Journal of Mathematical Research & Exposition
Jan., 2010, Vol. 30, No. 1, pp. 67–77
DOI:10.3770/j.issn:1000-341X.2010.01.006
Http://jmre.dlut.edu.cn

The Variational Lyapunov Method and Stability for Impulsive Delay Differential Systems

Kai En LIU[1,2,*], **Guo Wei YANG**[3]

1. *School of Mathematics, Qingdao University, Shandong* 266071*, P. R. China;*
2. *Center for Systems and Control, College of Engineering, Peking University, Beijing* 100871*, P. R. China;*
3. *School of Automation Engineering, Qingdao University, Shandong* 266071*, P. R. China*

Abstract By using the variational Lyapunov method and Razumikhin technique, the stability criteria in terms of two measures for impulsive delay differential systems are established. The known results are generalized and improved. An example is worked out to illustrate the advantages of the theorems.

Keywords impulsive delay differential system; stability; variational Lyapunov method; Razumikhin technique; two measures.

Document code A
MR(2000) Subject Classification 34D20; 34K45
Chinese Library Classification O175.13

1. Introduction

Recently, there has been a growing interest in the study of impulsive systems since they provide a natural framework for mathematical modeling of many real world phenomena. Significant progress on impulsive system has been made during the past 20 years, see [1–5] and references therein.

To unify a variety of stability concepts and to offer a general framework for investigation of stability theory, introducing the concept of stability in terms of two measures has been proven to be very useful [6,7].

In the study of nonlinear systems, the method of variation of parameters is an effective technique in the case that unperturbed terms are linear ones or of certain smoothness, though they might be nonlinear. On the other hand, Lyapunov second method is an indispensable tool in the theory of stability. By combining the two methods, the so-called variational Lyapunov method has been developed, see [8–12] and references therein. However, as for using variational Lyapunov method to investigate the stability for impulsive delay differential system, we only see the paper [13].

Received January 11, 2008; Accepted October 6, 2008
Supported by the National Natural Science Foundation of China (Grant No. 60973048), the Natural Science Foundation of Shandong Province (Grant No. Y2007G30) and the Young Science Foundation of Qingdao University.
* Corresponding author
E-mail address: kaienliu@yahoo.com.cn (K. E. LIU)

In this paper, we discuss stability in terms of two measures for impulsive delay differential systems by employing the variational Lyapunov method and Razumikhin technique. Several stability criteria are obtained for impulsive delay differential systems with fixed moments of impulsive effects. Our results improve and generalize some of the ones in [13]. The rest of this paper is organized as follows. In Section 2, we introduce some notations and definitions. In Section 3, we first give two Razumikhin type comparison lemmas. Then we establish several criteria on stability for impulsive delay differential systems. At last, an example is worked out to illustrate our results.

2. Preliminaries

Consider the impulsive delay differential system

$$\begin{cases} x'(t) = f(t, x_t),\ t \neq \tau_k, \\ x(\tau_k) = x(\tau_k^-) + I_k(x(\tau_k^-)), k \in \mathcal{N}, \\ x(t_0) = \varphi \end{cases} \tag{2.1}$$

and the ordinary differential system

$$\begin{cases} y'(t) = g(t, y),\ t \neq \tau_k, \\ y(t_0) = x_0, \end{cases} \tag{2.2}$$

where $\mathcal{N}$ is the set of all positive integers, $f : R_+ \times PC_\tau \to R^n,\ g : R_+ \times R^n \to R^n,\ I_k : R^n \to R^n$ for each $k \in \mathcal{N},\ R_+ = [0, \infty),\ PC_\tau = PC([-\tau, 0], R^n)$, where $\tau > 0$ and $PC([-\tau, 0], R^n) = \{\varphi : [-\tau, 0] \to R^n,\ \varphi(t)$ is continuous everywhere except at a finite number of points $\overline{t}$ at which $\varphi(\overline{t}^+)$ and $\varphi(\overline{t}^-)$ exist and $\varphi(\overline{t}^+) = \varphi(\overline{t})\ \}$. $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \tau_{k+1} < \cdots$ with $\tau_k \to \infty$ as $k \to \infty$ and $x'(t), y'(t)$ denote the right-hand derivatives of $x(t), y(t)$, respectively. For each $t \in R_+,\ x_t \in PC_\tau$ is defined by $x_t(\theta) = x(t + \theta),\ -\tau \leq \theta \leq 0$. We assume that $f(t, 0) = g(t, 0) = I_k(t, 0) = 0$ for all $t \in R_+$ and $k \in \mathcal{N}$ so that systems (2.1) and (2.2) admit trivial solutions.

Throughout this paper, we always assume f, g and I_k satisfy certain conditions to ensure the global existence and uniqueness of solutions of (2.1) and (2.2). Moreover, we assume that the solution $y(t) = y(t, t_0, x_0)$ is locally Lipschitzian in x_0 and depends continuously on initial data.

Definition 2.1 ([1]) *The function $V(t, x) : R_+ \times R^n \to R_+$ belongs to class v_0 if*

(A_1) The function V is continuous in each of the sets $[\tau_{k-1}, \tau_k) \times R^n,\ k \in \mathcal{N}$ and for each $x \in R^n,\ k \in \mathcal{N},\ \lim_{(t,y)\to(\tau_k^-,x)} V(t, y) = V(\tau_k^-, x)$ exists.

(A_2) $V(t, x)$ is locally Lipschitzian in $x \in R^n$ and $V(t, 0) \equiv 0$.

Definition 2.2 ([13]) *Given $V \in v_0,\ x(t) = x(t, t_0, \varphi)$ is the solution of (2.1) through (t_0, φ). For $t_0 \leq s \leq t$, the upper right hand derivative of variational Lyapunov function $V(s, y(t, s, x(s)))$ is defined by*

$$D^+V(s, y(t, s, x(s))) = \limsup_{h\to 0^+} \frac{1}{h}[V(s + h, y(t, s + h, x(s) + hf(s, x_s))) - V(s, y(t, s, x(s)))],$$

where $y(t) = y(t, s, x(s))$ is any solution of (2.2) satisfying $y(s, s, x(s)) = x(s)$.

We introduce the following notations for later use

$K = \{a(u) \in C[R_+, R_+]:$ strictly increasing and $a(0) = 0\}$;

$K_1 = \{a(u) \in K: \ a(u) \geq u\}$;

$PC = \{a : R_+ \to R_+:$ continuous on $[\tau_{k-1}, \tau_k)$ and $\lim_{t\to\tau_k^-} a(t) = a(\tau_k^-)$ exists, $k \in \mathcal{N}\}$;

$PCC = \{a : R_+ \times R_+ \to R_+ : \forall s \in R_+, \ a(\cdot, s) \in PC, \ \forall t \in R_+, a(t, \cdot) \in C[R_+, R_+]\}$;

$\Gamma = \{h : R_+ \times R^n, \ \forall x \in R^n, \ h(\cdot, x) \in PC, \ \forall t \in R_+, \ h(t, \cdot) \in C[R^n, R_+],$ and $\inf_x h(t, x) = 0\}$;

$\Omega = \{\psi(s) \in C[R_+, R_+], \ \psi(0) = 0, \ \psi(s) > 0$ for $s > 0\}$;

$S(h, \rho) = \{(t, x) \in R_+ \times R^n, \ h(t, x) < \rho,$ where $h \in \Gamma, \ \rho > 0\}$.

Definition 2.3 ([7]) *Let $h_0, h \in \Gamma$. We say that h_0 is uniformly finer than h if there exist a $\delta > 0$ and a function $c \in K$ such that $h_0(t, x) < \delta$ implies that $h(t, x) \leq c(h_0(t, x))$.*

Definition 2.4 ([7]) *Let $V \in v_0, \ h_0, h \in \Gamma$. $V(t, x)$ is said to be*

(i) h-positive definite if there exists a $\rho > 0$ and a function $a \in K$ such that $h(t, x) < \rho$ implies $a(h(t, x)) \leq V(t, x)$;

(ii) h_0-decrescent if there exist a $\delta > 0$ and a function $b \in K$ such that $h_0(t, x) < \delta$ implies

$$V(t, x) \leq b(h_0(t, x)).$$

Definition 2.5 ([13]) *Let $h_0 \in \Gamma$. For $\varphi \in PC_\tau$, we define*

$$\tilde{h}_0(t, \varphi) = \sup_{-\tau\leq\theta\leq 0} h_0(t + \theta, \varphi(\theta)).$$

Now, we introduce the definitions of stability in terms of two measures for system (2.1).

Definition 2.6 ([13]) *The system (2.1) is said to be*

(S_1) $(\tilde{h}_0, h)$-uniformly stable, if for any $\varepsilon > 0$ and $t_0 \in R_+$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\tilde{h}_0(t_0, \varphi) < \delta$ implies $h(t, x(t)) < \varepsilon, \ t \geq t_0$, where $x(t) = x(t, t_0, \varphi)$ is any solution of (2.1).

(S_2) $(\tilde{h}_0, h)$-uniformly asymptotically stable, if (S_1) holds and there exists a $\delta_0 > 0$ such that for any $\varepsilon > 0$ and $t_0 \in R_+$, there exists a $T = T(\varepsilon) > 0$ such that $\tilde{h}_0(t_0, \varphi) < \delta_0$ implies $h(t, x(t)) < \varepsilon, \ t \geq t_0 + T$, where $x(t) = x(t, t_0, \varphi)$ is any solution of (2.1).

3. Main Results

We shall state and prove our main results in this section. First, we give two Razumikhin type comparison lemmas.

Lemma 3.1 *Let $m \in PC, \ \omega \in PCC, \ \psi_k \in K_1$, satisfying*

(1) $D^+m(t) \leq \omega(t, m(t))$, whenever $m(t + \theta) \leq m(t), \ \theta \in [-\tau, 0]$;

(2) for all $k \in \mathcal{N}$ and $x \in R^n$, $m(\tau_k) \leq \psi_k(m(\tau_k^-))$.

Then we have

$$m(t) \leq \gamma(t, t_0, u_0) \text{ if } \sup_{-\tau\leq\theta\leq 0} m(t_0 + \theta) \leq u_0, \tag{3.1}$$

where $\gamma(t)=\gamma(t,t_0,u_0)$ *is the maximal solution of the impulsive differential system*

$$\begin{cases} u'=\omega(t,u),\ t\neq\tau_k, \\ u(\tau_k)=\psi_k(u(\tau_k^-)), \\ u(t_0)=u_0\geq 0. \end{cases} \tag{3.2}$$

Proof Assume $t_0\in[\tau_{m-1},\tau_m),\ m\in\mathcal{N}$. First, we prove that (3.1) holds for $t\in[t_0,\tau_m)$, that is

$$m(t)\leq\gamma(t),\ t\in[t_0,\tau_m). \tag{3.3}$$

If this is not true, there exist $t_0\leq t_1<t_2<\tau_m$ such that

(a) $m(t_1)=\gamma(t_1)$,

(b) $m(t+\theta)\leq m(t),\ \theta\in[-\tau,0],\ t\in[t_1,t_2]$, and

(c) $m(t_2)>\gamma(t_2)$.

By (1), (a) and (b), applying the classical comparison theorem, we have

$$m(t)\leq\gamma(t),\ t\in[t_1,t_2],$$

which contradicts (c). So (3.3) holds. Using the facts that $\psi_m\in K_1$ and (3.3), we obtain

$$m(\tau_m)\leq\psi_m(m(\tau_m^-))\leq\psi_m(\gamma(\tau_m^-))=\gamma(\tau_m),$$

$$\sup_{-\tau\leq\theta\leq 0} m(\tau_m+\theta)\leq\gamma(\tau_m).$$

By the same proof as for $t\in[t_0,\tau_m)$, we have $m(t)\leq\gamma(t),\ t\in[\tau_m,\tau_{m+1})$. By induction, (3.1) is correct. □

Lemma 3.2 *Assume there exist* $V\in v_0,\ \omega\in PCC$ *and* $\psi_k\in K_1$, *satisfying*

(1) for $t>t_0,\ V(s+\theta,y(t,s+\theta,x(s+\theta)))\leq V(s,y(t,s,x(s))),\ \theta\in[-\tau,0]$, *implies that*

$$D^+V(s,y(t,s,x(s)))\leq\omega(s,V(s,y(t,s,x(s)))),\ s\in[t_0,t],$$

where $x(t)=x(t,t_0,\varphi)$ *and* $y(t)=y(t,t_0,x_0)$ *are solutions of (2.1) and (2.2), respectively.*

(2) for all $k\in\mathcal{N}$ *and* $x\in R^n$,

$$V(\tau_k,\ y(t,\tau_k,x+I_k(x)))\leq\psi_k(V(\tau_k^-,y(t,\tau_k^-,x))).$$

Then we have

$$V(s,y(t,s,x(s)))\leq\gamma(s,t_0,u_0),\ s\in[t_0,t],\ \text{if}\ \sup_{-\tau\leq\theta\leq 0}V(t_0+\theta,y(t,t_0+\theta,x(t_0+\theta)))\leq u_0, \tag{3.4}$$

where $\gamma(s,t_0,u_0)$ *is the maximal solution of system*

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}s}=\omega(s,u), \\ u(\tau_k)=\psi_k(u(\tau_k^-)),\ k=\mathcal{N}, \\ u(t_0)=u_0\geq 0. \end{cases} \tag{3.5}$$

Moreover, when $s=t$, *by (3.4) we have*

$$V(t,x(t,t_0,\varphi))\leq\gamma(t,t_0,u_0)\ \text{if}\ \sup_{-\tau\leq\theta\leq 0}V(t_0+\theta,y(t,t_0+\theta,x(t_0+\theta)))\leq u_0. \tag{3.6}$$

Proof Set $m(s)=V(s,y(t,x,x(s)))$. By assumptions (1) and (2), we have

(1)* $m(s+\theta) \leq m(s),\ \theta \in [-\tau, 0]$, implies that $D^+m(s) \leq \omega(s, m(s)),\ s \in [t_0, t]$;

(2)* For all $k \in \mathcal{N}$ and $x \in R^n$, $m(\tau_k) \leq \psi_k(m(\tau_k^-))$.

Using Lemma 3.1, we can get

$$m(s) \leq \gamma(s, t_0, u_0),\ s \in [t_0, t],\ \text{if} \sup_{-\tau \leq \theta \leq 0} m(t_0 + \theta) \leq u_0,$$

which implies that (3.4) holds. It is obvious (3.4) becomes (3.6) when $s = t$. The proof is completed. □

Next, we give several theorems on stability for system (2.1).

Theorem 3.1 *Let $h_0, h^*, h \in \Gamma,\ V \in v_0,\ \omega \in PCC$ and $\psi_k \in K_1,\ x(t), y(t)$ are any solutions of (2.1) and (2.2), respectively. Suppose that*

(1) h^* is uniformly finer than $h,\ h^*(t, x)$ is nondecreasing in t;

(2) $V(t, x)$ is h-positive definite on $S(h, \rho)$ and h^*-decrescent, where $\rho > 0$;

(3) For $t > t_0$, $V(s+\theta, y(t, s+\theta, x(s+\theta))) \leq V(s, y(t, s, x(s))),\ \theta \in [-\tau, 0]$, implies that

$$D^+V(s, y(t, s, x(s))) \leq \omega(s, V(s, y(t, s, x(s)))),\ s \in [t_0, t];$$

also, for all $k \in \mathcal{N}$ and $(\tau_k, x) \in S(h, \rho)$,

$$V(\tau_k, y(t, \tau_k, x + I_k(x))) \leq \psi_k(V(\tau_k^-, y(t, \tau_k^-, x)));$$

(4) There exists a $\rho_0 \in (0, \rho)$ such that $(\tau_k, x) \in S(h, \rho_0)$ implies $h(\tau_k, x + I_k(x)) < \rho$;

(5) (3.5) is uniformly stable.

Then the (h_0, h^*)-uniformly asymptotic stability of (2.2) implies $(\tilde{h}_0, h)$-uniformly asymptotic stability of (2.1).

Proof Since $V(t, x)$ is h-positive definite on $S(h, \rho)$, there exists a function $a \in K$ such that

$$a(h(t, x)) \leq V(t, x),\ (t, x) \in S(h, \rho). \tag{3.7}$$

Because $V(t, x)$ is h^*-decrescent, there exist $\delta_0 > 0$ and $b \in K$ such that

$$V(t, x) \leq b(h^*(t, x)),\ (t, x) \in S(h^*, \delta_0). \tag{3.8}$$

Also, since h^* is uniformly finer than h, there exist $\delta_1 > 0$ and $c \in K$ $(c(\delta_1) < \rho)$ such that

$$h(t, x) \leq c(h^*(t, x)),\ (t, x) \in S(h^*, \delta_1). \tag{3.9}$$

Let $\varepsilon \in (0, \rho_0)$ and $t_0 \in [\tau_{m-1}, \tau_m),\ m \in \mathcal{N}$. From the uniform stability of (3.5), there exists $\delta_2 = \delta_2(\varepsilon) > 0$ $(\delta_2 \leq a(\varepsilon))$ such that $0 < u_0 < \delta_2$ implies

$$u(s) < a(\varepsilon), \tag{3.10}$$

where $u(s) = u(s, t_0, u_0)$ is any solution of (3.5). By the property of b, we can choose $0 < \eta = \eta(\varepsilon) < \min\{\delta_0, \delta_1\}$ such that

$$b(\eta) \leq u_0. \tag{3.11}$$

Assume (2.2) is (h_0, h^*)-uniformly stable. Then, for this η, there exists a $\delta = \delta(\eta) > 0$ such that $h_0(t_0, x_0) < \delta$ implies

$$h^*(t, y(t, t_0, x_0)) < \eta, \ t \geq t_0. \tag{3.12}$$

Assume that $x(t) = x(t, t_0, \varphi)$ is any solution of system (2.1) with $\tilde{h}_0(t_0, \varphi) < \delta$. It follows from (3.7)–(3.12) that

$$a(h(t_0 + \theta, x(t_0 + \theta))) \leq V(t_0 + \theta, x(t_0 + \theta)) \leq b(h^*(t_0 + \theta, x(t_0 + \theta))) < a(\varepsilon), \ \theta \in [-\tau, 0].$$

Thus, $h(t_0 + \theta, x(t_0 + \theta)) < \varepsilon$. We claim that $h(t, x(t)) < \varepsilon, \ t \geq t_0$. Otherwise, there exists a solution $x(t)$ with $\tilde{h}_0(t_0, \varphi) < \delta$ and a $t_1 > t_0$ such that $t_1 \in [\tau_k, \tau_{k+1})$ for some $k \in \mathcal{N}$, satisfying $\varepsilon \leq h(t_1, x(t_1))$ and $h(t, x(t)) < \varepsilon$ for $t \in [t_0, \tau_k)$. Since $0 < \varepsilon < \rho_0$, it follows from assumption (4) that $h(\tau_k, x(\tau_k)) < \rho$. Hence, we can find a $t^* \in [\tau_k, t_1]$ such that

$$\varepsilon \leq h(t^*, x(t^*)) < \rho \text{ and } h(t, x(t)) < \rho \text{ for } t \in [t_0, t^*]. \tag{3.13}$$

Note that (3.8), (3.11) and (3.12) imply that, for $t \geq t_0$,

$$\begin{aligned} V(t_0 + \theta, y(t, t_0 + \theta, x(t_0 + \theta))) &\leq b(h^*(t_0 + \theta, \ y(t, t_0 + \theta, x(t_0 + \theta)))) \\ &\leq b(h^*(t, y(t, t_0 + \theta, x(t_0 + \theta)))) \leq u_0, \ \theta \in [-\tau, 0]. \end{aligned}$$

By assumption (3), together with Lemma 2, we have

$$V(s, y(t^*, s, x(s))) \leq u(s, t_0, u_0), \ s \in [t_0, t^*]. \tag{3.14}$$

Together with (3.7) and (3.10), we have

$$a(h(t^*, x(t^*))) \leq V(t^*, x(t^*)) \leq u(t^*) < a(\varepsilon)$$

by choosing $s = t^*$ in (3.14). It contradicts (3.13). So (2.1) is $(\tilde{h}_0, h)$-uniformly stable.

Next, assume (2.2) is (h_0, h^*)-uniformly asymptotically stable. We prove (2.1) is $(\tilde{h}_0, h)$-uniformly asymptotically stable. By $(\tilde{h}_0, h)$-uniform stability of (2.1), for $\rho > 0$, there exists $\delta_2 > 0$ such that $\tilde{h}_0(t_0, \varphi) < \delta_2$ implies $h(t, x(t)) < \rho$. Also, since (3.5) is uniformly stable, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) \in (0, b(\delta_0))$ such that $0 \leq u_0 < \delta$ implies that

$$u(s) < a(\epsilon), \tag{3.15}$$

where $u(s) = u(s, t_0, u_0)$ is any solution of (3.5).

Because (2.2) is (h_0, h^*)-uniformly asymptotically stable, there exists a $\delta_3 > 0$ satisfying for the above u_0 and $t_0 \in R_+$, there exists $T = T(u_0) > 0$ such that

$$h^*(t, y(t, t_0, x_0)) \leq b^{-1}(u_0), \ t \geq t_0 + T, \tag{3.16}$$

where $y(t, t_0, x_0)$ is any solution of (2.2) with $h_0(t_0, x_0) < \delta_3$. Choose $\bar{\delta}_0 = \min\{\delta_2, \delta_3\}$. Then by (3.8) and (3.16), we have

$$\begin{aligned} V(t_0 + \theta, y(t, t_0 + \theta, x(t_0 + \theta))) &\leq b(h^*(t_0 + \theta, y(t, t_0 + \theta, x(t_0 + \theta)))) \\ &\leq b(h^*(t, y(t, t_0 + \theta, x(t_0 + \theta)))) \leq u_0, \ t \geq t_0 + T, \end{aligned}$$

where $x(t) = x(t, t_0, \varphi)$ is any solution of (2.1) with $\tilde{h}_0(t_0, \varphi) < \bar{\delta}_0$. From Lemma 3.2, $V(s, y(t, x(s))) \leq$

$u(s),\ s \in [t_0, t],\ t \geq t_0 + T$. Choosing $s = t$, together with (3.7) and (3.15), we have

$$a(h(t, x(t))) \leq V(t, x(t)) \leq u(t) < a(\varepsilon).$$

Hence, $h(t, x(t)) < \varepsilon,\ t \geq t_0 + T$. This shows that (2.1) is $(\tilde{h}_0, h)$-uniformly asymptotically stable. □

Remark 3.1 From the proof of Theorem 3.1, we can see if $V(t, x)$ is nondecreasing in t, the demand on monotone property for $h^*(t, x)$ in t is not necessary.

Corollary 3.1 *In Theorem 3.1, suppose* $\omega(s, u) = g(s)u,\ \psi_k(u) = (1 + d_k)u$ *for* $u \geq 0$, *where* $g \in C[R_+, R_+],\ \int_0^\infty g(s)\mathrm{d}s < \infty,\ d_k \geq 0$ *and* $\sum_{k=1}^\infty d_k < \infty$. *Then the conclusion of Theorem 3.1 holds.*

Proof For any $\varepsilon > 0$, by $\int_0^\infty g(s)\mathrm{d}s < \infty$ and $\sum_{k=1}^\infty d_k < \infty$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\int_0^\infty g(s)\mathrm{d}s + \sum_{k=1}^\infty d_k \leq \int_\delta^\varepsilon \frac{\mathrm{d}u}{u}.$$

Suppose that $u(s)$ is any solution of (3.5) through (t_0, u_0), where $u_0 > 0$. Let $t_0 \in [\tau_{m-1}, \tau_m),\ m \in \mathcal{N}$. Then if $u_0 < \delta$, we have

$$\begin{aligned}
&\int_{u_0}^{u(s)} \frac{\mathrm{d}u}{u} = \int_{u_0}^{u(\tau_m^-)} \frac{\mathrm{d}u}{u} + \int_{u(\tau_m^-)}^{u(\tau_m)} \frac{\mathrm{d}u}{u} + \cdots + \int_{u(\tau_{k-1})}^{u(\tau_k^-)} \frac{\mathrm{d}u}{u} + \int_{u(\tau_k^-)}^{u(\tau_k)} \frac{\mathrm{d}u}{u} + \int_{u(\tau_k)}^{u(s)} \frac{\mathrm{d}u}{u} \\
&= \int_{t_0}^{\tau_m} \frac{\mathrm{d}\xi}{g(\xi)} + \ln(1 + d_m) + \cdots + \int_{\tau_{k-1}}^{\tau_k} \frac{\mathrm{d}\xi}{g(\xi)} + \ln(1 + d_k) + \int_{\tau_k}^{s} \frac{\mathrm{d}\xi}{g(\xi)} \\
&\leq \int_{t_0}^{s} \frac{\mathrm{d}\xi}{g(\xi)} + \sum_{i=m}^{k} \ln(1 + d_i) \leq \int_{t_0}^{\infty} \frac{\mathrm{d}s}{g(s)} + \sum_{i=m}^{\infty} d_i \leq \int_\delta^\varepsilon \frac{\mathrm{d}u}{u} < \int_{u_0}^\varepsilon \frac{\mathrm{d}u}{u}.
\end{aligned}$$

Hence, $u(s) < \varepsilon$. This completes the proof. □

A much more easily usable conclusion which can be deduced from Theorem 3.1 is the next corollary.

Corollary 3.2 *In Theorem 3.1, suppose* $\omega(t, u) \equiv 0,\ \psi_k(u) = (1 + d_k)u$ *for* $u \geq 0$, *where* $d_k \geq 0$ *and* $\sum_{k=1}^\infty d_k < \infty$. *Then the conclusion of Theorem 3.1 holds.*

Remark 3.2 The assumptions of Corollary 3.2 are just the same as those of Theorem 3.2 in [13]. But the later only concludes that (h_0, h^*)-uniform stability of (2.2) implies $(\tilde{h}_0, h)$-uniform stability of (2.1). So our result is superior to the later.

Theorem 3.2 *Let* $h_0, h^*, h \in \Gamma,\ V \in v_0,\ a, b \in K,\ \omega, H \in \Omega,\ x(t), y(t)$ *are any solutions of (2.1) and (2.2), respectively. Assume that*

(1) $a(h(t, x)) \leq V(t, x),\ (t, x) \in S(h, \rho), V(t, x) \leq b(h^*(t, x)),\ (t, x) \in S(h^*, \rho)$;

(2) h^* *is uniformly finer than* $h,\ h^*(t, x)$ *is nondecreasing in* t;

(3) For all $k \in \mathcal{N}$ and $(\tau_k, x) \in S(h, \rho)$,

$$V(\tau_k, y(t, \tau_k, x + I_k(x))) \leq (1 + d_k)V(\tau_k^-, y(t, \tau_k^-, x)),$$

where $d_k \geq 0,\ \sum_{k=1}^{\infty} d_k < \infty$*;*

(4) For $t > t_0$, $V(s+\theta, y(t, s+\theta, x(s+\theta))) < P(V(s, y(t, s, x(s))))$, $\theta \in [-\tau, 0]$, *implies that*

$$D^+V(s, y(t,s,x(s))) \leq -\psi(s)\omega(V(s, y(t,s,x(s)))) + g(s)H(V(s, y(t,x,x(s)))),\ s \in [t_0, t],$$

where $P, \psi, g : R_+ \to R_+$ *are continuous,* $P(s) > Ms$ *for* $s > 0$*, where* $M = \prod_{k=1}^{\infty}(1 + d_k)$*, given* $\beta > 0$*, there exists* $\tilde{T} = \tilde{T}(\beta) > 0$ *such that* $\int_T^{T+\tilde{T}} \psi(s)\mathrm{d}s > (M'+1)\beta$ *for any* $T \in R_+$*, where* $M' = \sum_{k=1}^{\infty} d_k$*,* $\int_0^{\infty} g(s)\mathrm{d}s < \infty$*;*

(5) There exists a $\rho_0 \in (0, \rho)$ *such that* $(\tau_k, x) \in S(h, \rho_0)$ *implies* $h(\tau_k, x + I_k(x)) < \rho$*;*

(6) The following system

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}s} = g(s)H(u), \\ u(\tau_k) = (1 + d_k)u(\tau_k^-),\ k \in \mathcal{N}, \\ u(t_0) = u_0 \geq 0 \end{cases}$$

is uniformly stable.

Then the (h_0, h^*)*-uniform stability of (2.2) implies* $(\tilde{h}_0, h)$*-uniformly asymptotic stability of (2.1).*

Proof Assume that (2.2) is (h_0, h^*)-uniformly stable. Since $V(s+\theta, y(t, s+\theta, x(s+\theta))) \leq V(s, y(t,s,x(s)))$, $\theta \in [-\tau, 0]$, implies that $V(s+\theta, y(t, s+\theta, x(s+\theta))) < P(V(s, y(t,s,x(s))))$, $\theta \in [-\tau, 0]$, it is evident that (2.1) is $(\tilde{h}_0, h)$-uniformly stable by Theorem 3.1.

For given $\varepsilon_0 = \rho_0$, we can find the corresponding $\delta_0 > 0$ such that $\tilde{h}_0(t_0, \varphi) < \delta_0$ implies that $h(t, x(t)) < \varepsilon_0, V(s, y(t,s,x(s))) < A \triangleq a(\varepsilon_0),\ t \geq t_0$, by the proof of Theorem 3.1, where $x(t) = x(t, t_0, \varphi)$ is any solution of (2.1).

Given $\varepsilon > 0$ with $\varepsilon < \varepsilon_0$, let $B = \min_{M^{-1}\varepsilon^* \leq V \leq A} \omega(V)$, $C = \max_{0 \leq V \leq A} H(V)$, $\varepsilon^* \triangleq a(\varepsilon)$, $0 < d < \min_{M^{-1}\varepsilon^* \leq s \leq A}\{P(s) - Ms\}$. Let $N = N(\varepsilon)$ be the smallest positive integer such that $A \leq \varepsilon^* + Nd$. Since $\int_0^{\infty} g(s)\mathrm{d}s < \infty$, there exists $T > 0$ such that $C\int_T^{\infty} g(\xi)\mathrm{d}\xi < M^{-1}d/6$. Next, we prove that there exists $T_1 \geq T$ such that

$$V(T_1, y(t, T_1, x(T_1))) < M^{-1}[\varepsilon^* + (N-1)d],\ T_1 \leq t,$$

where $x(t) = x(t, t_0, \varphi)$ is any solution of (2.1) with $\tilde{h}_0(t_0, \varphi) < \delta_0$. Otherwise, for $s \geq T$,

$$V(s, y(t,s,x(s))) \geq M^{-1}[\varepsilon^* + (N-1)d],\ s \leq t.$$

Therefore,

$$\begin{aligned} P(V(s, y(t,s,x(s)))) &> MV(s, y(t,s,x(s))) + d \geq \varepsilon^* + Nd \\ &\geq A \geq V(s+\theta, y(t, s+\theta, x(s+\theta))),\ -\tau \leq \theta \leq 0. \end{aligned}$$

Then, by assumption (4) we have

$$D^+V(s, y(t,s,x(s))) \leq -\psi(s)B + g(s)C,\ T \leq s \leq t.$$

From assumptions (3) and (4), there exists $\tilde{T} > 0$ such that

$$V(T+\tilde{T}, y(t, T+\tilde{T}, x(T+\tilde{T}))) \leq V(T, y(t, T, x(T))) - B\int_T^{T+\tilde{T}} \psi(s)\mathrm{d}s + C\int_T^{T+\tilde{T}} g(s)\mathrm{d}s +$$

$$\sum_{T<\tau_j\le T+\tilde{T}}[V(\tau_j)-V(\tau_j^-)]\le A(1+M')-B\int_T^{T+\tilde{T}}\psi(s)\mathrm{d}s+M^{-1}d/6<0.$$

This contradicts $V(s,y(t,s,x(s)))\ge 0$. So we can choose $T_1=T+\tilde{T}$.

Next, we claim that

$$V(s,y(t,s,x(s)<[\varepsilon^*+(N-1)d]+d/2 \text{ for } T_1\le s\le t,$$

where $t\ge T+2N\hat{T}$ and $\hat{T}=\max\{\tilde{T},\tau\}$. Suppose $T_1\in[\tau_{j-1},\tau_j)$. We first prove that

$$V(s,y(t,s,x(s)))<M^{-1}[\varepsilon^*+(N-1)d]+M^{-1}d/6 \text{ for } s\in[T_1,\tau_j). \tag{3.17}$$

If (3.17) is not true, there must exist $T_1<t_1<t_2<\tau_j$ such that

$$V(t_1,y(t,t_1,x(t_1)))=M^{-1}[\varepsilon^*+(N-1)d], \tag{3.18}$$

$$V(t_2,y(t,t_2,x(t_2)))=M^{-1}[\varepsilon^*+(N-1)d]+M^{-1}d/6 \tag{3.19}$$

and

$$V(t_1,y(t,t_1,x(t_1)))\le V(s,y(t,s,x(s)))\le V(t_2,y(t,t_2,x(t_2))),\ \ s\in[t_1,t_2]. \tag{3.20}$$

From (3.18) and (3.20),

$$\begin{aligned}&P(V(s,y(t,s,x(s))))>MV(s,y(t,s,x(s)))+d\ge MV(t_1,y(t,t_1,x(t_1)))+d=\varepsilon^*+Nd\\&\quad\ge A\ge V(s+\theta,y(t,s+\theta,x(s+\theta))),-\tau\le s\le 0,\ t_1\le t\le t_2.\end{aligned}$$

Together with assumption (4), it follows that

$$V(t_2,y(t,t_2,x(t_2)))\le V(t_1,y(t,t_1,x(t_1)))+C\int_{t_1}^{t_2}g(s)\mathrm{d}s<M^{-1}[\varepsilon^*+(N-1)d]+M^{-1}d/6,$$

which contradicts (3.19). Then we have

$$\begin{aligned}V(\tau_j,y(t,\tau_j,x(\tau_j)))&\le(1+d_j)V(\tau_j^-,y(t,\tau_j^-,x(\tau_j^-)))\\&\le(1+d_j)\{M^{-1}[\varepsilon^*+(N-1)d]+M^{-1}d/6\}.\end{aligned}$$

Denote $\mu_m=\int_{\tau_m}^{\tau_{m+1}}g(s)\mathrm{d}s, m\ge j$. Then $\mu_m\ge 0,\ C\sum_{m=j}^{\infty}\mu_m<M^{-1}d/6$. Let $\{\nu_m\}, m\ge j$, be a sequence, satisfying $\nu_m>0,\ \sum_{m=j}^{\infty}\nu_m<M^{-1}d/6$. In a similar way as in the proof of (3.17), we can prove that

$$V(s,y(t,s,x(s)))<(1+d_j)\{M^{-1}[\varepsilon^*+(N-1)d]+M^{-1}d/6\}+C\mu_j+\nu_j,\ \ s\in[\tau_j,\tau_{j+1}).$$

By induction, we arrive at

$$\begin{aligned}V(s,y(t,s,x(s)))&<\prod_{k=j}^{l}(1+d_k)\{M^{-1}[\varepsilon^*+(N-1)d]+M^{-1}d/6\}+\prod_{k=j+1}^{l}(1+d_k)(C\mu_j+\nu_j)+\\&\quad\prod_{k=j+2}^{l}(1+d_k)(C\mu_{j+1}+\nu_{j+1})+\cdots+(C\mu_l+\nu_l),s\in[\tau_l,\tau_{l+1})\cap[T_1,t],\ \ l\ge j.\end{aligned}$$

Hence, by the definition of M,

$$V(s, y(t, s, x(s))) < \varepsilon^* + (N-1)d + d/6 + M \sum_{k=j}^{\infty} (C\mu_k + \nu_k) < \varepsilon^* + (N-1)d + d/2, \ s \in [T_1, t].$$

Similarly, we can prove there exists $T_2 = T_1 + \hat{T}$ such that

$$V(T_2, y(t, T_2, x(T_2))) < M^{-1}[\varepsilon^* + (N-2)d + d/2],$$

$$V(s, y(t, s, x(s))) < \varepsilon^* + (N-1)d, \ T_2 \leq s \leq t.$$

By induction, we obtain

$$V(s, y(t, s, x(s))) < \varepsilon^*, \ T + 2N\hat{T} \leq s \leq t,$$

which, together with assumption (1) and the definition of ε^*, yields

$$h(t, x(t)) < \varepsilon, \ t \geq T + 2N\hat{T}.$$

Thus $h(t, x(t)) < \varepsilon, t \geq t_0 + T + 2N\hat{T}$. This completes the proof. □

Remark 3.3 If $\psi(s) \equiv 1$, $g(s) \equiv 0$, Theorem 3.2 is just the same as Theorem 3.4 in [13]. Moreover, we omit the assumption that h is uniformly finer than h_0.

Finally, to illustrate the above results, we consider an example.

Example Consider the impulsive delay differential system

$$\begin{cases} x'(t) = a(t)x(t) + b(t) \displaystyle\int_{t-\tau}^{t} c(\xi)x(\xi)\mathrm{d}\xi, \ t \neq \tau_k, \\ x(\tau_k) = (1 + d_k)^{1/2} x(\tau_k^-), \ k \in \mathcal{N}, \\ x_{t_0} = \varphi \end{cases} \tag{3.21}$$

and the ordinary differential system

$$\begin{cases} y'(t) = a(t)y(t), \\ y(t_0) = x_0, \end{cases} \tag{3.22}$$

where $a(t) \in C[R_+, R_+]$ and $\int_0^\infty a(t)\mathrm{d}t < \infty, b(t), c(t) \in C[R_+, R]$, $\int_0^\infty |b(t)|\mathrm{d}t < \infty$ and $|c(t)| \leq K$, $K > 0$, $d_k \geq 0$ and $\sum_{k=1}^\infty d_k < \infty$. Denote by $x(t) = x(t, t_0, \varphi)$ and $y(t) = y(t, t_0, x_0)$ the solutions of (3.21) and (3.22), respectively. It is easy to see that $y(t) = x_0 \exp\{\int_{t_0}^t a(\eta)\mathrm{d}\eta\}$ and $y(t, s, x(s)) = x(s) \exp\{\int_s^t a(\eta)\mathrm{d}\eta\}$. Let $V(t, x) = (1/2)x^2$ and $h_0(t, x) = h^*(t, x) = h(t, x) = |x|$ for any $t \in R_+$ and $x \in R$. Then it is evident that V is h-positive definite and h^*-decrescent. Also, it is easy to see that (3.22) is (h_0, h^*)-uniformly stable. By direct calculation, we can get

$$\begin{aligned} D^+ V(s, y(t, s, x(s))) &= x(s) \exp\{2 \int_s^t a(\eta)\mathrm{d}\eta\}[x'(s) - a(s)x(s)] \\ &= b(s)x(s) \int_{s-\tau}^{s} c(\xi)x(\xi)\mathrm{d}\xi \cdot \exp\{2 \int_s^t a(\eta)\}, \\ V(\tau_k, y(t, \tau_k, x(\tau_k))) &= (1 + d_k)V(\tau_k^-, y(t, \tau_k^-, x(\tau_k^-))). \end{aligned}$$

If $V(s+\theta, y(t, s+\theta, x(s+\theta))) \leq V(s, y(t, s, x(s)))$, $-\tau \leq \theta \leq 0$, then $x^2(s+\theta) \exp\{2 \int_{s+\theta}^t a(\eta)\mathrm{d}\eta\} \leq x^2(s) \exp\{2 \int_s^t a(\eta)\mathrm{d}\eta\}$, and thus $|x(s)x(s+\theta)| \leq x^2(s) \exp\{\int_s^{s+\theta} a(\eta)\mathrm{d}\eta\}$ for $-\tau \leq \theta \leq 0$. In this case, we have

$$D^+V(s,y(t,s,x(s))) \le |b(s)|\int_{s-\tau}^{s}|c(\xi)||x(s)||x(\xi)|\mathrm{d}\xi \cdot \exp\{2\int_s^t a(\eta)\}$$
$$\le K|b(s)|\int_{s-\tau}^{s} x^2(s)\exp\{\int_s^{\xi} a(\eta)\mathrm{d}\eta\}\mathrm{d}\xi \cdot \exp\{2\int_s^t a(\eta)\}$$
$$= 2K|b(s)|V(s,y(t,s,x(s)))\int_{s-\tau}^{s}\exp\{\int_s^{\xi} a(\eta)\mathrm{d}\eta\}\mathrm{d}\xi \le 2K\tau|b(s)|V(s,y(t,s,x(s))).$$

Then it follows from Corollary 3.1 that (3.21) is $(\tilde{h}_0, h)$-uniformly stable.

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