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The Laplacian Spread of Bicyclic Graphs

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Abstract The Laplacian spread of a graph is defined to be the difference between the largest eigenvalue and the second smallest eigenvalue of the Laplacian matrix of the graph. In our recent work, we have determined the graphs with maximal Laplacian spreads among all trees of fixed order and among all unicyclic graphs of fixed order, respectively. In this paper, we continue the work on Laplacian spread of graphs, and prove that there exist exactly two bicyclic graphs with maximal Laplacian spread among all bicyclic graphs of fixed order, which are obtained from a star by adding two incident edges and by adding two nonincident edges between the pendant vertices of the star, respectively.

Keywords bicyclic graph; Laplacian matrix; spread.

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1. Introduction

Let G be a graph of order n with the vertex set $V = V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E = E(G)$. The adjacency matrix of the graph G is defined to be a matrix $A = A(G) = [a_{ij}]$ of order n, where $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise. Since A is symmetric and real, the eigenvalues of A can be arranged as follows:

$$\lambda_n(G) \le \lambda_{n-1}(G) \le \cdots \le \lambda_1(G).$$

The spread of the graph G is defined as

$$\mathscr{S}_A(G) = \lambda_1(G) - \lambda_n(G).$$

Generally, the spread of a square complex matrix M is defined to be $s(M) = \max_{i,j} |\lambda_i - \lambda_j|$, where the maximum is taken over all pairs of eigenvalues of M. There is a considerable literature on the spread of an arbitrary matrix [11, 15, 18, 20, 21].

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Recently, the spread of a graph has received much attention. In [19], Petrović determined all minimal graphs whose spread does not exceed 4. In [7], Gregory, Hershkowitz and Kirkland presented some lower and upper bounds for the spread of a graph. They showed that the path is the unique graph with minimal spread among all connected graphs of given order. However the graph(s) with maximal spread is still unknown, and some conjectures are presented in their paper. In [12], Li, Zhang and Zhou determined the unique graph with maximal spread among all unicyclic graphs with given order not less than 18, which is obtained from a star by adding an edge between two pendant vertices. In [17], Nikiforov considered a more general problem: what is the property of the linear combination of some extreme eigenvalues of a graph? He gave a theorem involving the limit of a certain combination as the order of a graph goes to infinity, and presented an upper bound for the sum of the largest eigenvalue and the second largest eigenvalue of all graphs of fixed order.

Here we consider another version of spread of a graph, i.e., the Laplacian spread of a graph, which is defined as follows. Let G be a graph as above. The Laplacian matrix of the graph G is defined as $L = L(G) = D(G) - A(G)$, where $D(G) = \mathrm{diag}\{d(v_1), d(v_2), \ldots, d(v_n)\}$ is a diagonal matrix, and $d(v)$ denotes the degree of the vertex v of G. It is known that L is symmetric and positive semidefinite so that its eigenvalues can be arranged as follows:

$$0 = \mu_n(G) \leq \mu_{n-1}(G) \leq \cdots \leq \mu_1(G),$$

where $\mu_n(G) = 0$ as each row sum of L is zero. There are a lot of results involved with the relations between the spectrum of $L(G)$ and numerous invariants of the graph G, such as connectivity, diameter, isoperimetric number, and expanding properties of a graph [5, 9, 10, 13, 16]. In particular, $\mu_{n-1}(G) > 0$ if and only if G is connected. Fiedler calls $\mu_{n-1}(G)$ the algebraic connectivity of the graph G, which is considered as an algebraic measurement of the connectivity of a graph. The corresponding eigenvectors of $\mu_{n-1}(G)$ are usually called Fiedler vectors, which have a beautiful structure property given by Fiedler [6, Theorem 3.14]. One can find that $\mu_1(G)$ is exactly the spectral radius of $L(G)$, which also has a lot of results (especially the upper bounds) for this eigenvalue [3]. We define the Laplacian spread of the graph G as

$$\mathscr{S}_L(G) = \mu_1(G) - \mu_{n-1}(G).$$

Note that in the definition we consider the largest eigenvalue and the second smallest eigenvalue, as the smallest eigenvalue always equals zero.

In our recent work we have shown that among all trees of fixed order, the star is the unique one with maximal Laplacian spread and the path is the unique one with the minimal Laplacian spread [4]. And among all unicyclic graphs of fixed order, the unique unicyclic graph with maximal Laplacian spread is obtained from a star by adding an edge between two pendant vertices [2]. In this paper, we continue the work on Laplacian spread of graphs, and prove that there exist exactly two bicyclic graphs with maximal Laplacian spread among all bicyclic graphs of fixed order, which are obtained from a star by adding two incident edges and by adding two nonincident edges between the pendant vertices of the star, respectively.

2. Results

We first introduce some preliminary results, which are needed in the following proofs. Let G be a graph and let v be a vertex of G. The neighborhood of v in G is denoted by $N(v)$, i.e., $N(v) = \{w : wv \in E(G)\}$. Denote by $\Delta(G)$ the maximum degree of all vertices of a graph G.

Lemma 2.1 ([1]) *Let G be a connected graph of order $n \geq 2$. Then $\mu_1(G) \leq n$, with equality holding if and only if the complement graph of G is disconnected.*

Lemma 2.2 ([3]) *Let G be a connected graph with vertex set $\{v_1, v_2, \dots, v_n\}$ $(n \geq 2)$. Then $\mu_1(G) \leq \max\{d(v_i) + d(v_j) - |N(v_i) \cap N(v_j)| : v_iv_j \in E(G)\}$.*

Lemma 2.3 ([14]) *Let G be a connected graph with vertex set $\{v_1, v_2, \dots, v_n\}$ $(n \geq 2)$. Then $\mu_1(G) \leq \max\{d(v_i) + m(v_i) : v_i \in V(G)\}$, where $m(v_i) = \frac{\Sigma_{v_j \in N(v_i)} d(v_j)}{d(v_i)}$ is the average of the degrees of the vertices adjacent to v_i.*

Lemma 2.4 ([10]) *Let G be graph of order $n \geq 2$ containing at least one edge. Then $\mu_1(G) \geq \Delta(G) + 1$. If G is connected, then the equality holds if and only if $\Delta(G) = n - 1$.*

Lemma 2.5 ([8]) *Let G be a connected graph of order n with a cutpoint v. Then $\mu_{n-1}(G) \leq 1$, with equality holding if and only if v is adjacent to every other vertex of G.*

We introduce ten bicyclic graphs of order n in Figure 2.1: the graphs $G_1(r, s; n)$, $r \geq s \geq 0$; $G_2(r, s; n)$, $r \geq 0$, $s \geq 0$; $G_3(r, s; n)$, $r \geq s \geq 0$; $G_4(r, s; n)$, $r \geq s \geq 0$; $G_5(r, s; n)$, $r \geq 0$, $s \geq 1$; $G_6(r, s; n)$, $r \geq 0$, $s \geq 1$; $G_7(r, s; n)$, $r \geq 0$, $s \geq 1$; $G_8(r, s; n)$, $r \geq 0$, $s \geq 0$; $G_9(r, s; n)$, $r \geq 0$, $s \geq 1$; $G_{10}(r, s; n)$, $r \geq s \geq 0$. Here r, s are nonnegative integers, which are respectively the numbers of pendant vertices adjacent to some vertices of the related graphs.

Lemma 2.6 *Let G be the graph $G_1(n-4, 0; n)$ or $G_2(n-5, 0; n)$. Then*

$$\mathscr{S}_L(G_1(n-4, 0; n)) = \mathscr{S}_L(G_2(n-5, 0; n)) = n - 1.$$

Proof By Lemmas 2.4 and 2.5, we can get the result easily. □

In the following, we will prove that the graphs $G_1(n-4, 0; n)$ and $G_2(n-5, 0; n)$ are the only two bicyclic ones with maximal Laplacian spread. We first narrow down the possibility of the bicyclic graphs with maximal Laplacian spread.

Lemma 2.7 *Let G be a graph with maximal Laplacian spread among all bicyclic graphs of order $n \geq 9$. Then G is among the graphs $G_1(n-4, 0; n)$, $G_1(n-5, 1; n)$, $G_2(n-5, 0; n)$, $G_2(n-6, 1; n)$, $G_3(n-5, 0; n)$, $G_5(n-6, 1; n)$, $G_6(n-7, 1; n)$, $G_8(n-6, 0; n)$, $G_9(n-5, 1; n)$, $G_9(0, n-4; n)$.*

Proof Let v_iv_j be an edge of G. Then

$$d(v_i) + d(v_j) - |N(v_i) \cap N(v_j)| = |N(v_i) \cup N(v_j)| \leq n,$$

with equality holding if and only if G is one graph in Figure 2.1 for some r or s. Therefore, if G is not a graph in Figure 2.1, then by Lemma 2.2, $\mu_1(G) \leq n - 1$ and hence $\mathscr{S}_L(G) = \mu_1(G) - \mu_{n-1}(G) < n - 1$ as $\mu_{n-1}(G) > 0$. However, by Lemma 2.6, $\mathscr{S}_L(G_1(n-4, 0; n)) =$

$\mathscr{S}_L(G_2(n-5,0;n)) = n-1$. So G must be one graph in Figure 2.1.

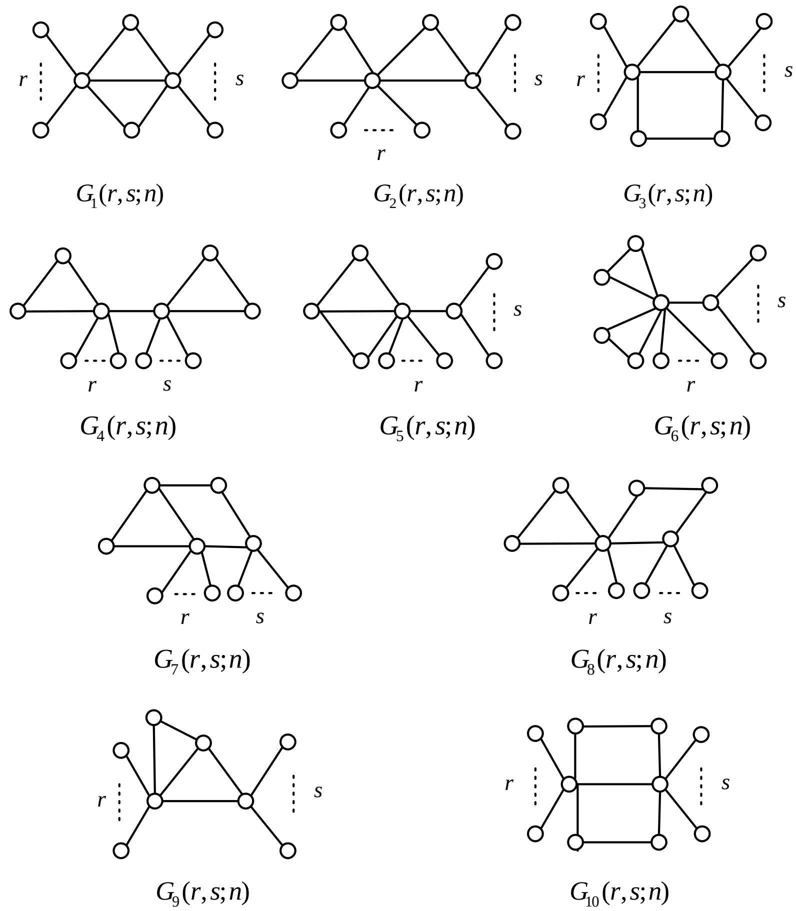

Figure 2.1 Ten bicyclic graphs on n vertices

For the graph $G_1(r,s;n)$ of Figure 2.1 with $0 \le s \le r \le n-4$, by Lemma 2.3,

$$\mu_1(G_1(r,s;n)) \le \max\{r+3+\frac{n+3}{r+3}, s+3+\frac{n+3}{s+3}\} =: \alpha.$$

Note that $r+3 \ge \frac{n-4}{2}+3 > \sqrt{n+3}$. If $r \le n-6$ and $n \ge 9$, then

$$\mu_1(G_1(r,s;n)) \le \alpha \le \max\{3+\frac{n+3}{3}, n-3+\frac{n+3}{n-3}\} \le n-1.$$

Hence, if $n \ge 9$ and $r \le n-6$, then $\mathscr{S}_L(G_1(r,s;n)) < n-1$ as $\mu_{n-1}(G) > 0$.

For the graph $G_2(r,s;n)$ of Figure 2.1 with $0 \le r,s \le n-5$, by Lemma 2.3,

$$\mu_1(G_2(r,s;n)) \le \max\{s+2+\frac{n+1}{s+2}, r+4+\frac{n+3}{r+4}\}.$$

For $n \ge 9$, $r \le n-7$, and an arbitrary s,

$$s+2+\frac{n+1}{s+2} \le \max\{2+\frac{n+1}{2}, n-3+\frac{n+1}{n-3}\} \le n-1,$$

$$r+4+\frac{n+3}{r+4}\leq \max\{4+\frac{n+3}{4},n-3+\frac{n+3}{n-3}\}\leq n-1,$$

and hence $\mu_1(G_2(r,s;n))\leq n-1$, $\mathscr{S}_L(G_2(r,s;n))<n-1$.

For the graph $G_3(r,s;n)$ of Figure 2.1 with $0\leq s\leq r\leq n-5$, by Lemma 2.3,

$$\mu_1(G_3(r,s;n))\leq \max\{r+3+\frac{n+2}{r+3},s+3+\frac{n+2}{s+3}\}.$$

For $n\geq 8$ and $r\leq n-6$,

$$\mu_1(G_3(r,s;n))\leq \max\{3+\frac{n+2}{3},n-3+\frac{n+2}{n-3}\}\leq n-1,$$

and hence $\mathscr{S}_L(G_3(r,s;n))<n-1$.

For the graph $G_4(r,s;n)$ of Figure 2.1 with $0\leq s\leq r\leq n-6$, by Lemma 2.3,

$$\mu_1(G_4(r,s;n))\leq \max\{r+3+\frac{n+1}{r+3},s+3+\frac{n+1}{s+3}\}.$$

For $n\geq 7$ and arbitrary r,s,

$$\mu_1(G_4(r,s;n))\leq \max\{3+\frac{n+1}{3},n-3+\frac{n+1}{n-3}\}\leq n-1,$$

and hence $\mathscr{S}_L(G_4(r,s;n))<n-1$.

For the graph $G_5(r,s;n)$ of Figure 2.1 with $1\leq s\leq n-5, 0\leq r\leq n-6$, by Lemma 2.3,

$$\mu_1(G_5(r,s;n))\leq \max\{s+1+\frac{n-1}{s+1},r+4+\frac{n+3}{r+4}\}.$$

For $n\geq 9$, $r\leq n-7$, and an arbitrary s,

$$s+1+\frac{n-1}{s+1}\leq \max\{2+\frac{n-1}{2},n-4+\frac{n-1}{n-4}\}\leq n-1,$$

$$r+4+\frac{n+3}{r+4}\leq \max\{4+\frac{n+3}{4},n-3+\frac{n+3}{n-3}\}\leq n-1,$$

and hence $\mathscr{S}_L(G_5(r,s;n))<n-1$.

For the graph $G_6(r,s;n)$ of Figure 2.1 with $1\leq s\leq n-6, 0\leq r\leq n-7$, by Lemma 2.3,

$$\mu_1(G_6(r,s;n))\leq \max\{s+1+\frac{n-1}{s+1},r+5+\frac{n+3}{r+5}\}.$$

For $n\geq 9$, $r\leq n-8$, and an arbitrary s,

$$s+1+\frac{n-1}{s+1}\leq \max\{2+\frac{n-1}{2},n-5+\frac{n-1}{n-5}\}\leq n-1,$$

$$r+5+\frac{n+3}{r+5}\leq \max\{5+\frac{n+3}{5},n-3+\frac{n+3}{n-3}\}\leq n-1,$$

and hence $\mathscr{S}_L(G_6(r,s;n))<n-1$.

For the graph $G_7(r,s;n)$ of Figure 2.1 with $1\leq s\leq n-5, 0\leq r\leq n-6$, by Lemma 2.3,

$$\mu_1(G_7(r,s;n))\leq \max\{s+2+\frac{n}{s+2},r+3+\frac{n+2}{r+3}\}.$$

For $n\geq 8$ and arbitrary r,s,

$$s+2+\frac{n}{s+2}\leq \max\{3+\frac{n}{3},n-3+\frac{n}{n-3}\}\leq n-1,$$

$$r+3+\frac{n+2}{r+3} \le \max\{3+\frac{n+2}{3}, n-3+\frac{n+2}{n-3}\} \le n-1,$$

and hence $\mathscr{S}_L(G_7(r,s;n)) < n-1$.

For the graph $G_8(r,s;n)$ of Figure 2.1 with $0 \le r, s \le n-6$, by Lemma 2.3,

$$\mu_1(G_8(r,s;n)) \le \max\{s+2+\frac{n}{s+2}, r+4+\frac{n+2}{r+4}\}.$$

For $n \ge 8$, $r \le n-7$, and an arbitrary s,

$$s+2+\frac{n}{s+2} \le \max\{2+\frac{n}{2}, n-4+\frac{n}{n-4}\} \le n-1,$$

$$r+4+\frac{n+2}{r+4} \le \max\{4+\frac{n+2}{4}, n-3+\frac{n+2}{n-3}\} \le n-1,$$

and hence $\mathscr{S}_L(G_8(r,s;n)) < n-1$.

For the graph $G_9(r,s;n)$ of Figure 2.1 with $0 \le r \le n-5, 1 \le s \le n-4$, by Lemma 2.3,

$$\mu_1(G_9(r,s;n)) \le \max\{r+3+\frac{n+3}{r+3}, s+2+\frac{n+2}{s+2}\}.$$

For $n \ge 9$, $r \le n-6$ and $s \le n-5$,

$$r+3+\frac{n+3}{r+3} \le \max\{3+\frac{n+3}{3}, n-3+\frac{n+3}{n-3}\} \le n-1,$$

$$s+2+\frac{n+2}{s+2} \le \max\{3+\frac{n+2}{3}, n-3+\frac{n+2}{n-3}\} \le n-1,$$

and hence $\mathscr{S}_L(G_9(r,s;n)) < n-1$.

For the graph $G_{10}(r,s;n)$ of Figure 2.1 with $0 \le s \le r \le n-6$, by Lemma 2.3,

$$\mu_1(G_{10}(r,s;n)) \le \max\{r+3+\frac{n+1}{r+3}, s+3+\frac{n+1}{s+3}\}.$$

For $n \ge 7$ and arbitrary r, s,

$$\mu_1(G_{10}(r,s;n)) \le \max\{3+\frac{n+1}{3}, n-3+\frac{n+1}{n-3}\} \le n-1,$$

and hence $\mathscr{S}_L(G_{10}(r,s;n)) < n-1$.

By the above discussion, if G is one with maximal Laplacian spread of all bicyclic graphs of order $n \ge 9$, then G is among the graphs $G_1(n-4,0;n)$, $G_1(n-5,1;n)$, $G_2(n-5,0;n)$, $G_2(n-6,1;n)$, $G_3(n-5,0;n)$, $G_5(n-6,1;n)$, $G_6(n-7,1;n)$, $G_8(n-6,0;n)$, $G_9(n-5,1;n)$, $G_9(0,n-4;n)$. The result follows. $\square$

We next show that except the graphs $G_1(n-4,0;n)$ and $G_2(n-5,0;n)$, the Laplacian spreads of the other graphs in Lemma 2.7 are all less than $n-1$ for a suitable n. Thus by a little computation for the graphs in Figure 2.1 of small order, $G_1(n-4,0;n)$ and $G_2(n-5,0;n)$ are proved to be the only two bicyclic graphs with maximal Laplacian spread among all bicyclic graphs of order $n \ge 5$.

In the following Lemmas 2.8–2.15, for convenience we simply write $\mu_1(G_i(r,s;n))$, $\mu_{n-1}(G_i(r,s;n))$ as μ_1, μ_{n-1} respectively if no confusions occur.

Lemma 2.8 *For $n \ge 7$, $\mathscr{S}_L(G_1(n-5,1;n)) < n-1$.*

Proof The characteristic polynomial $\det(\lambda I - L(G_1(n-5,1;n)))$ of $L(G_1(n-5,1;n))$ is

$$\lambda(\lambda-1)^{n-6}(\lambda-2)[\lambda^4-(n+6)\lambda^3+(7n+4)\lambda^2-(11n-6)\lambda+4n].$$

By Lemmas 2.1 and 2.4, $n > \mu_1 > n-1 \geq 6$, and by Lemma 2.5, $\mu_{n-1} < 1$. So μ_1, μ_{n-1} are both the roots of the following polynomial:

$$f_1(\lambda) =: \lambda^4-(n+6)\lambda^3+(7n+4)\lambda^2-(11n-6)\lambda+4n$$

with the derivative

$$f_1'(\lambda) = 4\lambda^3-3(n+6)\lambda^2+2(7n+4)\lambda-(11n-6),$$

and the second derivative

$$f_1''(\lambda) = 12\lambda^2-6(n+6)\lambda+2(7n+4).$$

Observe that

$$(n-1)-\mathscr{S}_L(G_1(n-5,1;n)) = (n-1)-(\mu_1-\mu_{n-1}) = (n-\mu_1)-(1-\mu_{n-1}).$$

If we can show $n-\mu_1 > 1-\mu_{n-1}$, the result will follow. By Lagrange Mean Value Theorem,

$$f_1(n)-f_1(\mu_1) = (n-\mu_1)f_1'(\xi_1)$$

for some $\xi_1 \in (\mu_1, n)$. As $f_1'(x)$ is positive and strictly increasing on the interval $(\mu_1, +\infty)$ and $\mu_1 < n$,

$$n-\mu_1 = \frac{f_1(n)-f_1(\mu_1)}{f_1(\xi_1)} > \frac{n^3-7n^2+10n}{f_1'(n)} = \frac{n(n-2)(n-5)}{(n-1)(n^2-3n-6)} > \frac{n-5}{n-1}.$$

By Lagrange Remainder Theorem,

$$f_1(\mu_{n-1}) = f_1(1)+f_1'(1)(\mu_{n-1}-1)+\frac{f_1''(\xi_2)}{2!}(\mu_{n-1}-1)^2$$

for some $\xi_2 \in (\mu_{n-1}, 1)$. As $f_1'(1) = 0$ and $f_1''(x)$ is positive and strictly decreasing on the open interval (0,1),

$$(1-\mu_{n-1})^2 = \frac{2(n-5)}{f_1''(\xi_2)} < \frac{2(n-5)}{f_1''(1)} = \frac{n-5}{4(n-2)}.$$

If $n \geq 7$, $\frac{n-5}{n-1} > \sqrt{\frac{n-5}{4(n-2)}}$, and hence $n-\mu_1 > 1-\mu_{n-1}$. The result follows. $\square$

Lemma 2.9 *For $n \geq 7$, $\mathscr{S}_L(G_2(n-6,1;n)) < n-1$.*

Proof The characteristic polynomial of $L(G_2(n-6,1;n))$ is

$$\lambda(\lambda-1)^{n-6}(\lambda-3)[\lambda^4-(n+5)\lambda^3+(6n+3)\lambda^2-(9n-5)\lambda+3n].$$

By a similar discussion to the proof of Lemma 2.8, both μ_1 and μ_{n-1} are the roots of the polynomial

$$f_2(\lambda) =: \lambda^4-(n+5)\lambda^3+(6n+3)\lambda^2-(9n-5)\lambda+3n,$$

$$f_2'(\lambda) = 4\lambda^3-3(n+5)\lambda^2+2(6n+3)\lambda-(9n-5), f_2''(\lambda) = 12\lambda^2-6(n+5)\lambda+6(2n+1).$$

and

$$n-\mu_1=\frac{f_2(n)-f_2(\mu_1)}{f_2'(\xi_1)}>\frac{n^3-6n^2+8n}{f_2'(n)}=\frac{n(n-2)(n-4)}{(n-1)(n^2-2n-5)}>\frac{n-4}{n-1},$$

for some $\xi_1\in(\mu_1,n)$. In addition,

$$f_2(\mu_{n-1})=f_2(1)+f_2'(1)(1-\mu_{n-1})+\frac{f_2''(\xi_2)}{2!}(1-\mu_{n-1})^2,$$

for some $\xi_2\in(\mu_{n-1},1)$. Noting $f_2'(1)=0$, we have

$$(1-\mu_{n-1})^2=\frac{2(n-4)}{f_2''(\xi_2)}<\frac{2(n-4)}{f_2''(1)}=\frac{n-4}{3(n-2)}.$$

If $n\geq 7$, $\frac{n-4}{n-1}>\sqrt{\frac{n-4}{3(n-2)}}$, and hence $n-\mu_1>1-\mu_{n-1}$. The result follows. $\square$

Lemma 2.10 *For $n\geq 6$, $\mathscr{S}_L(G_3(n-5,0;n))<n-1$.*

Proof The characteristic polynomial of $L(G_3(n-5,0;n))$ is

$$\lambda(\lambda-1)^{n-6}[\lambda^5-(n+8)\lambda^4+(9n+18)\lambda^3-(27n+6)\lambda^2+(31n-10)\lambda-11n].$$

So μ_1, μ_{n-1} are both the roots of the polynomial

$$f_3(\lambda)=:\lambda^5-(n+8)\lambda^4+9(n+2)\lambda^3-3(9n+2)\lambda^2+(31n-10)\lambda-11n,$$

and

$$n-\mu_1=\frac{f_3(n)-f_3(\mu_1)}{f_3'(\xi_1)}>\frac{n^4-9n^3+25n^2-21n}{f_3'(n)}=1-\frac{4n^3-25n^2+40n-10}{n^4-5n^3+19n-10},$$

for some $\xi_1\in(\mu_1,n)$. Note that the function

$$g_1(x)=:\frac{4x^3-25x^2+40x-10}{x^4-5x^3+19x-10}$$

is strictly decreasing for $x\geq 6$. Hence

$$(n-\mu_1)-(1-\mu_{n-1})=\mu_{n-1}-g_1(n)\geq\mu_{n-1}-g_1(6)=\mu_{n-1}-\frac{97}{160}.$$

Observe that a star of order n has eigenvalues: 0, n, 1 of multiplicity $n-2$, and hence has $(n-1)$ eigenvalues not less than 1. As $G_3(n-5,0;n)$ contains a star of order $n-1$, by eigenvalues interlacing theorem (that is, $\mu_i(G)\geq\mu_i(G-e)$ for $i=1,2,\ldots,n$ if we delete an edge e from a graph G of order n; or see [16]), $G_3(n-5,0;n)$ has $(n-2)$ eigenvalues not less than 1. Now $f_3(\frac{97}{160})\approx-5.2557-0.2595n<0$. So $\mu_{n-1}>\frac{97}{160}$; otherwise $\mu_{n-2}<\frac{97}{160}<1$, a contradiction. The result follows. $\square$

Lemma 2.11 *For $n\geq 6$, $\mathscr{S}_L(G_5(n-6,1;n))<n-1$.*

Proof The characteristic polynomial of $L(G_5(n-6,1;n))$ is

$$\lambda(\lambda-1)^{n-6}(\lambda-2)(\lambda-4)[\lambda^3-(n+2)\lambda^2+(3n-2)\lambda-n].$$

So μ_1, μ_{n-1} are both the roots of the polynomial

$$f_4(\lambda)=:\lambda^3-(n+2)\lambda^2+(3n-2)\lambda-n.$$

By Lagrange Mean Value Theorem,

$$n-\mu_1=\frac{f_4(n)-f_4(\mu_1)}{f_4'(\xi_1)}>\frac{n^2-3n}{f_4'(n)}=1-\frac{2n-2}{n^2-n-2},$$

for some $\xi_1\in(\mu_1,n)$. Note that the function $g_2(x)=:\frac{2x-2}{x^2-x-2}$ is strictly decreasing for all x. Hence

$$(n-\mu_1)-(1-\mu_{n-1})>\mu_{n-1}-g_2(n)\geq\mu_{n-1}-g_2(6)=\mu_{n-1}-\frac{5}{14}.$$

By a similar discussion to those in the last paragraph of the proof of Lemma 2.10, as $f_4(\frac{5}{14})=-\frac{2535}{2744}-\frac{11n}{196}<0$, $\mu_{n-1}>\frac{5}{14}$, and the result follows. □

Lemma 2.12 *For $n\geq 7$, $\mathscr{S}_L(G_6(n-7,1;n))<n-1$.*

Proof The characteristic polynomial of $L(G_6(n-7,1;n))$ is

$$\lambda(\lambda-1)^{n-6}(\lambda-3)^2[\lambda^3-(n+2)\lambda^2+(3n-2)\lambda-n].$$

So μ_1, μ_{n-1} are both the roots of the polynomial $\lambda^3-(n+2)\lambda^2+(3n-2)\lambda-n$, which is the same to $f_4(\lambda)$ in the proof of Lemma 2.11. Hence, for $n\geq 7$, $n-\mu_1>1-\mu_{n-1}$. The result follows. □

Lemma 2.13 *For $n\geq 6$, $\mathscr{S}_L(G_8(n-6,0;n))<n-1$.*

Proof The characteristic polynomial of $L(G_8(n-6,0;n))$ is

$$\lambda(\lambda-1)^{n-6}(\lambda-2)(\lambda-3)[\lambda^3-(n+3)\lambda^2+(4n-2)\lambda-2n].$$

So μ_1, μ_{n-1} are both the roots of the polynomial $f_5(\lambda)=:\lambda^3-(n+3)\lambda^2+(4n-2)\lambda-2n$. By Mean Value Theorem,

$$n-\mu_1=\frac{f_5(n)-f_5(\mu_1)}{f_5'(\xi_1)}=\frac{n^2-4n}{f_5'(\xi_1)},$$

$$1-\mu_{n-1}=\frac{f_5(1)-f_5(\mu_{n-1})}{f_5'(\xi_2)}=\frac{n-4}{f_5'(\xi_2)},$$

for some $\xi_1\in(\mu_1,n)$ and $\xi_2\in(\mu_{n-1},1)$. If we can show

$$\frac{n}{f_5'(\xi_1)}>\frac{1}{f_5'(\xi_2)},$$

the result will follow.

Note that $f_5'(\lambda)=3\lambda^2-2(n+3)\lambda+4n-2$. As $f_5'(\lambda)$ is positive and strictly decreasing on the interval (0,1), and is positive and strictly increasing on the interval (μ_1, $+\infty$),

$$nf_5'(\xi_2)>nf_5'(1)=n(2n-5).$$

$$f_5'(\xi_1)<f_5'(n)=n^2-2n-2.$$

Then

$$nf_5'(\xi_2)-f_5'(\xi_1)>n^2-3n+2>0.$$

The result follows. □

Lemma 2.14 *For $n \geq 6$, $\mathscr{S}_L(G_9(n-5,1;n)) < n-1$.*

Proof The characteristic polynomial of $L(G_9(n-5,1;n))$ is

$$\lambda(\lambda-1)^{n-6}[\lambda^5-(n+8)\lambda^4+(17+9n)\lambda^3-(2+26n)\lambda^2+(27n-13)\lambda-8n].$$

So μ_1, μ_{n-1} are both the roots of the polynomial $f_6(\lambda) =: \lambda^5-(n+8)\lambda^4+(17+9n)\lambda^3-(2+26n)\lambda^2+(27n-13)\lambda-8n$, and

$$n-\mu_1 = \frac{f_6(n)-f_6(\mu_1)}{f_6'(\xi_1)} > \frac{n^4-9n^3+25n^2-21n}{f'(n)} = 1-\frac{4n^3-26n^2+44n-13}{n^4-5n^3-n^2+23n-13}.$$

Noting that the function $g_3(x) =: \frac{4x^3-26x^2+44x-13}{x^4-5x^3-x^2+23x-13}$ is strictly decreasing for $x \geq 6$, we have

$$(n-\mu_1)-(1-\mu_{n-1}) \geq \mu_{n-1}-g_3(6) = \mu_{n-1}-\frac{179}{305}.$$

As $f(\frac{179}{305}) \approx -5.17633-0.405628n < 0$, $\mu_{n-1} > \frac{179}{305}$. The result follows. $\square$

Lemma 2.15 *For $n \geq 5$, $\mathscr{S}_L(G_9(0,n-4;n)) < n-1$.*

Proof The characteristic polynomial of $L(G_9(0,n-4;n))$ is

$$\lambda(\lambda-1)^{n-5}(\lambda-4)[\lambda^3-(n+3)\lambda^2+(4n-2)\lambda-2n].$$

So μ_1, μ_{n-1} are both the roots of the polynomial $f_7(\lambda) =: \lambda^3-(n+3)\lambda^2+(4n-2)\lambda-2n$. Then

$$n-\mu_1 > \frac{f_7(n)}{f_7'(n)} = 1-\frac{2n-2}{n^2-2n-2}.$$

Denote $g_4(x) = \frac{2x-2}{x^2-2x-2}$. Then

$$(n-\mu_1)-(1-\mu_{n-1}) \geq \mu_{n-1}-g_4(6) = \mu_{n-1}-\frac{5}{11}.$$

Noting that $f_7(\frac{5}{11}) = -\frac{1910}{1331}-\frac{47n}{121} < 0$, we have $\mu_{n-1} > \frac{5}{11}$. The result follows. $\square$

Let G be one with maximal Laplacian spread of all bicyclic graphs of order $n \geq 5$. From the first paragraph of the proof of Lemma 2.7, the graph G is necessarily among graphs in Figure 2.1. If the order $n \geq 9$, by Lemmas 2.7–2.15, G is the graph $G_1(n-4,0;n)$ or $G_2(n-5,0;n)$ of Figure 2.1. For the order $n \leq 8$, the graph(s) with maximal Laplacian spread are among the graphs in Figure 2.1, and can be identified by a little computation (through the software Mathematica), or by Lemmas of this paper; see Figure 2.2, where "Lemma" is abbreviated to "L" (e.g. "Lemma 2.15" is written as "L2.15"), and "Proof of Lemma 2.7" is abbreviated to "Pf. L2.7" for the explanation of the results on spreads of some graphs. From Figure 2.2 we find that in the case of $5 \leq n \leq 8$, the graph G is also $G_1(n-4,0;n)$ or $G_2(n-5,0;n)$.

Theorem 2.16 *For $n \geq 5$, $G_1(n-4,0;n)$ and $G_2(n-5,0;n)$ of Figure 2.1 are the only two graphs with maximal Laplacian spread among all bicyclic graphs of order n.*

$n = 5$

graph	$G_1(1,0;5)$	$G_2(0,0;5)$	$G_3(0,0;5)$	$G_9(0,1;5)$
spread	**4**	**4**	$1+\sqrt{5}$	< 4 (L2.15)

$n = 6$

graph	$G_1(2,0;6)$	$G_1(1,1;6)$	$G_2(1,0;6)$	$G_2(0,1;6)$	$G_3(1,0;6)$	$G_4(0,0;6)$
spread	**5**	$2\sqrt{5}$	**5**	≈ 4.476	< 5 (L2.10)	$\sqrt{17}$

graph	$G_5(0,1;6)$	$G_7(0,1;6)$	$G_8(0,0;6)$	$G_9(0,2;6)$	$G_9(1,1;6)$	$G_{10}(0,0;6)$
spread	< 5 (L2.11)	≈ 4.170	< 5 (L2.13)	< 5 (L2.15)	< 5 (L2.14)	4

$n = 7$

graph	$G_1(3,0;7)$	$G_1(2,1;7)$	$G_2(2,0;7)$	$G_2(1,1;7)$	$G_2(0,2;7)$	$G_3(2,0;7)$
spread	**6**	< 6 (L2.8)	**6**	< 6 (L2.9)	$2\sqrt{6}$	< 6 (L2.10)

graph	$G_3(1,1;7)$	$G_4(r,s;7)$	$G_5(0,2;7)$	$G_5(1,1;7)$	$G_6(0,1;7)$	$G_7(0,2;7)$
spread	≈ 4.863	< 6 (Pf. L2.7)	≈ 4.863	< 6 (L2.11)	< 6 (L2.12)	≈ 4.808

graph	$G_7(1,1;7)$	$G_8(1,0;7)$	$G_8(0,1;7)$	$G_9(0,3;7)$	$G_9(1,2;7)$	$G_9(2,1;7)$	$G_{10}(r,s;7)$
spread	≈ 4.808	< 6 (L2.13)	$2+2\sqrt{2}$	< 6 (L2.15)	≈ 4.884	< 6 (L2.14)	< 6 (Pf. L2.7)

$n = 8$

graph	$G_1(4,0;8)$	$G_1(3,1;8)$	$G_1(2,2;8)$	$G_2(3,0;8)$	$G_2(2,1;8)$	$G_2(1,2;8)$
spread	**7**	< 7 (L2.8)	$\sqrt{33}$	**7**	< 7 (L2.9)	≈ 5.681

graph	$G_3(3,0;8)$	$G_3(2,1;8)$	$G_4(r,s;8)$	$G_5(0,3;8)$	$G_5(1,2;8)$	$G_5(2,1;8)$
spread	< 7 (L2.10)	< 7 (Pf. L2.7)	< 7 (Pf. L2.7)	< 7 (L2.3)	≈ 5.768	< 7 (L2.11)

graph	$G_6(0,2;8)$	$G_6(1,1;8)$	$G_7(r,s;8)$	$G_8(0,2;8)$	$G_8(1,1;8)$	$G_8(2,0;8)$
spread	≈ 5.768	< 7 (L2.12)	< 7 (Pf. L2.7)	< 7 (Pf. L2.7)	< 7 (Pf. L2.7)	< 7 (L2.13)

graph	$G_9(0,4;8)$	$G_9(1,3;8)$	$G_9(2,2;8)$	$G_9(3,1;8)$	$G_{10}(r,s;8)$
spread	< 7 (L2.15)	< 7 (L2.3)	≈ 5.639	< 7 (L2.14)	< 7 (Pf. L2.7)

Figure 2.2 Laplacian spreads of graphs of order n in Figure 2.1 when $5 \le n \le 8$.

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