{\mu}_{\mu+1}(\Delta^{\mu}_{MS})$ $(\mu \geq 2)$ derived by Luo & Chen [17] naturally leads to some important concepts in algebraic geometry– Characteristic ratio of points on a line, Characteristic mapping, and Characteristic number of plane curves. [18] discovered that the characteristic number is an invariant of plane curves in a subspace of polynomial spaces known as Pascal space in the projective plane, and the characteristic mapping is a transformation which preserves some geometric properties or quantity to plane curves in Pascal space. By introducing the characteristic mapping and the characteristic number, another intrinsic property (Pascal's type Theorem) of a class of plane curves of degree $n \geq 3$ is proved.

The following definition is necessary to our discussion. Let $\mathbb{P}^2$ be the projective plane, and $\{\phi_1(x,y,z), \phi_2(x,y,z), \dots, \phi_N(x,y,z)\}$ be linearly independent functions over $\mathbb{P}^2$. The following set of points

$$\Gamma_{(N)} := \{(x,y,z) | \sum_{i=0}^{N} c_i \phi_i(x,y,z) = 0, c_i \in \mathbb{C}\}$$

is called a planar algebraic curve in $\mathbb{V} := \mathrm{span}\{\phi_1(x,y,z), \phi_2(x,y,z), \dots, \phi_N(x,y,z)\}$, where $\mathbb{C}$ is the complex field. The following point set, determined by

$$\Gamma_n := \{(x,y,z) | \sum_{i+j+k=n} a_{ijk} x^i y^j z^k = 0, \sum_{i+j+k=n} a_{ijk}^2 \neq 0\}, \quad a_{ijk} \in \mathbb{C}$$

is called a plane algebraic curve of degree n in the projective plane $\mathbb{P}^2$. Denoted by $\mathbb{P}_n$ the homogeneous polynomial space of total degree $\leq n$. In what follows we use the term plane curve always for plane algebraic curve in the projective plane.

This paper is organized as follows: In Section 2, we introduce and list the basic methods and some known results of multivariate spline functions related to the discussion of the paper. In Section 3, the concepts of characteristic ratio of points (or lines) on a line (or at point), characteristic mapping, and characteristic number of plane curves in the projective geometry are introduced and obtained. In Section 4, we obtained our main results in the geometric significance of the singularity in $S^{\mu}_{\mu+1}(\Delta^{\mu}_{MS})(\mu \geq 2)$. We also obtained a generalized Pascal's theorem and a geometric invariant of algebraic curves via the duality principle in Section 4.

2. Singularity of multivariate spline space and the relevant results

It is well known that spline is an important approximation tool in computational geometry, and it is widely used in CAGD, scientific computations and many engineering fields. Splines, i.e., piecewise polynomials, forms linear spaces that have a very simple structure in univariate case. However, the bivariate case is very complicated to determine the structure of a space of bivariate spline over arbitrary triangulation.

Bivariate spline is defined as follows [28]:

Definition 2.1 *Let Ω be a given planar polygonal region and Δ be a triangulation or partition of Ω, Denoted by $T_i, i = 1, 2, \dots, V$, all cells of Δ. For integer $k > \mu \geq 0$, the linear space*

$$S_k^\mu(\Delta) := \{s \mid s|_{T_i} \in \mathbb{P}_k, s \in C^\mu(\Omega), \forall T_i \in \Delta\}$$

is called the spline space of degree k with smoothness μ, where $\mathbb{P}_{\mathbf{k}}$ is the polynomial space of total degree less than or equal to k.

From the Smoothing Cofactor method [28], the fundamental theorem on bivariate splines was established.

Theorem 2.2 *$s(x, y) \in S_k^\mu(\Delta)$ if and only if the following conditions are satisfied:*

1) For each interior edge of Δ, which is defined by $\Gamma_i : l_i(x, y) = 0$, there exists a so-called smoothing cofactor $q_i(x, y)$, such that

$$p_{i1}(x, y) - p_{i2}(x, y) = l_i^{\mu+1}(x, y) q_i(x, y),$$

where the polynomials $p_{i1}(x, y)$ and $p_{i2}(x, y)$ are determined by the restriction of $s(x, y)$ on the two cells Δ_{i1} and Δ_{i2} with Γ_i as the common edge and $q_i(x, y) \in \mathbb{P}_{k-(\mu+1)}$.

2) For any interior vertex v_j of Δ, the following conformality conditions are satisfied

$$\sum [l_i^{(j)}(x, y)]^{\mu+1} q_i^{(j)}(x, y) \equiv 0, \tag{1}$$

where the summation is taken on all interior edges $\Gamma_i^{(j)}$ passing through v_j, and the sign of the smoothing cofactors $q_i^{(j)}$ are refixed in such a way that when a point crosses $\Gamma_i^{(j)}$ from Δ_{i1} to Δ_{i2}, it goes around v_j counter-clockwisely.

From Theorem 2.2, the dimension of the space $S_k^\mu(\Delta)$ can be expressed as

$$\dim S_k^\mu(\Delta) = \binom{k+2}{2} + \tau,$$

where τ is the dimension of the linear space defined by the conformality conditions (1).

However, for arbitrary given triangulation, the dimension of these spaces depends not only on the topology of the triangulation, but also on the geometry of the triangulation. In general case, no dimension formula is known. We say that a triangulation is singular to $S_k^\mu(\Delta)$ if the dimension of the spline space depends not only on the topology of the triangulation, but also on the geometric position of the vertices of Δ, and $S_k^\mu(\Delta)$ is singular when its dimension increases according to the geometric property of Δ. Hence, the singularity of multivariate spline spaces is an important object that is impossible to avoid in the research of the structure of multivariate spline spaces. For example, Morgan and Scott's triangulation Δ_{MS} ([20], see Figure 1.1) is a singular to $S_2^1(\Delta_{MS})$. For Morgan-Scott's triangulation, Shi [23] and Diener [12] independently obtained the geometric significance of the necessary and sufficient condition of $\dim(S_2^1(\Delta_{MS})) = 7$, respectively. Du [13] obtained another equivalent geometric necessary and sufficient condition of singularity of $S_2^1(\Delta_{MS})$ from the viewpoint of the projective geometry.

2.1 Singularity of $S_2^1(\Delta_{MS})$ and Pascal's Theorem

In Morgan-Scott's unpublished manuscripts of the 70's of last century, it was found that the dimension of spline spaces with degree 2 and smoothness 1 over the triangulation shown in Figure 3.1 heavily depends on the geometric property of the partition. Chou-Su-Wang [9] pointed out that the singularity of Morgan-Scott triangulation for $S_2^1(\Delta_{MS})$ does not need to be a symmetric partition and they obtained a sufficient condition for the dimension to be seven. Shi [23] and Diener [12] independently obtained the geometric necessary and sufficient condition of $\dim S_2^1(\Delta_{MS}) = 7$.

Theorem 2.3 ([23]) *The spline space $S_2^1(\Delta_{MS})$ is singular ($\dim S_2^1(\Delta_{MS}) = 7$) if and only if Aa, Bb, Cc are concurrent, otherwise $\dim S_2^1(\Delta_{MS}) = 6$ (see Figure 2.1).*

From the algebraic geometry viewpoint, Du [13] proved another type of equivalent condition.

Theorem 2.4 ([13]) *The spline space $S_2^1(\Delta_{MS})$ is singular ($\dim S_2^1(\Delta_{MS}) = 7$) if and only if 6 points lie on a conic when we regard $l_i(x, y) = 0$ $(i = 1, 2, \ldots, 6)$ as points in the projective space, otherwise $\dim S_2^1(\Delta_{MS}) = 6$.*

The following definition of a duality of figures consisting of lines and points in the projective plane is useful for our discussion.

Definition 2.5 (Duality of planar figure) *Let Δ be a planar figure consists of lines and points in the projective plane. If the points and lines of Δ are regarded as lines and points respectively and are drawn in the projective plane, then the new graphics from Δ is called Duality of Δ, denoted by Δ^*.*

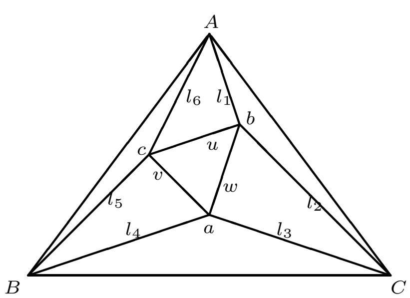


Figure 2.1 Morgan-Scott's partition Δ_{MS}

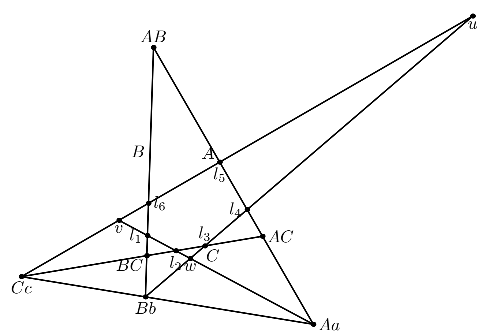


Figure 2.2 Duality of triangulation Δ_{MS}^*

Figure 2.2 shows the duality of Morgan-Scott's triangulation Δ_{MS}. Obviously, the Pascal's theorem plays pivotal role in the equivalence between Theorems 2.3 and 2.4.

This problem can also be solved via the Generator Basis method by Luo [15]. Let

$$\begin{cases} l_1 = a_1 u + b_1 w \\ l_2 = a_2 u + b_2 w \end{cases}, \begin{cases} l_3 = a_3 w + b_3 v \\ l_4 = a_4 w + b_4 v \end{cases} \text{ and } \begin{cases} l_5 = a_5 v + b_5 u \\ l_6 = a_6 v + b_6 u \end{cases}. \tag{2}$$

Then we have the following conclusion in algebraic form:

Theorem 2.6 ([15, 23]) *The spline space $S_2^1(\Delta_{MS})$ is singular ($\dim S_2^1(\Delta_{MS}) = 7$) if and only if*

$$\frac{b_1 b_2}{a_1 a_2} \cdot \frac{b_3 b_4}{a_3 a_4} \cdot \frac{b_5 b_6}{a_5 a_6} = 1. \tag{3}$$

It is evident that the result of Theorem 2.6 is also equivalent to the conclusions of Theorems 2.3 and 2.4.

2.2 Singularities of $S_3^2(\Delta_{MS}^2)$ and $S_{\mu+1}^{\mu}(\Delta_{MS}^{\mu})$ ($\mu \geq 3$)

The singularity of the space $S_3^2(\Delta_{MS}^2)$ (see Figure 2.3) was investigated by Luo & Wang [16] using the Generator Basis method. They obtained a necessary and sufficient condition in algebraic form and two sufficient geometric conditions for the singularity as follows: Let

$$\begin{cases} l_1 = a_1 u + b_1 w \\ l_2 = a_2 u + b_2 w \\ l_3 = a_3 u + b_3 w \end{cases}, \begin{cases} l_4 = a_4 w + b_4 v \\ l_5 = a_5 w + b_5 v \\ l_6 = a_6 w + b_6 v \end{cases} \text{ and } \begin{cases} l_7 = a_7 v + b_7 u \\ l_8 = a_8 v + b_8 u \\ l_9 = a_9 v + b_9 u \end{cases}. \tag{4}$$

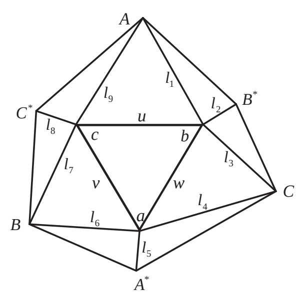


Figure 2.3 Morgan-Scott's type triangulation Δ_{MS}^2

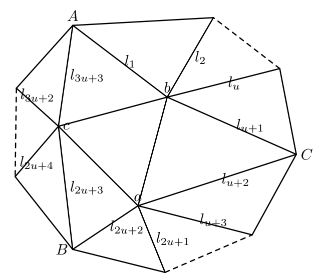


Figure 2.4 Morgan-Scott's type triangulation Δ_{MS}^{μ}

Then, the following conclusion in algebraic form holds.

Theorem 2.7 ([16]) *The spline space $S_3^2(\Delta_{MS}^2)$ is singular ($\dim S_3^2(\Delta_{MS}^2) = 11$) if and only if*

$$\frac{a_1 a_2 a_3}{b_1 b_2 b_3} \cdot \frac{a_4 a_5 a_6}{b_4 b_5 b_6} \cdot \frac{a_7 a_8 a_9}{b_7 b_8 b_9} = -1. \tag{5}$$

To get the geometric significance of (5) in Theorem 2.7, Luo & Wang [16] analyzed it using a similar method as in [13]. Denoted by $l_i : \alpha_i x + \beta_i y + \gamma_i z = 0, i = 1, 2, \ldots, 9$ in Δ_{Ms}^2 partition, λ_i ($i = 1, 2, \ldots, 9$) the corresponding smoothing cofactors and let $p_i = (\alpha_i, \beta_i, \gamma_i)$, $i = 1, 2, \ldots, 9$. Then $\dim S_3^2(\Delta_{MS}^2) = 11$ if and only if there exists nonzero solution of equation:

$$\sum_{i=1}^{9} \lambda_i l_i^3(x, y, z) = 0. \tag{6}$$

Hence, if we regard $p_i = (\alpha_i, \beta_i, \gamma_i)$ ($i = 1, 2, \ldots, 9$) as points in the projective space, then $p_i = (\alpha_i, \beta_i, \gamma_i)$ ($i = 1, 2, \ldots, 9$) lie on a cubic plane curve in the duality partition. Denote by $\overline{P_3}$ the cubic polynomial subspaces spanned by any nine monomials of $\{x^3, y^3, z^3, x^2y, xy^2, y^2z, yz^2, x^2z, xz^2, xyx\}$. Then the necessary and sufficient condition for nonzero solution of (6) to exist can be represented as:

Theorem 2.8 ([16]) *The spline space $S_3^2(\Delta_{MS}^2)$ is singular ($\dim S_3^2(\Delta_{MS}^2) = 11$) if and only if $p_i = (\alpha_i, \beta_i, \gamma_i)$ $(i = 1, 2, \ldots, 9)$ lie on a plane curve in $\overline{P_3}$ space.*

The following corollaries are direct results of Theorems 2.6 and 2.7.

Corollary 2.9 *Any $p_i = (\alpha_i, \beta_i, \gamma_i)$ $(i = 1, 2, \ldots, 9)$ of the form (5) lie on a plane curve in $\overline{P_3}$ if and only if (6) holds.*

Corollary 2.10 *If Aa, Bb, Cc in Δ_{Ms}^2 are concurrent, then the triangulation Δ_{MS}^2 is singular for S_3^2 if and only if A^*a, B^*b, C^*c are concurrent.*

Corollary 2.11 *If Aa, Bb, Cc in Δ_{Ms}^2 are not concurrent, and A^*a, B^*b, C^*c are prolongative lines of Aa, Bb, Cc, respectively, then the triangulation Δ_{MS}^2 is singular for S_3^2.*

For a general case of $\mu \geq 3$, Luo & Chen [17] gave the equivalent condition in an algebraic form to the singularity of $S_{\mu+1}^{\mu}(\Delta_{MS}^{\mu})$ $(\mu \geq 3)$ as follows: for a given triangulation Δ_{MS}^{μ} (see Figure 2.4), suppose

$$\begin{aligned} l_i &= a_i u + b_i w, \quad && i = 1, 2, \ldots, \mu + 1, \\ l_j &= a_j w + b_j v, && j = \mu + 2, \mu + 3, \ldots, 2\mu + 2, \\ l_k &= a_k v + b_k u, && k = 2\mu + 3, 2\mu + 4, \ldots, 3\mu + 3. \end{aligned} \tag{7}$$

Then we have the following theorem

Theorem 2.12 ([17]) *The spline space $S_{\mu+1}^{\mu}(\Delta_{MS}^{\mu})$ is singular if and only if*

$$\frac{a_1 \cdots a_{\mu+1}}{b_1 \cdots b_{\mu+1}} \cdot \frac{a_{\mu+2} \cdots a_{2\mu+2}}{b_{\mu+2} \cdots b_{2\mu+2}} \cdot \frac{a_{2\mu+3} \cdots a_{3\mu+3}}{b_{2\mu+3} \cdots b_{3\mu+3}} = (-1)^{\mu+1}. \tag{8}$$

The Corollaries 2.10 and 2.11 only provided sufficient geometric conditions to the singularity of the space $S_3^2(\Delta_{Ms}^2)$. What is the equivalent geometric condition for the singularity of $S_3^2(\Delta_{Ms}^2)$ (in general $S_{\mu+1}^{\mu}(\Delta_{MS}^{\mu})$ $(\mu \geq 3)$) which resembles Theorem 2.7? In this paper, we obtained the results to the problems and also provided some important concepts in algebraic geometry.

The following conclusion can serve to account for the intrinsic relationship of our results in the paper.

Proposition 2.13 ([19]) *For any given six points in $\mathbb{P}^2$, the six points lie on a conic if and only if the lines $\overline{AB}, \overline{BC}, \overline{CA}, \overline{DE}, \overline{EF}, \overline{FD}$, regarded as points in $\mathbb{P}^2$, lie on a conic.*

3. Characteristic ratio, characteristic mapping and characteristic number

In [18], a characteristic ratio was defined which quite differs from the cross ratio, and the concepts characteristic mapping and characteristic number were introduced to study intrinsic properties of algebraic plane curve. In what follows, we will use $u = \langle a, b \rangle$ for the intersection point of lines a and b, and $a = (u, v)$ for the line determined by the points u and v.

Definition 3.1 (characteristic ratio [18]) *Let $u, v \in \mathbb{P}^2$ be two lines (or points), $l_1, l_2, \ldots, l_k$ be distinct lines (or points) passing through $\langle u, v\rangle$ (on the line (u, v)), and $l_i = a_i u + b_i v, i = 1, 2, \ldots, k$. The ratio*

$$[u, v; l_1, \ldots, l_k] := \frac{b_1 b_2, \ldots, b_k}{a_1 a_2, \ldots, a_k}$$

is called the characteristic ratio of $l_1, l_2, \ldots, l_k$ to the basic lines (or basic points) u, v.

Definition 3.2 (characteristic mapping [18]) *Let $u, v, p, q \in \mathbb{P}^2$ be concurrent lines (or collinear points), the mapping $\chi_{(u,v)} : p \mapsto q$ is called a Characteristic mapping if*

$$[u, v; p, q] = 1$$

holds, and the characteristic mapping is denoted by $q = \chi_{(u,v)}(p)$.

It can be seen that if q is the characteristic mapping point (or line) of p, then p is the characteristic mapping of q as well, that is, the characteristic mapping is reflexive mapping, i.e., $\chi_{(u,v)} \circ \chi_{(u,v)} = I$ (identity mapping).

Definition 3.3 (characteristic number [18]) *Let Γ_n is a given plane curve of degree n, and L_1, L_2 and L_3 be any three distinct lines in $\mathbb{P}^2$ such that none of them is a component of Γ_n. Let $q_1 = \langle L_1, L_2\rangle, q_2 = \langle L_2, L_3\rangle$, $q_3 = \langle L_3, L_1\rangle$ and p_{ij} $(j = 1, 2, \ldots, n)$ be n points between L_i and curve Γ_n, $i = 1, 2, 3$. The following number determined by characteristic ratio:*

$$\mathcal{K}_n(\Gamma_n) := \prod_{i=1}^{3} [q_i, q_{i+1}; p_{i1}, p_{i2}, \ldots, p_{in}]$$

is called the characteristic number of the planar algebraic curve Γ_n. If there are multiple points in the intersection points, the corresponding characteristic number can be defined by their limit form.

It is not difficult to verify from the Definition 3.3 that if Γ_n is a reducible curve in $\mathbb{P}^2$ and their components are Γ_{n_1} and $\Gamma_{n_2}, n = n_1 + n_2$ then, $\mathcal{K}_n(\Gamma_n) = \mathcal{K}_{n_1}(\Gamma_{n_1}) \cdot \mathcal{K}_{n_1}(\Gamma_{n_1})$.

Next, we list some crucial properties of plane curves of lower degrees (for lines and conics).

Theorem 3.4 ([18]) *For any line Γ_1 in the projective plane $\mathbb{P}^2$,*

$$\mathcal{K}_1(\Gamma_1) = -1.$$

The following Corollary can be easily proved using the definition of characteristic mapping and Theorem 3.4.

Corollary 3.5 ([18]) *Three points P, Q and R in the projective plane $\mathbb{P}^2$ are collinear if and only if their characteristic mapping $\chi_{(q_3,q_1)}(P), \chi_{(q_1,q_2)}(Q)$ and $\chi_{(q_2,q_3)}(R)$ are also collinear.*

In the case of conic, we have

Theorem 3.6 ([18]) *For any given conic Γ_2 in the projective plane $\mathbb{P}^2$,*

$$\mathcal{K}_2(\Gamma_2) = 1.$$

Let $l_i \in \mathbb{P}^2$ $(i = 1, 2, \ldots, 6)$ be any six distinct points. Denoted by $a = (l_1, l_2), b = (l_3, l_4), c = (l_5, l_6)$ and $u = \langle a, c\rangle$, $v = \langle b, c\rangle$, $w = \langle a, b\rangle$. The following theorems can be easily shown by using the same arguments as in (2.4) and Theorem 3.6.

Theorem 3.7 ([18]) *Any six points in the projective plane lie on a conic if and only if (2.5) holds.*

Theorem 3.8 ([18]) *Any six distinct points $l_i \in \mathbb{P}^2$ $(i = 1, 2, \ldots, 6)$ lie on a conic if and only if their characteristic mapping $\chi_{(u,w)}(l_1)$, $\chi_{(u,w)}(l_2)$, $\chi_{(w,v)}(l_3)$, $\chi_{(w,v)}(l_4)$, $\chi_{(v,u)}(l_5)$ and $\chi_{(v,u)}(l_6)$ lie on a conic as well.*

Remark 3.9 Summarizing above discussions, we get the following systemic conclusions for the singularity of S_2^1 over Morgan-Scott's triangulation. The following statements are equivalent to each other:

1) $\dim S_2^1(\Delta_{MS}) = 7$;

2) $\dim S_2^1(\tilde{\Delta}_{MS}) = 7$, where $\tilde{\Delta}_{MS}$ consists of the lines u, w, v and $\chi_{(u,w)}(l_1)$, $\chi_{(u,w)}(l_2)$, $\chi_{(w,v)}(l_3)$, $\chi_{(w,v)}(l_4)$, $\chi_{(v,u)}(l_5)$, $\chi_{(v,u)}(l_6)$;

3) Aa, Bb, Cc in Figure 2.1 are concurrent;

4) $\chi_{(w,v)}(Aa), \chi_{(u,w)}(Bb), \chi_{(v,u)}(Cc)$ are concurrent;

5) The lines l_i $(i = 1, 2, 3, 4, 5, 6)$ regarded as points in the projective plane, lies on a conic;

6) The characteristic mapping lines $\chi_{(u,w)}(l_1)$, $\chi_{(u,w)}(l_2)$, $\chi_{(w,v)}(l_3)$, $\chi_{(w,v)}(l_4)$, $\chi_{(v,u)}(l_5)$, $\chi_{(v,u)}(l_6)$, regarded as the points in the projective plane, lie on a conic.

Generally,

Theorem 3.10 ([18]) *For any given plane curve Γ_n in $\bar{\mathbb{P}}_n$, the corresponding characteristic number is*

$$\mathcal{K}_n = (-1)^n,$$

where

$$\bar{\mathbb{P}}_n := \text{span}\{\{\textit{any 3 terms of } \{x^n, x^{n-1}y, \ldots, y^n\}\} \cup \{z\bar{\mathbb{P}}_{n-1}\}\}$$

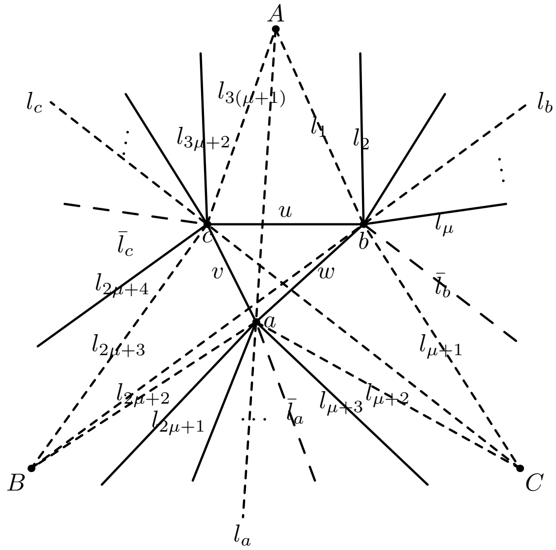
and $\bar{\mathbb{P}}_1 := \mathbb{P}_1$, $\bar{\mathbb{P}}_2 := \mathbb{P}_2$.

4. Main Results

In this section, we study the geometric significance of the singularity of the spline space $S^{\mu}_{\mu+1}$ over Δ^{μ}_{MS}. For any given three class of concurrent lines (which intersect at three distinct points a, b and c, resp.) in the projection plane, $\{l_1, l_2, \ldots, l_{\mu+1}\}$, $\{l_{\mu+2}, \ldots, l_{2(\mu+1)}\}$, $\{l_{2\mu+3}, \ldots, l_{3(\mu+1)}\}$, let $u = (b, c)$, $w = (a, b)$ and $v = (c, a)$. Then a Morgan-Scott's partition of order μ, Δ^{μ}_{MS}, is defined as a planar figure consisting of the edges $\{l_1, l_2, \ldots, l_{3(\mu+1)}$, u, w, v with Morgan-Scott's topology.

To clarify our argument clearly, for any positive integer μ and a Δ^{μ}_{MS} we define the three lines $l_a := (\langle l_1, l_{3(\mu+1)}\rangle, \langle l_{\mu+2}, l_{2(\mu+1)}\rangle)$, $l_b := (\langle l_1, l_{\mu+1}\rangle, \langle l_{2(\mu+1)}, l_{2\mu+3}\rangle)$ and $l_c := (\langle l_{\mu+1},\ l_{\mu+2}\rangle,$

$\langle l_{2\mu+3}, l_{3(\mu+1)}\rangle$) (see Aa, Bb, Cc in Figure 4.1) as the intrinsic lines of Δ^{μ}_{MS}. Suppose that Ξ^{μ}_{MS} denotes a set consists of all Δ^{μ}_{MS}.



For a given Morgan-Scott's type partition of order μ $\Delta^{\mu}_{MS} \in \Xi^{\mu}_{MS}$ shown in Figure 2.4, we define an inductive mapping

$$\phi : \Xi^{\mu}_{MS} \mapsto \Xi^{\mu-1}_{MS}$$

by defining the image of Δ^{μ}_{MS} as a new Morgan-Scott's type partition of order $\mu - 1$ whose edges consist of $l_2, \ldots, l_\mu, l_{\mu+3}, \ldots, l_{2\mu+1}$, $l_{2\mu+4}, \ldots, l_{3\mu+2}$, $\chi_{(w,v)}(l_a), \chi_{(u,w)}(l_b), \chi_{(v,u)}(l_c)$ and u, w, v (see the solid lines in Figure 4.1). The following equivalent relationship "$\sim$" is introduced in Ξ^{μ}_{MS}:

Figure 4.1 Inductive mapping of Δ^{μ}_{MS}: $\Delta^{\mu-1}_{MS}$

For $\bar{\Delta}^{\mu}_{MS}, \tilde{\Delta}^{\mu}_{MS} \in \Xi^{\mu}_{MS}$, we say $\bar{\Delta}^{\mu}_{MS} \sim \tilde{\Delta}^{\mu}_{MS}$, if they have the same the intrinsic lines. Obviously, the inductive mapping

$$\phi : \Xi^{\mu}_{MS}/ \sim \mapsto \Xi^{\mu-1}_{MS}/ \sim$$

becomes a bijection.

We will pay attention to the case of $\mu = 2$. For the triangulation Δ^2_{MS}, let $l_2^{'}, l_5^{'}, l_8^{'}$ be the characteristic mappings of the lines l_2, l_5, l_8, resp. That is, $l_2^{'} := \chi_{(u,w)}(l_2) = b_2 u + a_2 w$, $l_5^{'} := \chi_{(w,v)}(l_5) = b_5 w + a_5 v$, $l_8^{'} := \chi_{(v,u)}(l_8) = b_8 v + a_8 u$.

Without loss of generality, we assume that the six points determined by intersections of Aa, Bb, Cc and intersections of $l_2^{'}, l_5^{'}, l_8^{'}$ are distinct from each other in Figure 2.3.

Under this assumption, we now prove the following important conclusion.

Theorem 4.1 *For a given partition $\Delta^2_{MS} \in \Xi^{\mu}_{MS}/ \sim$, the spline space $S^2_3(\Delta^2_{MS})$ is singular if and only if the six points determined by the intersections of the intrinsic lines Aa, Bb, Cc and the intersections of $\chi_{(u,w)}(l_2), \chi_{(w,v)}(l_5)$, $\chi_{(v,w)}(l_8)$ lie on a conic. Furthermore, the intersections of $\chi_{(w,v)}(Aa), \chi_{(u,w)}(Bb), \chi_{(v,u)}(Cc)$ and the intersections of l_2, l_5, l_8 lie on a conic as well.*

Proof Without loss of generality, we regard the lines u, v, w as basic lines, and let $u = (1, 0, 0), w = (0, 1, 0), v = (0, 0, 1)$ and $l_2^{'} = \chi_{(u,w)}(l_2), l_5^{'} = \chi_{(w,v)}(l_5), l_8^{'} = \chi_{(v,w)}(l_8)$. From (2.4), we have

$$\begin{array}{lllll} l_1 = (a_1, b_1, 0) & & l_4 = (0, a_4, b_4) & & l_7 = (b_7, 0, a_7) \\ l_3 = (a_3, b_3, 0) & , & l_6 = (0, a_6, b_6) & \text{and} & l_9 = (b_9, 0, a_9) \\ l_2^{'} = (b_2, a_2, 0) & & l_5^{'} = (0, b_5, a_5) & & l_8^{'} = (a_8, 0, b_8), \end{array}$$

and

$$A = l_1 \times l_9 = (b_1a_9, -a_1a_9, -b_1b_9), B = l_6 \times l_7 = (a_6a_7, b_6b_7, -a_6b_7),$$
$$C = l_3 \times l_4 = (b_3b_4, -a_3b_4, a_3a_4), a = w \times v = (1, 0, 0), b = u \times w$$
$$= (0, 0, 1), c = u \times v = (0, -1, 0).$$

So the lines Aa, Bb and Cc can be expressed as follows

$$Aa = A \times a = (0, -b_1b_9, a_1a_9),\ \ Bb = B \times b = (b_6b_7, -a_6a_7, 0),$$
$$Cc = C \times c = (a_3a_4, 0, -b_3b_4).$$

By direct calculations, the intersections of Aa, Bb, Cc and the intersections of $l_2^{'}, l_5^{'}, l_8^{'}$ are formed to be

$$v1 = Aa \times Bb = (a_1a_9a_6a_7, a_1a_9b_6b_7, b_1b_9b_6b_7),$$
$$v2 = Bb \times Cc = (a_6a_7b_3b_4, b_6b_7b_3b_4, a_6a_7a_3a_4),$$
$$v3 = Cc \times Aa = (-b_3b_4b_1b_9, -a_3a_4a_1a_9, -a_3a_4b_1b_9),$$
$$v_4 = l_2^{'} \times l_5^{'} = (a_2a_5, -b_2a_5, b_2b_5),$$
$$v_5 = l_5^{'} \times l_8^{'} = (b_5b_8, a_5a_8, -b_5a_8),$$
$$v_6 = l_8^{'} \times l_2^{'} = (-b_8a_2, b_8b_2, a_8a_2).$$

We now give an equivalent condition that $v_1, v_2, \ldots, v_6$ lie on a conic by Pascal's Theorem. To do this, the three intersection points of three subtense of the hexagon with vertices $v_1, v_2, \ldots, v_6$ are

$$B1 = (v_1 \times v_5) \times (v_2 \times v_6) =$$
$$((b_1b_9b_6b_7b_5b_8 + a_1a_9a_6a_7b_5a_8)(a_6a_7b_3b_4b_8b_2 + b_6b_7b_3b_4b_8a_2) - (a_1a_9a_6a_7a_5a_8$$
$$- a_1a_9b_6b_7b_5b_8)(-a_6a_7a_3a_4b_8a_2 - a_6a_7b_3b_4a_8a_2), (a_1a_9a_6a_7a_5a_8 - a_1a_9b_6b_7b_5b_8)$$
$$(b_6b_7b_3b_4a_8a_2 - a_6a_7a_3a_4b_8b_2) - (-a_1a_9b_6b_7b_5a_8 - b_1b_9b_6b_7a_5a_8)(a_6a_7b_3b_4b_8b_2$$
$$+ b_6b_7b_3b_4b_8a_2), (-a_1a_9b_6b_7b_5a_8 - b_1b_9b_6b_7a_5a_8)(-a_6a_7a_3a_4b_8a_2 - a_6a_7b_3b_4a_8a_2)$$
$$- (b_1b_9b_6b_7b_5b_8 + a_1a_9a_6a_7b_5a_8)(b_6b_7b_3b_4a_8a_2 - a_6a_7a_3a_4b_8b_2)),$$
$$B2 = (v_1 \times v_4) \times (v_3 \times v_6) =$$
$$((b_1b_9b_6b_7a_2a_5 - a_1a_9a_6a_7b_2b_5)(-b_3b_4b_1b_9b_8b_2 - a_3a_4a_1a_9b_8a_2) - (-a_1a_9a_6a_7b_2a_5$$
$$- a_1a_9b_6b_7a_2a_5)(a_3a_4b_1b_9b_8a_2 + b_3b_4b_1b_9a_8a_2), (-a_1a_9a_6a_7b_2a_5 - a_1a_9b_6b_7a_2a_5)$$
$$(-a_3a_4a_1a_9a_8a_2 + a_3a_4b_1b_9b_8b_2) - (a_1a_9b_6b_7b_2b_5 + b_1b_9b_6b_7b_2a_5)(-b_3b_4b_1b_9b_8b_2$$
$$- a_3a_4a_1a_9b_8a_2), (a_1a_9b_6b_7b_2b_5 + b_1b_9b_6b_7b_2a_5)(a_3a_4b_1b_9b_8a_2 + b_3b_4b_1b_9a_8a_2)$$
$$- (b_1b_9b_6b_7a_2a_5 - a_1a_9a_6a_7b_2b_5)(-a_3a_4a_1a_9a_8a_2 + a_3a_4b_1b_9b_8b_2)),$$
$$B3 = (v_2 \times v_4) \times (v_3 \times v_5)$$
$$= ((a_6a_7a_3a_4a_2a_5 - a_6a_7b_3b_4b_2b_5)(-b_3b_4b_1b_9a_5a_8 + a_3a_4a_1a_9b_5b_8) - (-a_6a_7b_3b_4b_2a_5$$
$$- b_6b_7b_3b_4a_2a_5)(-a_3a_4b_1b_9b_5b_8 - b_3b_4b_1b_9b_5a_8), (-a_6a_7b_3b_4b_2a_5 - b_6b_7b_3b_4a_2a_5)$$

$$(a_3a_4a_1a_9b_5a_8 + a_3a_4b_1b_9a_5a_8) - (b_6b_7b_3b_4b_2b_5 + a_6a_7a_3a_4b_2a_5)(-b_3b_4b_1b_9a_5a_8 + a_3a_4a_1a_9b_5b_8), (b_6b_7b_3b_4b_2b_5 + a_6a_7a_3a_4b_2a_5)(-a_3a_4b_1b_9b_5b_8 - b_3b_4b_1b_9b_5a_8) - (a_6a_7a_3a_4a_2a_5 - a_6a_7b_3b_4b_2b_5)(a_3a_4a_1a_9b_5a_8 + a_3a_4b_1b_9a_5a_8)).$$

The directed area of triangle determined by B_1, B_2 and B_3 is

$$\begin{aligned}(B_1, B_2, B_3) = &-(b_5b_8b_2 + a_2a_5a_8)^2(b_7b_6a_2 + b_2a_6a_7)(b_1b_9a_5 + a_1a_9b_5)\\ &(b_3b_4a_8 + b_8a_3a_4)(b_1b_3b_4b_6b_7b_9 - a_1a_3a_4a_6a_7a_9)^2(b_1b_2b_3b_4b_5b_6b_7b_8b_9+\\ &a_1a_2a_3a_4a_5a_6a_7a_8a_9).\end{aligned}$$

Since the six points $v_1, v_2, \ldots, v_6$ are all distinct, we have

$$\begin{aligned}&(b_5b_8b_2 + a_2a_5a_8)^2(b_7b_6a_2 + b_2a_6a_7)(b_1b_9a_5 + a_1a_9b_5)(b_3b_4a_8 + b_8a_3a_4)\\ &(b_1b_3b_4b_6b_7b_9 - a_1a_3a_4a_6a_7a_9)^2 \neq 0.\end{aligned} \tag{9}$$

Hence, it follows from Pascal's Theorem (stating that $v_1, v_2, \ldots, v_6$ lie on a conic if an only if $(B_1, B_2, B_3) = 0$) that the necessary and sufficient condition that $v_1, v_2, \ldots, v_6$ lie on a conic is

$$\frac{a_1a_2a_3}{b_1b_2b_3} \cdot \frac{a_4a_5a_6}{b_4b_5b_6} \cdot \frac{a_7a_8a_9}{b_7b_8b_9} = -1.$$

It follows from Theorem 2.8 that the proof of the first statement is completed. The second statement of the theorem can be obtained immediately from Theorem 3.8.

From Proposition 2.13 and Theorem 4.1, we can easily prove the following corollary.

Corollary 4.2 *The spline space $S_3^2(\Delta_{MS}^2)$ is singular ($\dim S_3^2(\Delta_{MS}^2) = 11$) if and only if the lines Aa, Bb, Cc and $\chi_{(u,w)}(l_2), \chi_{(w,v)}(l_5), \chi_{(v,w)}(l_8)$ regarded as points in $\mathbb{P}^2$ lie on a conic.*

The following theorem follows from Theorems 2.7, 2.8 and 3.8.

Theorem 4.3 *For a given partition $\Delta_{MS}^2 \in \Xi_{MS}^2/\sim$, the spline space $S_3^2(\Delta_{MS}^2)$ is singular ($\dim S_3^2(\Delta_{MS}^2) = 11$) if and only if the space $S_2^1(\phi(\Delta_{MS}^2))$ is singular as well.*

We provide two examples to illustrate our conclusions more clearly. We consider a given triangulation shown in Figure 4.2, where

$$\begin{aligned}&A = (1/2, 2, 1), B = (-4, -2, 1), C = (4, -2, 1), a = (0, -1, 1), b = (1, 0, 1),\\ &c = (-1, 0, 1), \quad u : -y = 0 \quad v : -x - y - z = 0 \quad w : x - y - z = 0\\ &l_1 : -4x - y + \frac{4}{3}z = 0, \quad l_2 : 2x - y - 2z = 0, \quad l_3 : -\frac{2}{3}x - y + \frac{2}{3}z = 0,\\ &l_4 : -\frac{1}{4}x - y - z = 0, \quad l_5 : -\frac{1}{2}x - y - z = 0, \quad l_6 : \frac{1}{4}x - y - z = 0,\\ &l_7 : \frac{2}{3}x - y + \frac{2}{3}z = 0, l_8 : -\frac{42}{47}x - y - \frac{42}{47}z = 0, l_9 : \frac{4}{3}x - y + \frac{4}{3}z = 0.\end{aligned}$$

It can be proved that the spline space $S_3^2(\Delta_{MS}^2)$ of piecewise polynomial of degree three with smoothness two is singular. The characteristic mappings of l_2, l_5, l_8 corresponding to $(u, w), (w, v)$ and (v, u) are

$$l_2' := \chi_{(u,w)}(l_2) : -x - y + z = 0, \quad l_5' := \chi_{(w,v)}(l_5) : 1/2x - y - z = 0,$$

$$l_8^{'} := \chi_{(v,u)}(l_8) : -5/47x - y - 5/47z = 0.$$

The conic corresponding to Theorem 4.1, which is determined by the following six points, the intersections of the intrinsic lines Aa, Bb, Cc and the intersections of the characteristic mappings of l_2, l_5, l_6, is of the form

$$302x^2 - 2861xy + 15800y^2 - 1421x + 12880y + 2624 = 0,$$

which forms elliptic conic and is shown in Figure 4.2. One can straightly prove that the $S_2^1(\phi(\Delta^2_{MS}))$ is singular, where $\phi(\Delta^2_{MS})$ is an inductive Morgan-Scott's partition (or triangulation) whose edges consists of $Aa, Bb, Cc, \xi_{(u,w)}(l_2), \xi_{(w,v)}(l_5)$, $\xi_{(v,u)}(l_8)$ and u, w, v (all dashed lines and u, w, v in Figure 4.2) (or $\xi_{(w,v)}(Aa), \xi_{(u,w)}(Bb)$, $\xi_{(v,u)}(Cc), l_2, l_5, l_8$ and u, w, v).

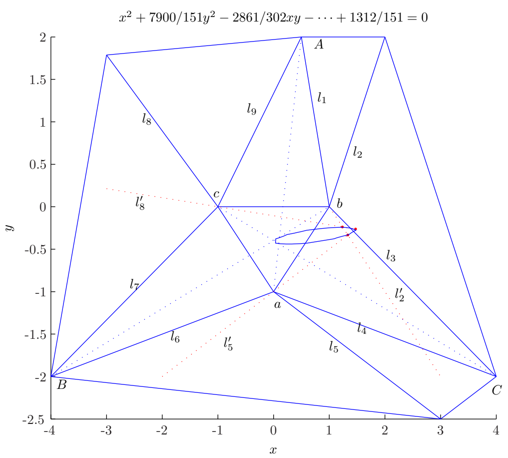


Figure 4.2 Singular $\triangle^2_{MS}$ and the elliptic conic

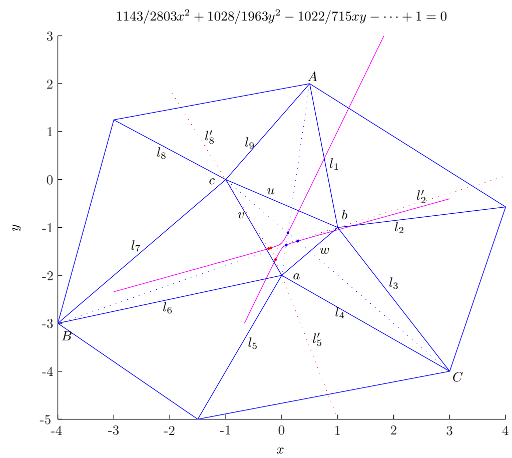


Figure 4.3 Singular $\triangle^2_{MS}$ and hyperbolic conic

The next example of singular Morgan-Scott's partition Δ^2_{MS} to S_3^2 is shown in Figure 4.3, where

$$\begin{aligned}
&A = (1/2, 2, 1), B = (-4, -3, 1), C = (3, -4, 1), a = (0, -2, 1), b = (1, -1, 1),\\
&c = (-1, 0, 1), \quad u : -1/2x - y - 1/2z = 0 \quad v : -2x - y - 2z = 0\\
&w : x - y - 2z = 0, \quad l_1 : -6x - y + 5z = 0, \quad l_2 : 1/7x - y - 8/7z = 0,\\
&l_3 : -3/2x - y + 1/2z = 0, \quad l_4 : -2/3x - y - 2z = 0, \quad l_5 : 2x - y - 2z = 0,\\
&l_6 : 1/4x - y - 2z = 0, \quad l_7 : x - y + z = 0, \quad l_8 : -38/61x - y - 38/61z = 0,\\
&l_9 : -4/3x - y + 4/3z = 0,
\end{aligned}$$

and

$$\begin{aligned}
&l_7^{'} := \chi_{(u,w)}(l_7) : \frac{5}{14}x - \frac{19}{14} - y = 0, \quad l_8^{'} := \chi_{(u,w)}(l_8) : -3x - 2 - y = 0,\\
&l_9^{'} := \chi_{(u,w)}(l_9) : \ -\frac{229}{122}x - \frac{229}{122} - y = 0.
\end{aligned}$$

The corresponding conic, which forms hyperbola, uniquely determined by the six points (the intersections of the intrinsic lines Aa, Bb, Cc and the intersections of the characteristic mappings of l_2, l_5, l_6) is

$$\frac{1143}{2803}x^2 + \frac{1028}{1963}y^2 - \frac{1022}{715}xy - \frac{503}{256}xz + \frac{1161}{1112}yz + z = 0,$$

and is shown in Figure 4.3. From Theorem 4.3, the spline space S_2^1 over the inductive partition $\phi(\Delta_{MS}^2)$ is singular as well.

For the case of $\mu \geq 3$, we have the following main result.

Theorem 4.4 *Let $\Delta_{MS}^\mu \in \Xi_{MS}^\mu / \sim$ be a Morgan-Scott's partition as shown in Figure 2.4. The spline space $S_{\mu+1}^\mu(\Delta_{MS}^\mu)$ is singular if and only if the spline space $S_\mu^{\mu-1}(\phi(\Delta_{MS}^\mu))$ is singular as well.*

Proof Without loss of generality, suppose that $u = (1,0,0), w = (0,1,0), v = (0,0,1)$ and let

$$\begin{aligned} l_i &= (a_i, b_i, 0), \quad && i = 1, 2, \ldots, n, \\ l_j &= (0, a_j, b_j), && j = n+1, n+2 \ldots, 2n, \\ l_k &= (b_k, 0, a_k), && k = 2n+1, 2n+2, \ldots, 3n. \end{aligned}$$

Since

$$\begin{aligned} Aa :&= (\langle l_1, l_{3n}\rangle, \langle l_{n+1}, l_{2n}\rangle) = (0, -b_1 b_{3n}, a_1 a_{3n}) = -b_1 b_{3n} w + a_1 a_{3n} v \\ Bb :&= (\langle l_1, l_n\rangle, \langle l_{2n}, l_{2n+1}\rangle) = (b_{2n} b_{2n+1}, -a_{2n} a_{2n+1}, 0) = b_{2n} b_{2n+1} u - a_{2n} a_{2n+1} w \\ Cc :&= (\langle l_n, l_{n+1}\rangle, \langle l_{2n+1}, l_{3n}\rangle) = (a_n a_{n+1}, 0, -b_n b_{n+1}) = -b_n b_{n+1} v + a_n a_{n+1} u, \end{aligned}$$

we have

$$\begin{aligned} \chi_{(w,v)}(Aa) &= a_1 a_{3n} w - b_1 b_{3n} v \\ \chi_{(u,w)}(Bb) &= -a_{2n} a_{2n+1} u + b_{2n} b_{2n+1} w \\ \chi_{(v,u)}(Cc) &= a_n a_{n+1} v - b_n b_{n+1} u. \end{aligned}$$

It follows from Theorem 2.13 that the necessary and sufficient condition for the singularity of $S_\mu^{\mu-1}(\phi(\Delta_{MS}^\mu))$ is that

$$\begin{aligned} &\frac{a_2 \cdots a_{n-1} \cdot (-a_{2n} a_{2n+1})}{b_2 \cdots b_{n-1} \cdot (b_{2n} b_{2n+1})} \cdot \frac{a_{n+2} \cdots a_{2n-1} \cdot (a_1 a_{3n})}{b_{n+2} \cdots b_{2n-1} \cdot (-b_1 b_{3n})} \cdot \frac{a_{2n+2} \cdots a_{3n-1} \cdot (a_n a_{n+1})}{b_{2n+2} \cdots b_{3n-1} \cdot (-b_n b_{n+1})} \\ &= (-1)^{n-1}, \end{aligned}$$

which is equivalent to (2.10), and it follows from Theorem 3.10 that the proof is completed. $\square$

Using the duality principle to Theorem 4.4, some interesting results on algebraic curves can be derived easily. They are listed in the paper without proofs.

Let a, b, c be three distinct non-infinity lines in $\mathbb{P}^2$ and Γ_n be a given plane curve in $\mathbb{P}^2$. Denote the points of intersection between lines a, b, c and the curve Γ_n by l_i $(i = 1, 2, \ldots, n)$, l_j $(j = n+1, \ldots, 2n)$ and l_k $(k = 2n+1, \ldots, 3n)$, and let $u = \langle b, c\rangle$, $v = \langle c, a\rangle$, $w = \langle a, b\rangle$, respectively.

It follows from the definition of duality of figure that the planar figure consisting of lines a, b, c and points $\{l_i$ $(i = 1, 2, \ldots, n)$, l_j $(j = n+1, \ldots, 2n)$, l_k $(k = 2n+1, \ldots, 3n)$, $u, v, w\}$ yields a dual figure in the same projective plane. Fortunately, the generated dual figure is in the form of a Morgan-Scott's type triangulation as shown in Figure 2.4.

Theorem 4.5 *Let $a, b, c \in \mathbb{P}^2$ be three distinct lines, and $u = \langle b, c\rangle$, $v = \langle c, a\rangle$, $w = \langle a, b\rangle$. Suppose that $p_i = a_i u + b_i w$ $(i = 1, 2, \ldots, n)$, $p_j = a_j w + b_j v$ $(j = n+1, \ldots, 2n)$ and $p_k = a_k v + b_k u$ $(j = 2n+1, \ldots, 3n)$ are three classes of points lying on the lines a, b and c, respectively. Then the $3n$ points p_i $(i = 1, 2, \ldots, 3n)$ lie on a plane curve in $\bar{\mathbb{P}}_n$ if and only if the following $3(n-1)$ points $\{\chi_{(w,v)}(\langle\langle(p_1, p_{3n}), (p_{n+1}, p_{2n})\rangle\rangle),\ \chi_{(u,w)}(\langle\langle(p_1, p_n), (p_{2n}, p_{2n+1})\rangle\rangle),$ $\chi_{(v,u)}(\langle\langle(p_n, p_{n+1}), (p_{2n+1}, p_{3n})\rangle\rangle),\ p_2, \ldots, p_{n-1}, p_{n+2}, \ldots, p_{2n-1}, p_{2n+2}, \ldots,\ p_{3n-1}\}$ lie on a plane curve in $\bar{\mathbb{P}}_{n-1}$.*

The following corollary is directly from the definition of the characteristic mapping and Theorem 4.5.

Corollary 4.6 *Let $a, b, c \in \mathbb{P}^2$ be three distinct lines, and $u = \langle b, c\rangle$, $v = \langle c, a\rangle$, $w = \langle a, b\rangle$. Suppose that $p_i = a_i u + b_i w$ $(i = 1, 2, \ldots, n)$, $p_j = a_j w + b_j v$ $(j = n+1, \ldots, 2n)$ and $p_k = a_k v + b_k u$ $(j = 2n+1, \ldots, 3n)$ are three classes of points lying on the lines a, b and c, respectively. Then the $3n$ points p_i $(i = 1, 2, \ldots, 3n)$ lie on a plane curve in $\bar{\mathbb{P}}_n$ if and only if the following $3(n-1)$ points $\{(\langle\langle(p_1, p_{3n}),\ (p_{n+1}, p_{2n})\rangle\rangle), (\langle\langle(p_1, p_n),\ (p_{2n}, p_{2n+1})\rangle\rangle), (\langle\langle(p_n, p_{n+1}),\ (p_{2n+1}, p_{3n})\rangle\rangle),$ $\chi_{(w,v)}(p_2), \ldots, \chi_{(u,w)}(p_{n-1}), \chi_{(w,v)}(p_{n+2}), \ldots, \chi_{(u,w)}(p_{2n-1}), \chi_{(v,u)}(p_{2n+2}), \ldots, \chi_{(v,u)}(p_{3n-1})\}$ lie on a plane curve in $\bar{\mathbb{P}}_{n-1}$.*

Theorem 4.5 and Corollary 4.6 can be regarded as a direct generalizations of the famous Pascal's theorem in algebraic geometry since the Pascal's theorem is the case of $n = 2$ in Theorem 4.5 or Corollary 4.6. This generalization of the Pascal's theorem is quite different from the known generalizations as Chasles's Theorem and Cayley-Bacharach Theorem in algebraic geometry.

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