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The Root Operator on Invariant Subspaces of the Weighted Bergman Space

Xiao Yang ZHOU[1], **Xiu Ying SHI**[2], **Yu Feng LU**[1,*]
1. *School of Mathematical Science, Dalian University of Technology, Liaoning* 116024*, P. R. China;*
2. *School of Continuing Education, Chifeng University, Inner Mongolia* 024000*, P. R. China*

Abstract In this paper, for an invariant subspace I of the weighted Bergman space, the weighted root operator is defined. We study the weighted root operator and obtain its fundamental properties when the invariant subspace I has finite index. Furthermore, we give some examples of the root operator and estimate ranks of the operators.

Keywords root operator; index; weighted Bergman space; invariant subspace.

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1. Introduction

Let $\mathrm{d}A$ denote Lebesgue area measure on the unit disk D, normalized so that the measure of D equals 1. For a nonnegative integer α, we denote by $\mathrm{d}A_\alpha$ the measure $\mathrm{d}A_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \mathrm{d}A(z)$. The weighted Bergman space A^2_α consists of analytic functions f in the unit disk D such that

$$\|f\|^2_\alpha = \int_D |f(z)|^2 \mathrm{d}A_\alpha(z) < \infty.$$

If $\alpha = 0$, we write $A^2 = A^2_\alpha$ and call A^2 the Bergman space. We use $K_\alpha(z,w)$ to denote the reproducing kernel of the weighted Bergman space,

$$K_\alpha(z,w) = \frac{1}{(1-\overline{w}z)^{2+\alpha}}.$$

It is easy to see that A^2_α is closed in $L^2(D, \mathrm{d}A_\alpha)$, and so is a Hilbert space with inner product

$$\langle f, g\rangle = \int_D f(z)\overline{g(z)}\mathrm{d}A_\alpha(z), \quad f, g \in A^2_\alpha.$$

For the general theory of the weighted Bergman space, we refer to [1] and [2].

We use B to denote the weighted Bergman shift as

$$Bf(z) = zf(z), \quad f \in A^2_\alpha.$$

A closed subspace of A^2_α is called an invariant subspace if it is invariant for B.

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* Corresponding author
E-mail address: lyfdlut@dlut.edu.cn (Y. F. LU)

We give two examples of invariant subspaces in A^2_α. First, for any $f \in A^2_\alpha$, let I_f denote the smallest subspace containing f. Then I_f is an invariant subspace, called the invariant subspace generated by f. It is clear that I_f is simply the closure in A^2_α of the set of polynomial multiples of f.

Next, if $A = \{a_n\}_n$ is a sequence of points from D, and if I_A consists of all functions in A^2_α whose zero sets contain A (counting multiplicities), then I_A is an invariant subspace of A^2_α. We call such spaces zero-based invariant subspace.

One of the reasons why invariant subspaces in Bergman spaces A^2_α have attracted so much attention in recent years is that they are closely related to an old open problem in Operator Theory. More specifically, the invariant subspace problem (of whether every bounded linear operator on a separable Hilbert space of infinite dimension has a nontrivial invariant subspace) is equivalent to the following problem about invariant subspaces of the Bergman space A^2: Given two invariant subspaces I and J of A^2 with $I \subset J$ and $\dim(J \ominus I) = \infty$, does there exist another invariant subspace M of A^2 lying strictly between I and J? See [3] and the references therein for an explanation.

Throughout the paper we fix an invariant subspace I of A^2_α and let the operator T be the restriction of the Bergman shift B to the invariant subspace I of A^2_α. Let Z_I denote the zero set of I which consists of the common zeros of all functions in I. The operator T contains much information about invariant subspace I and has been a subject of many studies.

We denote the reproducing kernel of I by $K^I_\alpha(z,w)$. The root function R^I_α of I is defined on $D \times D$ as

$$R^I_\alpha(z,w) = \frac{K^I_\alpha(z,w)}{K_\alpha(z,w)},$$

and the associated root operator on I is defined as

$$C^I_\alpha(f)(z) = \int_D \frac{K^I_\alpha(z,w)}{K_\alpha(z,w)} f(w) \mathrm{d}A_\alpha(w).$$

When no confusion is likely, we shall suppress the superscript I and simply write $R_\alpha(z,w)$ for the root function and C_α for the root operator.

In [5], Yang and Zhu studied the root operator on an invariant subspace of the Bergman space. Motivated by paper [5], in this paper we study the root operator on an invariant subspace of the weighted Bergman space. While our method is partially adapted from [5], a substantial amount of extra work is necessary for the setting of weighted Bergman spaces.

2. The representation of the root operator

In this paper, most of our analysis is based on the following explicit relationship between the operator C_α and T.

Lemma 2.1 *For every invariant subspace I of A^2_α, with α being a nonnegative integer, we have*

$$C_\alpha = \sum_{j=0}^{\alpha+2} (-1)^j C^j_{\alpha+2} T^j T^{*j},$$

where $C^j_{\alpha+2} = \frac{(\alpha+2)\cdots(\alpha-j+3)}{j!}$.

Proof Note that for $f \in I$, we have $Tf(z) = zf(z)$ and $T^*f(z) = \int_D \overline{w}f(w)K^I_\alpha(z,w)\mathrm{d}A_\alpha(w)$. Then $T^jf(z) = z^jf(z)$, and

$$T^{*j}f(z) = \int_D \overline{w}^j f(w)K^I_\alpha(z,w)\mathrm{d}A_\alpha(w), \quad j = 0, 1, 2, \ldots .$$

By Fubini's Theorem, the above observation and $\|T\| \le 1$, we have

$$\begin{aligned} C_\alpha f(z) &= \int_D (1 - z\overline{w})^{2+\alpha} K^I_\alpha(z,w)f(w)\mathrm{d}A_\alpha(w) \\ &= \int_D \sum_{j=0}^{\alpha+2}(-1)^j C^j_{\alpha+2} z^j\overline{w}^j K^I_\alpha(z,w)f(w)\mathrm{d}A_\alpha(w) \\ &= \sum_{j=0}^{\alpha+2}(-1)^j C^j_{\alpha+2} z^j \int_D \overline{w}^j K^I_\alpha(z,w)f(w)\mathrm{d}A_\alpha(w) \\ &= (\sum_{j=0}^{\alpha+2}(-1)^j C^j_{\alpha+2} T^jT^{*j})(f(z)). \ \square \end{aligned}$$

Lemma 2.2 *For every invariant subspace I of A^2_α, with α being a nonnegative integer, we have*

$$C_\alpha = \sum_{j=0}^{\alpha+1}(-1)^j C^j_{\alpha+1} T^j(1 - TT^*)T^{*j},$$

where $C^j_{\alpha+1} = \frac{(\alpha+1)\cdots(\alpha-j+2)}{j!}$ *for* $0 \le j \le \alpha+1$, *and* $C^j_{\alpha+1} = 0$ *for* $j > \alpha+1$.

Proof It follows from Lemma 2.1 that

$$\begin{aligned} C_\alpha &= \sum_{j=0}^{\alpha+2}(-1)^j C^j_{\alpha+2} T^jT^{*j} \\ &= \sum_{j=0}^{\alpha+2}(-1)^j (C^j_{\alpha+1} + C^j_{\alpha+2} - C^j_{\alpha+1}) T^jT^{*j} \\ &= \sum_{j=0}^{\alpha+2}(-1)^j (C^j_{\alpha+1} + C^{j-1}_{\alpha+1}) T^jT^{*j} \\ &= \sum_{j=0}^{\alpha+1}(-1)^j C^j_{\alpha+1} T^jT^{*j} + \sum_{j=1}^{\alpha+2}(-1)^j C^{j-1}_{\alpha+1} T^jT^{*j} \\ &= \sum_{j=0}^{\alpha+1}(-1)^j C^j_{\alpha+1} T^jT^{*j} + \sum_{j=0}^{\alpha+1}(-1)^{j+1} C^j_{\alpha+1} T^{j+1}T^{*j+1} \\ &= \sum_{j=0}^{\alpha+1}(-1)^j C^j_{\alpha+1} T^j(1 - TT^*)T^{*j}. \ \square \end{aligned}$$

It is easy to see that $K^I_\alpha(z,z) \le K_\alpha(z,z)$, so that $0 \le R(z,z) \le 1$. If m is the smallest non-negative integer such that $f^{(m)}(0) \ne 0$ for some $f \in I$, then the following extremal problem has a unique solution:

$$\sup\{\operatorname{Re} f^m(0) : f \in I, \|f\| \leq 1\}.$$

This solution is called the extremal function for I and denoted by $G(z)$.

Lemma 2.3 *For every invariant subspace I, we have $(1-|z|^2)^{\alpha+1}|G(z)|^2 \leq R(z,z)$, $z \in D$.*

Proof Without loss of generality, we may assume $m_I = 0$. The general case will then follow from an approximation argument.

Since I is an invariant subspace of A^2_α, the function z is a contractive multiplier on I. By Theorem 2.2 in [4], the function $(1-z\overline{w})K^I_\alpha(z,w)$ is a reproducing kernel. Applying the Cauchy-Schwarz inequality to this new kernel, we obtain

$$\frac{|K^I_\alpha(z,w)|^2}{K^I_\alpha(z,z)K^I_\alpha(w,w)} \leq \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2},$$

for all z and w not in Z_I. Set $w=0$ and observe that

$$G(z) = K^I_\alpha(z,0)/\sqrt{K^I_\alpha(0,0)}$$

in the case $m_I = 0$. This lead to

$$|G(z)|^2 \leq (1-|z|^2)K^I_\alpha(z,z),$$

that is

$$(1-|z|^2)^{\alpha+1}|G(z)|^2 \leq R_\alpha(z,z). \ \square$$

Proposition 2.1 *If α is a nonnegative integer, then the root operator on the invariant subspace I always has 1 as an eigenvalue. Furthermore, $\|C_\alpha\| \leq 2^{\alpha+2}$, and for the eigenvalue 1 of C_α, its corresponding eigenspace $E_1 \supseteq I \ominus zI$.*

Proof By Lemma 2.1, $C_\alpha = \sum_{j=0}^{\alpha+2}(-1)^j C^j_{\alpha+2} T^j T^{*j}$, we have

$$\begin{aligned}\|C_\alpha\| &\leq \sum_{j=0}^{\alpha+2} |(-1)^j C^j_{\alpha+2}| \cdot \|T^jT^{*j}\| \leq \sum_{j=0}^{\alpha+2} C^j_{\alpha+2} \cdot \|T^j\| \cdot \|T^{*j}\| \\ &\leq \sum_{j=0}^{\alpha+2} C^j_{\alpha+2} = 2^{\alpha+2}.\end{aligned}$$

Since $\ker(T^*) = I \ominus zI$, Lemma 2.1 shows that $C_\alpha f(z) = f(z)$ on $I \ominus zI$, so $I \ominus zI \subseteq E_1$. $\square$

3. Membership of C_α in the algebra of all compact operators

For any $0 < p < \infty$, the Schatten p-class, or the Schatten p-ideal, denoted by S_p, consists of all compact operators S on a Hilbert space H such that the eigenvalue sequence $\{\lambda_n\}$ of $(S^*S)^{\frac{1}{2}}$ satisfies

$$\|S\|_p = (\sum_n |\lambda_n|^p)^{\frac{1}{p}} < \infty.$$

It is well known that for $1 \leq p < \infty$ the Schatten p-class is a Banach space with the above norm and is actually a two-sided ideal in the algebra of bounded linear operators on H.

When $p=1$, the Schatten p-class is called the trace class. If S is in the trace class and $\{e_n\}$ is an orthonormal basis for H, then the series $\sum_n \langle Se_n, e_n\rangle$ converges and the sum is independent of the choice of the orthonormal basis. This sum is called the trace of S and is denoted by $\mathrm{tr}(S)$. It is easy to see that if S is a self-adjoint operator in the trace class, and if the eigenvalue sequence of S is $\{\lambda_n\}$, then $\mathrm{tr}(S)=\sum_n \lambda_n$.

When $p=2$, the Schatten p-class is called the Hilbert-Schmidt class. If S is in the Hilbert-Schmidt class and if $\{e_n\}$ is an orthonormal basis for H, then the series $\sum_n \|Se_n\|^2$ converges and the sum is independent of the choice of the orthonormal basis. The square root of this sum is equal to the Hilbert-Schmidt norm $\|S\|_2$, or the Schatten 2-norm, of S.

For $z\in D-Z_I$, we define

$$k^I_{\alpha,z}(w)=\frac{K^I_\alpha(w,z)}{\sqrt{K^I_\alpha(z,z)}},\quad w\in D,$$

and call it the normalized reproducing kernel of I at z.

For any bounded linear operator S on I we define a bounded function $\tilde{S}$ in $L^\infty(D,\mathrm{d}A_\alpha)$ by

$$\tilde{S}(z)=\langle Sk^I_{\alpha,z},k^I_{\alpha,z}\rangle,\quad z\in D-Z_I.$$

This function is called the Berezin transform of S.

Lemma 3.1 *Let S be a bounded linear operator on I. If S is either positive or in the trace class, then*

$$\mathrm{tr}(S)=\int_D \tilde{S}(z)K^I_\alpha(z,z)\mathrm{d}A_\alpha(z).$$

In particular, a positive operator S on I belongs to the trace class if and only if the above integral is finite.

Proof Fix an orthonormal basis $\{e_n\}_n$ of I, then

$$\begin{aligned}\mathrm{tr}(S)&=\sum_{n=1}^\infty\langle Se_n,e_n\rangle=\sum_{n=1}^\infty\int_D(Se_n)(z)\overline{e_n(z)}\mathrm{d}A_\alpha(z)\\&=\sum_{n=1}^\infty\int_D\langle Se_n,K^I_{\alpha,z}\rangle\overline{e_n(z)}\mathrm{d}A_\alpha(z)=\int_D\langle S\sum_{n=1}^\infty e_n\overline{e_n(z)},K^I_{\alpha,z}\rangle\mathrm{d}A_\alpha(z)\\&=\int_D\langle SK^I_{\alpha,z},K^I_{\alpha,z}\rangle\mathrm{d}A_\alpha(z)=\int_D\langle Sk^I_{\alpha,z},k^I_{\alpha,z}\rangle K^I_\alpha(z,z)\mathrm{d}A_\alpha(z)\\&=\int_D\tilde{S}(z)K^I_\alpha(z,z)\mathrm{d}A_\alpha(z).\ \square\end{aligned}$$

Lemma 3.2 *A bounded linear operator S on I is Hilbert-Schmidt if and only if*

$$\int_D\|Sk^I_{\alpha,z}\|^2K^I_\alpha(z,z)\mathrm{d}A_\alpha(z)<\infty.$$

Furthermore, the square root of the above integral is equal to the Hilbert-Schmidt norm of S.

Proof The operator S is Hilbert-Schmidt if and only if the positive operator S^*S is in the trace class. Since

$$\widetilde{S^*S}(z) = \langle S^*Sk^I_{\alpha,z}, k^I_{\alpha,z}\rangle = \|Sk^I_{\alpha,z}\|^2,$$

the desired result then follows from Lemma 3.1. □

Lemma 3.3 *If the root operator C_α is in the trace class, then*

$$\mathrm{tr}(C_\alpha) = \int_D R^I_\alpha(z,z)\mathrm{d}A_\alpha(z),$$

and the Hilbert-Schmidt norm of C_α is

$$\|C_\alpha\|_2 = \Big[\int_D\int_D |R^I_\alpha(z,w)|^2\mathrm{d}A_\alpha(z)\mathrm{d}A_\alpha(w)\Big]^{\frac{1}{2}}.$$

In particular, if C_α is in the trace class, then $0 < \mathrm{tr}(C_\alpha) \le 1$, and $\mathrm{tr}(C_\alpha) = 1$ holds only for $I = A^2_\alpha$.

Proof The proof is similar to that of Corollary 6 in [5]. □

Lemma 3.4 *Suppose I is an invariant subspace of $A^2_\alpha(D)$, α is a nonnegative integer. Then the self-commutator $[T^*,T]$ is compact, if and only if,* $\mathrm{index}(I) < \infty$.

Proof First assume that $\mathrm{index}(I) = \infty$. In the following, we will use the fact that $\|Bf\| \ge \frac{1}{\sqrt{2+\alpha}}\|f\|$ for every $f \in A^2_\alpha(D)$. In fact, for any function $f = \sum_{n=0}^\infty a_n z^n \in A^2_\alpha(D)$,

$$\|B(f)\|^2 = \|\sum_{n=0}^\infty a_n z^{n+1}\|^2 = \sum_{n=0}^\infty |a_n|^2\omega_{n+1} = \sum_{n=0}^\infty |a_n|^2\omega_n\frac{n+1}{n+2+\alpha} \ge \frac{1}{2+\alpha}\|f\|^2,$$

where $\omega_n = \frac{n!(2+\alpha)!}{(n+2+\alpha)!}$.

Let $\{f_n\}$ be an orthonormal basis for $I \ominus zI$. Then $f_n \to 0$ weakly in I as $n \to \infty$. And

$$\langle [T^*,T]f_n, f_n\rangle = \langle (T^*T - TT^*)f_n, f_n\rangle = \langle T^*Tf_n, f_n\rangle = \|Tf_n\|^2 \ge \frac{1}{2+\alpha}.$$

Since $\langle [T^*,T]f_n, f_n\rangle = \|[T^*,T]^{\frac{1}{2}}f_n\|^2$, we have $\|[T^*,T]f_n\|^2 \ge \frac{1}{2+\alpha}$, which implies that $[T^*,T]$ is not compact.

Next, we assume that *index*$(I) = n < +\infty$.

Under the standard orthonormal basis of $A^2_\alpha(D)$, the operator B^*B can be represented as

$$\begin{pmatrix} \omega_1 & 0 & \cdots & 0 & \cdots \\ 0 & \frac{\omega_2}{\omega_1} & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\omega_{n+1}}{\omega_n} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix},$$

then

$$I - B^*B = \begin{pmatrix} \frac{1+\alpha}{2+\alpha} & 0 & \cdots & 0 & \cdots \\ 0 & \frac{1+\alpha}{3+\alpha} & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1+\alpha}{n+2+\alpha} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to see that $B^*B = I + K$, where K is compact on $A^2_\alpha(D)$, and

$$K = \begin{pmatrix} \frac{1+\alpha}{2+\alpha} & 0 & \cdots & 0 & \cdots \\ 0 & \frac{1+\alpha}{3+\alpha} & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1+\alpha}{n+2+\alpha} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \ddots \end{pmatrix}.$$

The Bergman shift B has the block matrix representation with respect to the decomposition $A^2_\alpha(D) = I \oplus I^\perp$:

$$B = \begin{pmatrix} T & P_I B \\ 0 & P_{I^\perp} B \end{pmatrix},$$

where P_I and $P_{I^\perp}$ are the orthogonal projections from $A^2_\alpha(D)$ onto I and $I^\perp$, respectively. Let

$$S = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$[S^*, S] = \begin{pmatrix} [T^*, T] & 0 \\ 0 & 0 \end{pmatrix},$$

and $[T^*, T]$ is compact on I, if and only if, $[S^*, S]$ is compact on $A^2_\alpha(D)$.

For $S = BP_I$, we have

$$\begin{aligned} [S^*, S]S = (S^*S - SS^*)S &= P_I B^* B P_I B P_I - B P_I B^* B P_I \\ &= P_I(I + K)P_I B P_I - B P_I (I + K) P_I \\ &= P_I K P_I B P_I - B P_I K P_I. \end{aligned}$$

Then $[S^*, S]S$ is compact on $A^2_\alpha(D)$, so $[T^*, T]T$ is compact on I.

It is easy to check that T has closed range, and $\ker T^* = I \ominus zI$. So T is Fredholm since $\mathrm{index}(I) < +\infty$.

Let $\mathcal{B}(I)$ and $\mathcal{B}_0(I)$ denote the set of bounded linear operators and the set of compact operators, on I, respectively. And let π be the natural homomorphism from $\mathcal{B}(I)$ onto the Calkin algebra $\mathcal{B}(I)/\mathcal{B}_0(I)$.

Since $[T^*, T]T$ is compact on I, $\pi([T^*, T]T) = \pi([T^*, T])\pi(T) = 0$. It follows from that T is Fredholm, $\pi(T)$ is invertible in $\mathcal{B}(I)/\mathcal{B}_0(I)$. So we have $\pi([T^*, T]) = 0$, that is, $[T^*, T]$ is compact. □

Lemma 3.5 *The following conditions are equivalent:*

(1) The operator $1 - TT^*$ *is compact;*

(2) $\mathrm{index}(I) < \infty$.

Proof First observe that the operator $1 - T^*T$ is always Hilbert-Schmidt, regardless of the index of I. In fact, it is a Toeplitz-type operator, and its integral representation is given by

$$(1 - T^*T)f(z) = \int_D (1 - |w|^2) K_\alpha^I(z, w) f(w) \mathrm{d}A_\alpha(w), \quad f \in I.$$

Therefore, we have the following estimate for the Hilbert-Schmidt norm.

$$\begin{aligned}\|1 - T^*T\|_2^2 &= \int_D \int_D (1 - |z|^2)^2 |K_\alpha^I(z, \omega)|^2 \mathrm{d}A_\alpha(z) \mathrm{d}A_\alpha(\omega) \\ &= \int_D (1 - |z|^2)^2 \mathrm{d}A_\alpha(z) \int_D |K_\alpha^I(z, \omega)|^2 \mathrm{d}A_\alpha(\omega) \\ &= \int_D (1 - |z|^2)^2 K_\alpha^I(z, z) \mathrm{d}A_\alpha(z) \\ &= \int_D (\alpha + 1)(1 - |z|^2)^{2+\alpha} K_\alpha^I(z, z) \mathrm{d}A(z) \\ &= (\alpha + 1) \int_D R_\alpha(z, z) \mathrm{d}A(z) \\ &\le \alpha + 1.\end{aligned}$$

So $1 - T^*T$ is Hilbert-Schmidt. In particular, $1 - T^*T$ is compact.

Next observe that $1 - TT^* = 1 - T^*T + [T^*, T]$. If $\mathrm{index}(I) < \infty$, then by Lemma 3.4, the self-commutator $[T^*, T]$ is compact. Thus $1 - TT^*$ is compact whenever I has finite index.

If $1 - TT^*$ is compact, then the fact $(1 - TT^*)f = f$, $f \in I \ominus zI$, implies that $I \ominus zI$ is finite dimensional, or that I has finite index. □

Theorem 3.1 *If* α *is a nonnegative integer, then the following conditions are equivalent:*

(1) $\mathrm{index}(I) < \infty$*;*

(2) The root operator C_α *is compact.*

Proof (2)⇒(1). If the root operator C_α is compact. From Proposition 2.1, it is easy to see that $\mathrm{index}(I) < \infty$.

(1)⇒(2). If the index of I is finite, then by Lemma 3.5, $1 - TT^*$ is compact. Since the algebra of all compact operators is actually a closed two-sided ideal in the algebra of all bounded linear operators on I, the operator $T^j(1 - TT^*)T^{*j}$ is compact for $j = 0, 1, 2, \ldots$.

By Lemma 2.2, the root operator

$$C_\alpha = \sum_{j=0}^{\alpha+1} (-1)^j C_{\alpha+1}^j T^j (1 - TT^*) T^{*j},$$

then C_α is compact. □

4. Examples of the root operator

In this section we give some examples of the root operator. In particular, we compute the rank of the root operator, which is the dimension of the range of the root operator.

Proposition 4.1 *If I is an invariant subspace generated by $(z-a)^N$, $a \in D$, N is a positive integer, and α is a non-negative integer, then the root function of I is*

$$R_\alpha(z,w) = 1 - (1-z\overline{w})^{\alpha+2} \sum_{k=0}^{N-1} \frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)} (\varphi_a(z)\overline{\varphi_a(w)})^k.$$

In particular, the root operator C_α has finite rank, and its rank is at most $N+\alpha+1$. Moreover, except for 1, the sum λ of the other eigenvalues of C_α is

$$\lambda = \int_D -(1-z\overline{w})^{\alpha+2} \sum_{k=0}^{N-1} \frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)} |\varphi_a(z)|^{2k} \mathrm{d}A_\alpha(z).$$

Proof If $a=0$, then

$$\begin{aligned} K_\alpha^I(z,w) &= \frac{1}{(1-z\overline{w})^{\alpha+2}} - \sum_{k=0}^{N-1} \frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)} z^k \overline{w}^k \\ &= \frac{1}{(1-z\overline{w})^{\alpha+2}} [1 - (1-z\overline{w})^{\alpha+2} \sum_{k=0}^{N-1} \frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)} z^k \overline{w}^k] \\ &= \frac{1}{(1-z\overline{w})^{\alpha+2}} [1 - (\sum_{j=0}^{\alpha+2} (-1)^j C_{\alpha+2}^j (z\overline{w})^j)(\sum_{k=0}^{N-1} \frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)} (z\overline{w})^k)]. \end{aligned}$$

In general, $a \neq 0$, we use the Möbius invariance of the reproducing kernel to obtain

$$\begin{aligned} K_\alpha^I(z,w) &= \frac{1}{(1-z\overline{w})^{\alpha+2}} - \sum_{k=0}^{N-1} \frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)} \varphi_a(z)^k \overline{\varphi_a(w)}^k \\ &= \frac{1}{(1-z\overline{w})^{\alpha+2}} [1 - (\sum_{j=0}^{\alpha+2} (-1)^j C_{\alpha+2}^j (\varphi_a(z)\overline{\varphi_a(w)})^j)(\sum_{k=0}^{N-1} \frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)} (\varphi_a(z)\overline{\varphi_a(w)})^k)], \\ R_\alpha(z,w) &= 1 - (1-z\overline{w})^{\alpha+2} \sum_{k=0}^{N-1} \frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)} (\varphi_a(z)\overline{\varphi_a(w)})^k \\ &= 1 - (\sum_{j=0}^{\alpha+2} (-1)^j C_{\alpha+2}^j (\varphi_a(z)\overline{\varphi_a(w)})^j)(\sum_{k=0}^{N-1} \frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)} (\varphi_a(z)\overline{\varphi_a(w)})^k). \end{aligned}$$

Let $\varphi_a(z)\overline{\varphi_a(w)} = x$, $M = \max\{\alpha+2, N-1\}$. Then

$$R_\alpha(z,w) = 1 - (\sum_{j=0}^{M} a_j x^j)(\sum_{k=0}^{M} b_k x^k),$$

where $a_j = (-1)^j C_{\alpha+2}^j$, $b_k = \frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)}$ for $j \le \alpha+2$, $k \le N-1$, and $a_j = 0$, $b_k = 0$ for $j > \alpha+2$, $k > N-1$. So it follows from the multiplication rule of series that

$$R_\alpha(z,w) = 1 - \sum_{n=0}^{2M} \sum_{j+k=n} a_j b_k x^{j+k} = 1 - \sum_{n=0}^{N+\alpha+1} \sum_{j+k=n} a_j b_k x^{j+k}$$

$$= -\sum_{n=1}^{N+\alpha+1}\sum_{j+k=n} a_j b_k x^{j+k}.$$

So the root operator C_α has finite rank, and its rank is at most $N+\alpha+1$.

Since C_α is self-adjoint, we have $1+\lambda = \mathrm{tr}(C_\alpha) = \int_D R_\alpha(z,z)\mathrm{d}A_\alpha(z)$, and it follows that

$$\lambda = \int_D -(1-z\overline{w})^{\alpha+2}\sum_{k=0}^{N-1}\frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)}|\varphi_a(z)|^{2k}\mathrm{d}A_\alpha(z). \ \Box$$

Proposition 4.2 *Suppose $N = \dim(A_\alpha^2 \ominus I) < \infty$, where α is a non-negative integer. Then the rank of C_α is finite, and its rank is at most $N+\alpha+3$.*

Proof It is well known that I has finite codimension in A_α^2 if and only if it is generated by a finite Blaschke product $B(z)$, and the codimension of I in A_α^2 is equal to the number of zeros of $B(z)$.

In fact, let $[B]$ denote the closure in the Bergman space A_α^p of the set of all multiples of B by polynomials, N^p denote the set of functions in A_α^p that vanish on the zero-set of B. It is clear that $[B] \subset N^p$, so we have to show that each $f \in N^p$ can be approximated by polynomial multiples of B. Write $B = B_nR_n$, where B_n is the partial product consisting of the first n factors of B. Then $R_n(z) \to 1$ as $n \to \infty$ for each $z \in D$, while $\|R_n\|_\infty = 1$. By the Lebesgue dominated convergence theorem, this shows that $\|R_nf - f\|_{\alpha,p} \to 0$ for each $f \in A_\alpha^p$. Given $\varepsilon > 0$, choose n so large that $\|R_nf - f\|_{\alpha,p} < \frac{\varepsilon}{2}$. Because $f/B_n \in A_\alpha^p$, there is a polynomial Q such that $\|Q - f/B_n\|_{\alpha,p} < \frac{\varepsilon}{2}$. Thus $\|BQ - R_nf\|_{\alpha,p} < \frac{\varepsilon}{2}$, since $\|Bg\|_{\alpha,p} < \|g\|_{\alpha,p}$ for every function $g \in A_\alpha^p$. Now for $1 \le p < \infty$, the triangle inequality is not strictly valid but can be replaced by the inequality $\|f+g\|_{\alpha,p}^p \le \|f\|_{\alpha,p}^p + \|g\|_{\alpha,p}^p$. So $N^p \subset [B]$, then $N^p = [B]$.

So the case $N = 1$ follows from Proposition 4.1.

Let $\{e_1, \dots, e_N\}$ be an orthonormal basis for $A_\alpha^2 \ominus I$. Then

$$K_\alpha^I(z,w) = K_\alpha(z,w) - \sum_{k=1}^N e_k(z)\overline{e_k(w)}.$$

So the root function is given by

$$\begin{aligned} R_\alpha^I(z,w) &= 1 - (1-z\overline{w})^{\alpha+2}\sum_{k=1}^N e_k(z)\overline{e_k(w)} \\ &= 1 - (1-z\overline{w})^{\alpha+2}\sum_{k=1}^N \frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)}(z\overline{w})^k \\ &= 1 - \Big(\sum_{j=0}^{\alpha+2}(-1)^jC_{\alpha+2}^j(z\overline{w})^j\Big)\Big(\sum_{k=1}^N \frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)}(z\overline{w})^k\Big). \end{aligned}$$

According to the multiplication rule of series, we have

$$R_\alpha^I(z,w) = 1 - \sum_{n=1}^{N+\alpha+2}\sum_{j+k=n} a_jb_k(z\overline{w})^{j+k},$$

where

$$a_j = (-1)^j C_{\alpha+2}^j,\ b_k = \frac{\Gamma(k+2+\alpha)}{k!\Gamma(2+\alpha)},\ \text{for } j \le \alpha+2,\ k \le N,$$

and

$$a_j = 0,\ \ b_k = 0,\ \ \text{for } j > \alpha+2,\ \ k > N.$$

So the rank of C_α^I is finite, and its rank is at most $N+\alpha+3$. □

Proposition 4.3 *If the invariant subspace I is generated by n zeros $a_1, a_2, \ldots, a_n$, and α is a nonnegative integer, then the root function C_α^I has finite rank, and its rank is at most $1 + \frac{n(n+1)(\alpha+3)}{2}$.*

Proof (1) For I is generated by one zero a_1, we have

$$K_\alpha^{(1)}(z,w) = K_\alpha(z,w) - (1-|a_1|^2)^{2+\alpha} K_\alpha(z,a_1) K_\alpha(a_1,w),$$

$$R_\alpha^{(1)}(z,w) = 1 - (1-|a_1|^2)^{2+\alpha} \frac{K_\alpha(a_1,w)}{K_\alpha(z,w)} K_\alpha(z,a_1).$$

So

$$\begin{aligned} C_\alpha^{(1)} f(z) &= \int R_\alpha^{(1)}(z,w) f(w) \mathrm{d}A_\alpha(w) \\ &= f(0) - (1-|a_1|^2)^{2+\alpha} K_\alpha(z,a_1) \int \frac{K_\alpha(a_1,w)}{K_\alpha(z,w)} f(w) \mathrm{d}A_\alpha(w) \\ &= f(0) - (1-|a_1|^2)^{2+\alpha} M_1(z) K_\alpha(z,a_1), \end{aligned}$$

where

$$\begin{aligned} M_1(z) &= \int \frac{K_\alpha(a_1,w)}{K_\alpha(z,w)} f(w) \mathrm{d}A_\alpha(w) \\ &= \int (1-z\overline{w})^{\alpha+2} K_\alpha(a_1,w) f(w) \mathrm{d}A_\alpha(w) \\ &= \int \sum_{j=0}^{\alpha+2} (-1)^j C_{\alpha+2}^j z^j \overline{w}^j K_\alpha(a_1,w) f(w) \mathrm{d}A_\alpha(w) \\ &= \sum_{j=0}^{\alpha+2} (-1)^j C_{\alpha+2}^j z^j \int \overline{w}^j K_\alpha(a_1,w) f(w) \mathrm{d}A_\alpha(w). \end{aligned}$$

Thus the rank of $C_\alpha^{(1)}$ is at most $\alpha+4$.

(2) For I is generated by two zeros a_1, a_2, we have

$$K_\alpha^{(2)}(z,w) = K_\alpha^{(1)}(z,w) - \frac{K_\alpha^{(1)}(z,a_2) K_\alpha^{(1)}(a_2,w)}{K_\alpha^{(1)}(a_2,a_2)},$$

$$K_\alpha^{(1)}(z,a_2) = K_\alpha(z,a_2) - (1-|a_1|^2)^{2+\alpha} K_\alpha(z,a_1) K_\alpha(a_1,a_2),$$

$$R_\alpha^{(2)}(z,w) = R_\alpha^{(1)}(z,w) - \frac{K_\alpha^{(1)}(a_2,w)}{K_\alpha^{(1)}(a_2,a_2) K_\alpha(z,w)} [K_\alpha(z,a_2) - (1-|a_1|^2)^{2+\alpha} K_\alpha(z,a_1) K_\alpha(a_1,a_2)],$$

$$C_\alpha^{(2)} f(z) = \int R_\alpha^{(1)}(z,w) f(w) \mathrm{d}A_\alpha(w) -$$

$$\int \frac{K_\alpha^{(1)}(a_2,w)}{K_\alpha^{(1)}(a_2,a_2)K_\alpha(z,w)} f(w)\mathrm{d}A_\alpha(w)[K_\alpha(z,a_2)-(1-|a_1|^2)^{2+\alpha}K_\alpha(z,a_1)K_\alpha(a_1,a_2)].$$

Since

$$\int \frac{K_\alpha^{(1)}(a_2,w)}{K_\alpha^{(1)}(a_2,a_2)K_\alpha(z,w)} f(w)\mathrm{d}A_\alpha(w) = \sum_{j=0}^{\alpha+2}(-1)^j C_{\alpha+2}^j z^j \int \overline{w}^j \frac{K_\alpha^{(1)}(a_2,w)}{K_\alpha^{(1)}(a_2,a_2)} f(w)\mathrm{d}A_\alpha(w) = M_2(z),$$

we have

$$C_\alpha^{(2)}f(z) = \int R_\alpha^{(1)}(z,w)f(w)\mathrm{d}A_\alpha(w) - M_2(z)[K_\alpha(z,a_2)-(1-|a_1|^2)^{2+\alpha}K_\alpha(z,a_1)K_\alpha(a_1,a_2)],$$

and then the rank of $C_\alpha^{(2)}$ is at most $\alpha+4+2(\alpha+3)=3\alpha+10$.

(3) For I is generated by n zeros $a_1,a_2,\ldots,a_n$, we have

$$K_\alpha^{(n)}(z,w) = K_\alpha^{(n-1)}(z,w) - \frac{K_\alpha^{(n-1)}(z,a_n)K_\alpha^{(n-1)}(a_n,w)}{K_\alpha^{(n-1)}(a_n,a_n)},$$

$$R_\alpha^{(n)}(z,w) = R_\alpha^{(n-1)}(z,w) - \frac{K_\alpha^{(n-1)}(a_n,w)}{K_\alpha(z,w)K_\alpha^{(n-1)}(a_n,a_n)} K_\alpha^{(n-1)}(z,a_n),$$

$$\begin{aligned} K_\alpha^{(n-1)}(z,a_n) = & q_0^{(n-1)}(a_1,\ldots,a_n)K_\alpha(z,a_n) + q_1^{(n-1)}(a_1,\ldots,a_n)K_\alpha(z,a_{n-1}) + \cdots + \\ & q_{n-1}^{(n-1)}(a_1,\ldots,a_n)K_\alpha(z,a_1), \end{aligned}$$

where $q_i^{(n-1)}(a_1,\ldots,a_n)$ $(i=0,\ldots,n-1)$ are the constants. So

$$\begin{aligned} C_\alpha^{(n)}f(z) &= \int R_\alpha^{(n-1)}(z,w)f(w)\mathrm{d}A_\alpha(w) - K_\alpha^{(n-1)}(z,a_n)\int \frac{K_\alpha^{(n-1)}(a_n,w)}{K_\alpha(z,w)K_\alpha^{(n-1)}(a_n,a_n)} f(w)\mathrm{d}A_\alpha(w) \\ &= \int R_\alpha^{(n-1)}(z,w)f(w)\mathrm{d}A_\alpha(w) - M_n(z)K_\alpha^{(n-1)}(z,a_n), \end{aligned}$$

where

$$\begin{aligned} M_n(z) &= \int \frac{K_\alpha^{(n-1)}(a_n,w)}{K_\alpha(z,w)K_\alpha^{(n-1)}(a_n,a_n)} f(w)\mathrm{d}A_\alpha(w) \\ &= \sum_{j=0}^{\alpha+2}(-1)^j C_{\alpha+2}^j z^j \int \overline{w}^j \frac{K_\alpha^{(n-1)}(a_n,w)}{K_\alpha^{(n-1)}(a_n,a_n)} f(w)\mathrm{d}A_\alpha(w). \end{aligned}$$

So the rank of $C_\alpha^{(n)}$ is at most $1+\frac{n(n+1)(\alpha+3)}{2}$.

(4) For I is generated by $n+1$ zeros $a_1,a_2,\ldots,a_{n+1}$, we have

$$K_\alpha^{(n+1)}(z,w) = K_\alpha^{(n)}(z,w) - \frac{K_\alpha^{(n)}(z,a_{n+1})K_\alpha^{(n)}(a_{n+1},w)}{K_\alpha^{(n)}(a_{n+1},a_{n+1})},$$

$$R_\alpha^{(n+1)}(z,w) = R_\alpha^{(n)}(z,w) - \frac{K_\alpha^{(n)}(a_{n+1},w)}{K_\alpha(z,w)K_\alpha^{(n)}(a_{n+1},a_{n+1})} K_\alpha^{(n)}(z,a_{n+1}),$$

$$K_\alpha^{(n)}(z,a_{n+1}) = K_\alpha^{(n-1)}(z,a_{n+1}) - \frac{K_\alpha^{(n-1)}(z,a_n)K_\alpha^{(n-1)}(a_n,a_{n+1})}{K_\alpha^{(n-1)}(a_n,a_n)}.$$

According to (3), we have

$$\begin{aligned}
&K_\alpha^{(n)}(z,a_{n+1})\\
&=q_0^{(n-1)}(a_1,\dots,a_{n-1},a_{n+1})K_\alpha(z,a_{n+1})+q_1^{(n-1)}(a_1,\dots,a_{n-1},a_{n+1})K_\alpha(z,a_{n-1})+\cdots+\\
&\quad q_{n-1}^{(n-1)}(a_1,\dots,a_{n-1},a_{n+1})K_\alpha(z,a_1)-\frac{K_\alpha^{(n-1)}(a_n,a_{n+1})}{K_\alpha^{(n-1)}(a_n,a_n)}K_\alpha^{(n-1)}(z,a_n)\\
&=q_0^{(n-1)}(a_1,\dots,a_{n-1},a_{n+1})K_\alpha(z,a_{n+1})-\frac{K_\alpha^{(n-1)}(a_n,a_{n+1})}{K_\alpha^{(n-1)}(a_n,a_n)}q_0^{(n-1)}(a_1,\dots,a_n)K_\alpha(z,a_n)+\\
&\quad [q_1^{(n-1)}(a_1,\dots,a_{n-1},a_{n+1})-\frac{K_\alpha^{(n-1)}(a_n,a_{n+1})}{K_\alpha^{(n-1)}(a_n,a_n)}q_1^{(n-1)}(a_1,\dots,a_n)]K_\alpha(z,a_{n-1})+\\
&\quad [q_{n-1}^{(n-1)}(a_1,\dots,a_{n-1},a_{n+1})-\frac{K_\alpha^{(n-1)}(a_n,a_{n+1})}{K_\alpha^{(n-1)}(a_n,a_n)}q_{n-1}^{(n-1)}(a_1,\dots,a_n)]K_\alpha(z,a_1).
\end{aligned}$$

We can write

$$\begin{aligned}
K_\alpha^{(n)}(z,a_{n+1})=&q_0^{(n)}(a_1,\dots,a_{n+1})K_\alpha(z,a_{n+1})+q_1^{(n)}(a_1,\dots,a_{n+1})K_\alpha(z,a_n)+\cdots+\\
&q_n^{(n)}(a_1,\dots,a_{n+1})K_\alpha(z,a_1).
\end{aligned}$$

Then

$$\begin{aligned}
C_\alpha^{(n+1)}f(z)&=\int R_\alpha^{(n)}(z,w)f(w)\mathrm{d}A_\alpha(w)-K_\alpha^{(n)}(z,a_{n+1})\int\frac{K_\alpha^{(n)}(a_{n+1},w)}{K_\alpha(z,w)K_\alpha^{(n)}(a_{n+1},a_{n+1})}f(w)\mathrm{d}A_\alpha(w)\\
&=\int R_\alpha^{(n)}(z,w)f(w)\mathrm{d}A_\alpha(w)-M_{n+1}(z)K_\alpha^{(n)}(z,a_{n+1}),
\end{aligned}$$

where

$$\begin{aligned}
M_{n+1}(z)&=\int\frac{K_\alpha^{(n)}(a_{n+1},w)}{K_\alpha(z,w)K_\alpha^{(n)}(a_{n+1},a_{n+1})}f(w)\mathrm{d}A_\alpha(w)\\
&=\sum_{j=0}^{\alpha+2}(-1)^jC_{\alpha+2}^jz^j\int\overline{w}^j\frac{K_\alpha^{(n)}(a_{n+1},w)}{K_\alpha^{(n)}(a_{n+1},a_{n+1})}f(w)\mathrm{d}A_\alpha(w).
\end{aligned}$$

So the rank of $C_\alpha^{(n+1)}$ is at most $1+\frac{n(n+1)(\alpha+3)}{2}+(n+1)(\alpha+3)=1+\frac{(n+1)(n+2)(\alpha+3)}{2}$.

The desired result is thus proved by induction. □

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