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Existence of Solutions for Nonlinear Neumann Boundary Value Problems

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Abstract Using perturbation theories on sums of ranges of nonlinear accretive mappings of Calvert and Gupta, we present the abstract results on the existence of solutions of one kind nonlinear Neumann boundary value problems related to p-Laplacian operator. The equation discussed in this paper and the method used here extend and complement some of the previous work.

Keywords maximal monotone operator; accretive mapping; hemi–continuous mapping.

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1. Introduction

Nonlinear boundary value problems involving p-Laplacian operator $-\Delta_p$ occur in a variety of physical phenomena. And, many mathematicians do their researches from different angles on $-\Delta_p$ and its generalized forms. Some significant work has been done by us too, see [1–7].

In 2005, we studied in [6] the following equation (1.1):

$$\begin{aligned}&-\mathrm{div}(\alpha(\mathrm{grad}\, u)) + |u|^{p-2}u + g(x, u(x)) = f(x), \quad \text{a.e. in } \Omega\\ &-\langle \vartheta, \alpha(\mathrm{grad}\, u)\rangle \in \beta_x(u(x)), \quad \text{a.e. on } \Gamma\end{aligned} \tag{1.1}$$

which had a solution in $L^2(\Omega)$, where $\frac{2N}{N+1} < p < +\infty$ and $N \geq 1$. Here $\alpha : R^N \to R^N$ was a function satisfying some conditions and was related to p, and ϑ was the exterior normal derivative of Γ. Moreover, in [7], we showed that Eq(1.1) had a solution in $L^p(\Omega)$, where $2 \leq p < +\infty$.

We note that if $\alpha(\xi) = |\xi|^{p-2}\xi$, for $\forall \xi \in R^N$, then Eq(1.1) is reduced to the case involving the p-Laplacian operator.

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We also need to mention that although Eq(1.1) was similar to that discussed in [8] from the appearance, however, their discussion in [8] did not include the case of p-Laplacian operator.

In this paper, we'll continue to study Eq(1.1) in a more general space $L^s(\Omega)$, where $\frac{2N}{N+1} < p \leq s < +\infty$ and $N \geq 1$. Necessary details of Eq(1.1) will be provided in Section 3.

2. Preliminaries

Let X be a real Banach space with a strictly convex dual space X^*. We use "$\rightarrow$" and "$w-\lim$" to denote strong and weak convergence, respectively. For any subset G of X, we denote by $intG$ its interior and $\overline{G}$ its closure, respectively. Let "$X \hookrightarrow Y$" denote the space X embedded continuously in space Y and "$X \hookrightarrow\hookrightarrow Y$" denote that X is embedded compactly in Y. A mapping $T : D(T) = X \to X^*$ is said to be hemi-continuous on X if $w - \lim_{t\to 0} T(x+ty) = Tx$, for any $x, y \in X$. A mapping $T : D(T) = X \to X^*$ is said to be demi–continuous on X if $w - \lim_{n\to\infty} Tx_n = Tx$, for any sequence $\{x_n\}$ strongly convergent to x in X.

Let J denote the duality mapping from X into 2^{X^*} defined by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\| \cdot \|f\|, \|f\| = \|x\|\}, \quad \forall x \in X,$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing between X and X^*. Since X^* is strictly convex, J is a single-valued mapping.

A multi-valued mapping $A : X \to 2^X$ is said to be accretive if $(v_1 - v_2, J(u_1 - u_2)) \geq 0$, for any $u_i \in D(A)$ and $v_i \in Au_i$, $i = 1, 2$. The accretive mapping A is said to be m-accretive if $R(I + \lambda A) = X$ for some $\lambda > 0$. We say that $A : X \to 2^X$ is boundedly-inversely-compact if, for any pair of bounded subsets G and G' of X, the subset $G \bigcap A^{-1}(G')$ is relatively compact in X.

A multi-valued mapping $B : X \to 2^{X^*}$ is said to be monotone if its graph $G(B)$ is a monotone subset of $X \times X^*$ in the sense that $(u_1 - u_2, w_1 - w_2) \geq 0$, for any $[u_i, w_i] \in G(B)$, $i = 1, 2$. The monotone operator B is said to be maximal monotone if $G(B)$ is maximal among all monotone subsets of $X \times X^*$ in the sense of inclusion. The mapping B is said to be coercive if $\lim_{n\to+\infty} (x_n, x_n^*)/\|x_n\| = +\infty$ for all $[x_n, x_n^*] \in G(B)$ such that $\lim_{n\to+\infty} \|x_n\| = +\infty$.

Definition 2.1 ([8]) *The duality mapping $J : X \to X^*$ is said to satisfy Condition (I) if there exists a function $\eta : X \to [0, +\infty)$ such that for $u, v \in X$,*

$$\|Ju - Jv\| \leq \eta(u - v). \tag{I}$$

Lemma 2.1 ([8]) *Let Ω be a bounded domain in R^N and let $J_p : L^p(\Omega) \to L^{p'}(\Omega)$ denote the duality mapping. Then, J_p satisfies Condition (I). Moreover, for $2 \leq p < +\infty$, $J_p u = |u|^{p-1}\mathrm{sgn}\, u\|u\|_p^{2-p}$, $\forall u \in L^p(\Omega)$; for $1 < p \leq 2$, $J_p u = |u|^{p-1}\mathrm{sgn}\, u$, $\forall u \in L^p(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.*

Definition 2.2 ([8]) *Let $A : X \to 2^X$ be an accretive mapping and $J : X \to X^*$ be a duality mapping. We say that A satisfies Condition $(*)$ if, for any $f \in R(A)$ and $a \in D(A)$, there exists a constant $C(a, f)$ such that, for any $u \in D(A)$, $v \in Au$,*

$$(v - f, J(u - a)) \geq C(a, f). \tag{$*$}$$

Lemma 2.2 ([9]) *Let Ω be a bounded conical domain in R^N. If $mp > N$, then $W^{m,p}(\Omega) \hookrightarrow C_B(\Omega)$; if $mp < N$ and $q = \frac{Np}{N-mp}$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$; if $mp = N$ and $p > 1$, then for $1 \le q < +\infty$, $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$.*

Lemma 2.3 ([9]) *Let Ω be a bounded conical domain in R^N. If $mp > N$, then $W^{m,p}(\Omega) \hookrightarrow\hookrightarrow C_B(\Omega)$; if $0 < mp \le N$ and $q_0 = \frac{Np}{N-mp}$, then $W^{m,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, where $1 \le q < q_0$.*

Lemma 2.4 ([8]) *Let Ω be a bounded domain in R^N and $g : \Omega \times R \to R$ be a function satisfying Carathéodory's conditions such that*

(i) $g(x, \cdot)$ is monotonically increasing on R;

(ii) The mapping $u \in L^p(\Omega) \to g(x, u(x)) \in L^p(\Omega)$, $1 < p < +\infty$, is well defined.

Then, the mapping $B : L^p(\Omega) \to L^p(\Omega)$ defined by $(Bu)(x) = g(x, u(x))$, for any $x \in \Omega$, satisfies Condition (*).

Theorem 2.1 ([8]) *Let X be a real Banach space with a strictly convex dual X^*. Let $J : X \to X^*$ be a duality mapping on X satisfying Condition (I). Let A, $C_1 : X \to 2^X$ be accretive mappings such that*

(i) Either both A and C_1 satisfy Condition $(*)$, or $D(A) \subset D(C_1)$ and C_1 satisfies Condition $(*)$;

(ii) $A + C_1$ is m-accretive and boundedly-inversely-compact.

If $C_2 : X \to X$ is a bounded continuous mapping such that, for any $y \in X$, there is a constant $C(y)$ satisfying $(C_2(u + y), Ju) \ge -C(y)$ for any $u \in X$, then:

(a) $\overline{[R(A) + R(C_1)]} \subset \overline{R(A + C_1 + C_2)}$;

(b) $\operatorname{int}[R(A) + R(C_1)] \subset \operatorname{int} R(A + C_1 + C_2)$.

3. Main results

3.1 Explanation of Equation (1.1)

In this paper, unless otherwise stated, we assume that $\frac{2N}{N+1} < p \le s < +\infty$ where $N \ge 1$. Moreover, assume that $\frac{1}{p} + \frac{1}{p'} = 1$, and $\frac{1}{s} + \frac{1}{s'} = 1$.

In Equation (1.1), Ω is a bounded conical domain of an Euclidean space R^N with its boundary $\Gamma \in C^1$ (see [1]). We shall assume that Green's Formula is available. $f \in L^s(\Omega)$ is a given function, and ϑ denotes the exterior normal derivative of Γ.

$\alpha : R^N \to R^N$ is a given monotone and continuous function, and there exist positive constants k_1, k_2 and k_3 such that for $\forall \xi$, $\xi' \in R^N$, the following conditions are satisfied:

(i) $|\alpha(\xi)| \le k_1 |\xi|^{p-1}$;

(ii) $|\alpha(\xi) - \alpha(\xi')| \le k_2 ||\xi|^{p-2}\xi - |\xi'|^{p-2}\xi'|$;

(iii) $\langle \xi, \alpha(\xi) \rangle \ge k_3 |\xi|^p$.

Let $\varphi : \Gamma \times R \to R$ be a given function such that, for each $x \in \Gamma$, $\varphi_x = \varphi(x, \cdot) : R \to R$ is a proper, convex and lower-semi-continuous function with $\varphi_x(0) = 0$. Let β_x be the subdifferential of φ_x, i.e., $\beta_x \equiv \partial\varphi_x$. Suppose that $0 \in \beta_x(0)$, β_x is continuous and for each $t \in R$, the function

$x \in \Gamma \to (I+\lambda\beta_x)^{-1}(t) \in R$ is measurable for $\lambda > 0$. $g : \Omega \times R \to R$ is a given function satisfying Carathéodory's conditions such that the mapping $u \in L^s(\Omega) \to g(x, u(x)) \in L^s(\Omega)$ is defined. Suppose that there is a function $T(x) \in L^s(\Omega)$ such that $g(x,t)t \geq 0$, for $|t| \geq T(x)$ and $x \in \Omega$.

3.2 Main ideas of the discussion of Equation(1.1)

First, we shall construct a mapping A_s and prove that it is m-accretive and boundedly-inversely-compact. Then, we shall construct two mappings C_1 and C_2 and show that these mappings satisfy the conditions of Theorem 2.1. Next, we shall find conditions when $f \in \mathrm{int}[R(A_s) + R(C_1)]$, so that we can use Theorem 2.1 to prove that $f \in \mathrm{int}\, R(A_s + C_1 + C_2)$. Finally, we will show that if $f \in \mathrm{int}\, R(A_s+C_1+C_2)$, then Equation (1.1) has solutions in $L^s(\Omega)$.

3.3 Details

Lemma 3.1 ([6]) *Define the mapping $B_p : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ by*

$$(v, B_p u) = \int_\Omega \langle \alpha(\mathrm{grad}\, u), \mathrm{grad}\, v\rangle \mathrm{d}x + \int_\Omega |u(x)|^{p-2}u(x)v(x)\mathrm{d}x$$

for any $u, v \in W^{1,p}(\Omega)$. Then, B_p is everywhere defined, monotone, hemi-continuous and coercive.

Lemma 3.2 ([6]) *The mapping $\Phi_p : W^{1,p}(\Omega) \to R$ defined by $\Phi_p(u) = \int_\Gamma \varphi_x(u|_\Gamma(x))\mathrm{d}\Gamma(x)$, for any $u \in W^{1,p}(\Omega)$, is proper, convex and lower-semi-continuous on $W^{1,p}(\Omega)$.*

Lemma 3.3 ([6]) *Define a mapping $A : L^2(\Omega) \to 2^{L^2(\Omega)}$ as follows:*

$$D(A) = \{u \in L^2(\Omega) |\ \text{ there exists an } f \in L^2(\Omega) \text{ such that } f \in B_p u + \partial\Phi_p(u)\}.$$

For $u \in D(A)$, $Au = \{f \in L^2(\Omega) | f \in B_p u + \partial\Phi_p(u)\}$. Then A is an m-accretive mapping.

Definition 3.1 *Define a mapping $A_s : L^s(\Omega) \to 2^{L^s(\Omega)}$ as follows:*

(i) If $s \geq 2$, then

$$D(A_s) = \{u \in L^s(\Omega) |\ \text{ there exists an } f \in L^s(\Omega) \text{ such that } f \in B_p u + \partial\Phi_p(u)\}.$$

For $u \in D(A_s)$, we set $A_s u = \{f \in L^s(\Omega) | f \in B_p u + \partial\Phi_p(u)\}$;

(ii) If $1 < s < 2$, then define $A_s : L^s(\Omega) \to 2^{L^s(\Omega)}$ as the L^s-closure of $A : L^2(\Omega) \to 2^{L^2(\Omega)}$ defined in Lemma 3.3.

Remark 3.1 Compared to our previous work, a new definition of A_s is given in the case of $1 < s < 2$ to prove our main results.

Lemma 3.4 *If $f, g \in L^2(\Omega)$, and there exist $u, v \in L^2(\Omega)$ such that $u + \lambda Au = f$, $v + \lambda Av = g$, for $\lambda > 0$. Then $\int_\Omega |u - v|^s \mathrm{d}x \leq \int_\Omega |f - g|^s \mathrm{d}x$, where $1 < s < +\infty$.*

Proof Similar to the proof of Lemma 2.5 in [4], the result is true. □

Lemma 3.5 *If $\frac{2N}{N+1} < p \leq s \leq 2$, then $R(I + \lambda A_s) = L^s(\Omega)$, $\forall \lambda > 0$.*

Proof For $\forall f \in L^s(\Omega)$, we can choose a sequence $f_n \in L^2(\Omega)$, such that $f_n \to f$ in $L^s(\Omega)$, as

$n \to \infty$. By Lemma 3.3, $f_n = u_n + \lambda A u_n$, for $n \geq 1$. Then Lemma 3.4 implies that

$$\int_\Omega |u_n - u_m|^s \mathrm{d}x \leq \int_\Omega |f_n - f_m|^s \mathrm{d}x.$$

Therefore, there exists a $u \in L^s(\Omega)$, such that $u_n \to u$ in $L^s(\Omega)$, and then $f = u + \lambda A_s u$. Hence $R(I + \lambda A_s) = L^s(\Omega)$, $\forall \lambda > 0$. □

Lemma 3.6 *The mapping* $A_s : L^s(\Omega) \to 2^{L^s(\Omega)}$ *is accretive if* $\frac{2N}{N+1} < p \leq s \leq 2$, *for* $N \geq 1$.

Proof To show that A_s is accretive, it suffices to prove that $(I + \lambda A_s)^{-1} : L^s(\Omega) \to L^s(\Omega)$ is non-expansive.

To this end, let $f = u + \lambda A_s u$ and $g = v + \lambda A_s v$. Then there exist $u_n, v_n \in L^2(\Omega)$, such that $u_n \to u, v_n \to v$ in $L^s(\Omega)$, and there exist $f_n, g_n \in L^2(\Omega)$, such that $f_n \to f, g_n \to g$ in $L^s(\Omega)$, as $n \to \infty$. Moreover, $f_n = u_n + \lambda A u_n$ and $g_n = v_n + \lambda A v_n$, for $\forall n \geq 1$.

Then from Lemma 3.4, we know that $\|u - v\|_s \leq \|f - g\|_s$. So A_s is accretive. □

Proposition 3.1 *The mapping* A_s *is* m*-accretive.*

Proof Lemmas 3.5 and 3.6 imply that A_s is m-accretive if $\frac{2N}{N+1} < p \leq s \leq 2$, for $N \geq 1$. Similarly to the proof of Proposition 2.1 in [4], A_s is also m-accretive if $s \geq 2$. □

Proposition 3.2 *(i) If* $\frac{2N}{N+1} < p \leq s < 2$, *then* $A_s : L^s(\Omega) \to 2^{L^s(\Omega)}$ *has a compact resolvent;*

(ii) If $s \geq 2$ *and* $\frac{2N}{N+1} < p \leq 2$, *then* $A_s : L^s(\Omega) \to 2^{L^s(\Omega)}$ *has a compact resolvent.*

Proof (i) It suffices to prove that if $f \in L^2(\Omega)$, $u + \lambda A u = f$ $(\lambda > 0)$, $u \in L^2(\Omega)$ and $\{f\}$ is bounded in $L^s(\Omega)$, then $\{u\}$ is relatively compact in $L^s(\Omega)$. For this, we define functions $\chi_n,\ \xi_n : R \to R$ by

$$\chi_n(t) = \begin{cases} |t|^{p-1} \operatorname{sgn} t, & \text{if } |t| \geq 1/n \\ (1/n)^{p-2} t, & \text{if } |t| \leq 1/n \end{cases}$$

and

$$\xi_n(t) = \begin{cases} |t|^{2-(2/p)} \operatorname{sgn} t, & \text{if } |t| \geq 1/n \\ (1/n)^{1-(2/p)} t, & \text{if } |t| \leq 1/n. \end{cases}$$

Now as in the proof of Lemma 2.6 in [5], it follows that $\{|u|^{2-(2/p)} \operatorname{sgn} u\}$ is bounded in $W^{1,p}(\Omega)$. Next notice that $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^{\frac{ps}{2(p-1)}}(\Omega)$ when $N \geq 2$ and $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C_B(\Omega)$ when $N = 1$, hence $\{|u|^{2-(2/p)} \operatorname{sgn} u\}$ is relatively compact in $L^{\frac{ps}{2(p-1)}}(\Omega)$. Therefore, $\{u\}$ is relatively compact in $L^s(\Omega)$ since the Nemytskii mapping $u \in L^{\frac{ps}{2(p-1)}}(\Omega) \to |u|^{\frac{p}{2(p-1)}} \operatorname{sgn} u \in L^s(\Omega)$ is continuous.

(ii) Similarly to the proof of Lemma 2.8 in [4], the result holds. □

Proposition 3.3 ([8]) *Define* $g_+(x) = \liminf_{t \to +\infty} g(x,t)$ *and* $g_-(x) = \limsup_{t \to -\infty} g(x,t)$. *Further, define a function* $g_1 : \Omega \times R \to R$ *by*

$$g_1(x,t) = \begin{cases} (\inf_{s \geq t} g(x,s)) \bigwedge (t - T(x)), & \forall t \geq T(x) \\ 0, & \forall t \in [-T(x), T(x)] \\ (\sup_{s \leq t} g(x,s)) \bigvee (t + T(x)), & \forall t \leq -T(x). \end{cases}$$

Then, the mapping $C_1 : L^s(\Omega) \to L^s(\Omega)$ *defined by* $(C_1u)(x) = g_1(x, u(x))$ *for any* $u \in L^s(\Omega)$ *and* $x \in \Omega$, *is bounded, continuous and* m*-accretive. Also* $C_2 : L^s(\Omega) \to L^s(\Omega)$ *defined by* $(C_2u)(x) = g_2(x, u(x)) = g(x, u(x)) - g_1(x, u(x))$ *satisfies the condition*

$$(C_2(u+y), J_s u) \geq -C(y), \tag{3.1}$$

for any $u, y \in L^s(\Omega)$, *where* $C(y)$ *is a constant depending on* y *and* $J_s : L^s(\Omega) \to L^{s'}(\Omega)$ *denotes the duality mapping.*

Remark 3.2 ([5]) If $\beta_x \equiv 0$, $\forall x \in \Gamma$, then $\partial\Phi_p(u) \equiv 0$, $\forall u \in W^{1,p}(\Omega)$.

Lemma 3.7 ([10]) *Let* X_0 *denote the closed subspace of the all constant functions in* $W^{1,p}(\Omega)$. *Let* X *be the quotient space* $W^{1,p}(\Omega)/X_0$. *For* $u \in W^{1,p}(\Omega)$, *define the mapping* $P : W^{1,p}(\Omega) \to X_0$ *by* $Pu = \frac{1}{\mathrm{meas}(\Omega)}\int_\Omega u\mathrm{d}x$. *Then, there is a constant* $C > 0$, *such that* $\forall u \in W^{1,p}(\Omega)$,

$$\|u - Pu\|_p \leq C\|\nabla u\|_{(L^p(\Omega))^N}.$$

Lemma 3.8 *If* $\beta_x \equiv 0$, $\forall x \in \Gamma$, *then we have*

(i) $\{f \in L^2(\Omega) | \int_\Omega f\mathrm{d}x = 0\} \subset R(A)$, *for* $\frac{2N}{N+1} < p < +\infty$ *and* $N \geq 1$ *(see [6]);*

(ii) $\{f \in L^s(\Omega) | \int_\Omega f\mathrm{d}x = 0\} \subset R(A_s)$, *for* $s \geq 2$ *and* $\frac{2N}{N+1} < p \leq 2$ *where* $N \geq 1$;

(iii) $\{f \in L^s(\Omega) | \int_\Omega f\mathrm{d}x = 0\} \subset R(A_s)$, *for* $2 \leq p \leq s < +\infty$;

(iv) $\{f \in L^s(\Omega) | \int_\Omega f\mathrm{d}x = 0\} \subset R(A_s)$, *for* $\frac{2N}{N+1} < p \leq s < 2$ *and* $N \geq 1$.

Proof (ii) Similarly to the proof of Proposition 2.3 in [4], the result is true.

(iii) For $f \in L^s(\Omega)$ with $\int_\Omega f\mathrm{d}x = 0$, from (i) we know that there exists $u \in L^2(\Omega)$ such that $f = B_pu + \partial\Phi_p(u)$. Therefore, if $u \in L^s(\Omega)$, from the definition of A_s, it will follow that $f = A_su$. To show $u \in L^s(\Omega)$, let $2 \leq p \leq r \leq s$. For $k > 0$, define a function $\chi_k : R \to R$ by

$$\chi_k(t) = |(t \bigwedge k) \bigvee (-k)|^{r-1}\mathrm{sgn}\, t.$$

Then, we have

$$\begin{aligned}\|f\|_s\|u\|_{p+r-2}^{r-1} &\geq \|f\|_s\|u\|_r^{r-1} \geq \|f\|_s\|u\|_{(r-1)s'}^{r-1} \\ &\geq (|u|^{r-1}\mathrm{sgn}\, u, f) \geq (|u|^{r-1}\mathrm{sgn}\, u, B_pu) \geq (r-1)k_3\int_\Omega |\nabla u|^p|u|^{r-2}\mathrm{d}x \\ &\geq \mathrm{const}\int_\Omega \left|\mathrm{grad}(|u|^{1+\frac{r-2}{p}}\mathrm{sgn}\, u)\right|^p\mathrm{d}x,\end{aligned} \tag{3.2}$$

where $\frac{1}{s} + \frac{1}{s'} = 1$. Thus if $u \in L^{p+r-2}(\Omega)$, from (3.2), we have $|u|^{1+\frac{r-2}{p}}\mathrm{sgn}\, u \in W^{1,p}(\Omega)$. Therefore, in view of Lemma 3.7, it follows from (3.2) that

$$\|f\|_s\|u\|_{p+r-2}^{r-1} \geq \mathrm{const}\left\||u|^{1+\frac{r-2}{p}}\mathrm{sgn}u - \frac{1}{\mathrm{meas}(\Omega)}\int_\Omega |u|^{1+\frac{r-2}{p}}\mathrm{sgn}u\mathrm{d}x\right\|_{1,p}^p. \tag{3.3}$$

Now we need to discuss the following four cases:

Case 1 If $N \geq 3$ and $2 \leq p < N$, then in view of Lemma 2.2, we have $W^{1,p}(\Omega) \hookrightarrow L^{\frac{Np}{N-p}}(\Omega)$.

Thus from (3.3), it follows that

$$\|f\|_s\|u\|_{p+r-2}^{r-1} \geq \mathrm{const}\Big(\int_\Omega \Big||u|^{1+\frac{r-2}{p}}\operatorname{sgn} u - \frac{1}{\mathrm{meas}(\Omega)}\int_\Omega |u|^{1+\frac{r-2}{p}}\operatorname{sgn} u\mathrm{d}x\Big|^{\frac{Np}{N-p}}\mathrm{d}x\Big)^{\frac{N-p}{N}}.$$

Therefore, $u \in L^{p+r-2}(\Omega)$ implies that $u \in L^{(1+\frac{r-2}{p})\frac{Np}{N-p}}(\Omega)$. Hence, if $u \in L^2(\Omega)$, after finite steps we find that $u \in L^s(\Omega)$.

Case 2 If $p > N \geq 3$, then in view of Lemma 2.2 we have $W^{1,p}(\Omega) \hookrightarrow C_B(\Omega)$. Thus from (3.3), it follows that

$$\|f\|_s\|u\|_{p+r-2}^{r-1} \geq \mathrm{const}\Big\||u|^{1+\frac{r-2}{p}}\operatorname{sgn} u - \frac{1}{\mathrm{meas}(\Omega)}\int_\Omega |u|^{1+\frac{r-2}{p}}\operatorname{sgn} u\mathrm{d}x\Big\|^p_{\frac{2pr}{p+r-2}}.$$

Therefore, $u \in L^{p+r-2}(\Omega)$ implies that $u \in L^{2r}(\Omega)$.

Case 3 If $p = N \geq 3$, then in view of Lemma 2.2 we have $W^{1,p}(\Omega) \hookrightarrow L^{\frac{2pr}{p+r-2}}(\Omega)$. The rest of the proof is the same as that in Case 2. Hence after finite steps we find that $u \in L^s(\Omega)$. Then the result is true.

Case 4 If $N = 1$ or $N = 2$, then $p \geq N$, repeating the proof of Case 2, the result holds.

(iv) Let $f \in L^s(\Omega)$ with $\int_\Omega f\mathrm{d}x = 0$. Choose a sequence $\{f_n\}$ in $L^2(\Omega)$ such that $\int_\Omega f_n\mathrm{d}x = 0$, for every n and $f_n \to f$ in $L^s(\Omega)$, as $n \to \infty$. Now by (i), there exists $u_n \in L^2(\Omega)$ such that $Au_n = f_n$, for each n. We now define functions $\chi : R \to R$ and $\xi : R \to R$ by

$$\chi(t) = \begin{cases} |t|^{p-1}\operatorname{sgn} t, & \text{if } |t| \geq 1 \\ t, & \text{if } |t| \leq 1 \end{cases}$$

and

$$\xi(t) = \begin{cases} |t|^{2-\frac{2}{p}}\operatorname{sgn} t, & \text{if } |t| \geq 1 \\ t, & \text{if } |t| \leq 1. \end{cases}$$

Note that for $u \in L^2(\Omega)$, the function $t \in R \to \int_\Omega \chi(u+t)\mathrm{d}x \in R$ is continuous on R and $\lim_{t\to\pm\infty}\int_\Omega \chi(u+t)\mathrm{d}x = \pm\infty$. So $\exists t_u \in R$ such that $\int_\Omega \chi(u+t_u)\mathrm{d}x = 0$. Using this, we can assume that $u_n \in L^2(\Omega)$ are such that $\int_\Omega \chi(u_n)\mathrm{d}x = 0$ and $Au_n = f_n$ for each n. Since, now $\chi'(t) \geq c[\xi'(t)]^p$ for every $t \in R$, where c is a positive constant, we have from $Au_n = f_n$ on multiplication by $\chi(u_n)$ that

$$\begin{aligned}
&\|f_n\|_p\Big(\int_{|u_n|\leq 1}|u_n|^{p'}\mathrm{d}x + \int_{|u_n|\geq 1}|u_n|^2\mathrm{d}x\Big)^{\frac{p-1}{p}}\\
&\geq \|f_n\|_p\Big(\int_{|u_n|\leq 1}|u_n|^{p'}\mathrm{d}x + \int_{|u_n|\geq 1}|u_n|^p dx\Big)^{\frac{p-1}{p}}\\
&\geq \|f_n\|_p\|\chi(u_n)\|_{p'} \geq (\chi(u_n), f_n) = (\chi(u_n), Au_n)\\
&\geq (\chi(u_n), B_pu_n) \geq \mathrm{const}\int_\Omega |\nabla u_n|^p\chi'(u_n)\mathrm{d}x\\
&\geq \mathrm{const}\int_\Omega |\mathrm{grad}(\xi(u_n))|^p\mathrm{d}x. \qquad (3.4)
\end{aligned}$$

From Lemma 3.7, we know that

$$\int_\Omega |\mathrm{grad}(\xi(u_n))|^p \mathrm{d}x \geq \mathrm{const}\|\xi(u_n)\|_{1,p}^p. \tag{3.5}$$

From Lemma 2.3, we have

$$\begin{aligned}\|\xi(u_n)\|_{1,p}^p \geq \mathrm{const}\|\xi(u_n)\|_{p'}^p &= \mathrm{const}\Big(\int_{|u_n|\leq 1} |u_n|^{p'}\mathrm{d}x + \int_{|u_n|\geq 1} |u_n|^{(2-\frac{2}{p})p'}\mathrm{d}x\Big)^{\frac{p}{p'}} \\ &= \mathrm{const}\Big(\int_{|u_n|\leq 1} |u_n|^{p'}\mathrm{d}x + \int_{|u_n|\geq 1} |u_n|^{2}\mathrm{d}x\Big)^{\frac{p}{p'}}. \end{aligned} \tag{3.6}$$

From (3.4), (3.5) and (3.6), we obtain

$$\|f_n\|_p \geq \mathrm{const}\Big(\int_{|u_n|\leq 1} |u_n|^{p'}\mathrm{d}x + \int_{|u_n|\geq 1} |u_n|^{2}\mathrm{d}x\Big)^{\frac{p-1}{p'}}.$$

Since $f_n \to f$ in $L^s(\Omega)$, $\{\xi(u_n)\}$ is bounded in $W^{1,p}(\Omega)$ and hence compact in $L^{p'}(\Omega)$.

Notice that the Nemytskii mapping $u \in L^{p'}(\Omega) \to \xi^{-1}(u) \in L^s(\Omega)$ is continuous. We see that $\{u_n\}$ is a compact sequence in $L^s(\Omega)$. This immediately gives that $f \in R(A_s)$ from the definition of A_s. This completes the proof. □

Remark 3.3 Some new techniques are employed in proving Lemma 3.8.

From Lemma 3.8, the following result is immediate.

Proposition 3.4 *If $\beta_x \equiv 0$ for any $x \in \Gamma$, then $\{f \in L^s(\Omega) | \int_\Omega f\mathrm{d}x = 0\} \subset R(A_s)$, for $\frac{2N}{N+1} < p \leq s < +\infty$ and $N \geq 1$.*

Definition 3.2 ([8]) *For $t \in R$ and $x \in \Gamma$, let $\beta_x^0(t) \in \beta_x(t)$ be the element with least absolute value if $\beta_x(t) \neq \emptyset$ and $\beta_x^0(t) = \pm\infty$, where $t > 0$ or < 0, respectively, in case $\beta_x(t) = \emptyset$. Finally, let $\beta_\pm(x) = \lim_{t\to\pm\infty} \beta_x^0(t)$ (in the extended sense) for $x \in \Gamma$. Then, $\beta_\pm(x)$ define measurable functions on Γ.*

Lemma 3.9 ([6]) *Assume that $f \in L^2(\Omega)$ satisfies*

$$\int_\Gamma \beta_-(x)\mathrm{d}\Gamma(x) < \int_\Omega f\mathrm{d}x < \int_\Gamma \beta_+(x)\mathrm{d}\Gamma(x). \tag{3.7}$$

Then, $f \in \mathrm{int}\, R(A)$, for $\frac{2N}{N+1} < p < +\infty$ and $N \geq 1$.

Lemma 3.10 *Let $f \in L^s(\Omega)$ satisfy (3.7). Then, the following results hold:*

(i) If $s \geq 2$ and $\frac{2N}{N+1} < p \leq 2$ for $N \geq 1$, then we have $f \in \mathrm{int}R(A_s)$;

(ii) If $2 \leq p \leq s < +\infty$, then we have $f \in \mathrm{int}\, R(A_s)$;

(iii) If $\frac{2N}{N+1} < p \leq s < 2$ for $N \geq 1$, then we have $f \in \mathrm{int}\, R(A_s)$.

Proof (i) Similarly to the proof of Proposition 2.4 in [4], the result holds.

(ii) Let $f \in L^s(\Omega)$ satisfy (3.7). Then, by Lemma 3.9, we have $f \in \mathrm{int}R(A)$. Now using the similar arguments to that of (iii) in Lemma 3.8, we find that $f \in \mathrm{int}\, R(A_s)$.

(iii) Now $f \in L^s(\Omega)$ implies that there is a sequence $\{f_n\}$ in $L^2(\Omega)$ such that $f_n \to f$ in $L^s(\Omega)$, as $n \to \infty$. By Proposition 3.1, there exists $u_n \in L^2(\Omega)$, such that for each $n \geq$

1, $f_n = \frac{1}{n}u_n + Au_n$. Now , it suffices to show that $\|u_n\|_s \leq$ const, for $\forall n \geq 1$. Indeed, suppose to the contrary that $1 \leq \|u_n\|_s \to \infty$, as $n \to \infty$. Let $v_n = u_n/\|u_n\|_s$. And, let $\psi : R \to R$ be defined by $\psi(t) = |t|^p$, $\partial\psi : R \to R$ be its subdifferential, and for $\mu > 0$, $\partial\psi_\mu : R \to R$ be the Yosida-approximation of $\partial\psi$. Further, let $\theta_\mu : R \to R$ be the indefinite integral of $[(\partial\psi_\mu)']^{\frac{1}{p}}$ with $\theta_\mu(0) = 0$ so that $(\theta'_\mu)^p = (\partial\psi_\mu)'$. By using similar arguments to those for Proposition 2.4 in [1], we have

$$\int_\Omega |\mathrm{grad}(\theta_\mu(v_n))|^p \mathrm{d}x \leq \frac{C}{\|u_n\|_s^{p-1}}, \quad \text{for} \quad \mu > 0 \quad \text{and} \quad n \geq 1, \tag{3.8}$$

where C is a constant which does not depend on n or μ. Now since $(\theta'_\mu)^p = (\partial\psi_\mu)' \to (\partial\psi)'$, as $\mu \to 0$, a.e., on R. Letting $\mu \to 0$, we see from Fatou's lemma and (3.8) that

$$\int_\Omega |\mathrm{grad}(|v_n|^{2-(2/p)}\mathrm{sgn}\, v_n)|^p \mathrm{d}x \leq \frac{C}{\|u_n\|_s^{p-1}}. \tag{3.9}$$

From (3.9), it follows that $|v_n|^{2-(2/p)}\mathrm{sgn}\, v_n \to k$ (a constant) in $L^p(\Omega)$, as $n \to +\infty$. Next, we will prove that $k \neq 0$ in $L^p(\Omega)$. Since $2N/(N+1) < p \leq s \leq 2$, $\||v_n|^{2-(2/p)}\mathrm{sgn}\, v_n\|_p = \|v_n\|_{2p-2}^{2-(2/p)} \leq \|v_n\|_s^{2-(2/p)} = 1$, and hence $\{|v_n|^{2-(2/p)}\mathrm{sgn}\, v_n\}$ is bounded in $W^{1,p}(\Omega)$. By Lemma 2.3, $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C_B(\Omega)$ when $N = 1$ and $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^{\frac{ps}{2(p-1)}}(\Omega)$ when $N \geq 2$. Thus $\{|v_n|^{2-(2/p)}\mathrm{sgn}\, v_n\}$ is relatively compact in $L^{\frac{ps}{2(p-1)}}(\Omega)$. Therefore, there exists a subsequence of $\{|v_n|^{2-(2/p)}\mathrm{sgn}\, v_n\}$. For simplicity, we denote it by $\{|v_n|^{2-(2/p)}\mathrm{sgn}\, v_n\}$, satisfying $|v_n|^{2-(2/p)}\mathrm{sgn}\, v_n \to g$ in $L^{\frac{ps}{2(p-1)}}(\Omega)$. Noticing that $p \leq \frac{ps}{2(p-1)}$ when $\frac{2N}{N+1} < p \leq s \leq 2$ for $N \geq 1$, it follows that $k = g$, a.e., on Ω. Finally, since

$$\begin{aligned} 1 = \|v_n\|_s^s &= \int_\Omega ||v_n|^{2-(2/p)}\mathrm{sgn}\, v_n|^{\frac{ps}{2(p-1)}} \mathrm{d}x \\ &\leq \mathrm{const} \int_\Omega ||v_n|^{2-(2/p)}\mathrm{sgn}\, v_n - g|^{\frac{ps}{2(p-1)}} \mathrm{d}x + \mathrm{const}\|g\|_{\frac{ps}{2(p-1)}}^{\frac{ps}{2(p-1)}}, \end{aligned}$$

it follows that $g \neq 0$ in $L^{\frac{ps}{2(p-1)}}(\Omega)$, and hence $k \neq 0$ in $L^p(\Omega)$. The following argument is the same as Proposition2.4 in [1]. □

From Lemma 3.10, the following result is immediate.

Proposition 3.5 *Let $f \in L^s(\Omega)$ satisfy (3.7). Then $f \in \mathrm{int}\, R(A_s)$, where $\frac{2N}{N+1} < p \leq s < +\infty$ for $N \geq 1$.*

Remark 3.4 Since $\Phi_p(u+\alpha) = \Phi_p(u)$ for any $u \in D(A_s)$ and $\alpha \in C_0^\infty(\Omega)$, we find $f \in A_s u$ ($s \geq 2$) implies that $f = B_p u$ in the sense of distributions.

Lemma 3.11 *If $f \in L^2(\Omega)$ and $u \in L^2(\Omega)$ are such that $f \in Au$, then the following results hold*

(a) $-\mathrm{div}(\alpha(\mathrm{grad}\, u)) + |u|^{p-2}u = f(x)$, a.e., $x \in \Omega$;

(b) $-\langle \vartheta, \alpha(\mathrm{grad}\, u)\rangle \in \beta_x(u(x))$, a.e., $x \in \Gamma$.

Proof Similarly to the proof of Proposition 2.2 in [6], the result is valid. □

Lemma 3.12 *If $\frac{2N}{N+1} < p \leq s < 2$, let $f \in L^s(\Omega)$ and $u \in L^s(\Omega)$ be such that $f \in A_s u$. Then, the results of (a) and (b) in Lemma 3.11 are also true.*

Proof (a) For $\frac{2N}{N+1} < p \leq s < 2$, we see that $u \in L^s(\Omega)$ and $f \in A_s u$ imply the existence of sequences $\{u_n\}, \{f_n\}$ in $L^2(\Omega)$ such that $u_n \to u, f_n \to f$ in $L^s(\Omega)$, as $n \to \infty$, and $f_n \in Au_n$ for all n.

Lemma 3.11 implies that $f_n = -\mathrm{div}(\alpha(\mathrm{grad}\, u_n)) + |u_n|^{p-2}u_n$, a.e. in Ω. From the fact that α is continuous, we know that $f = -\mathrm{div}(\alpha(\mathrm{grad}\, u)) + |u|^{p-2}u$, a.e. in Ω. This completes the proof of (a).

(b) From (a) we know that for $f \in A_s u$, $f = -\mathrm{div}(\alpha(\mathrm{grad}\, u))$, a.e. in Ω and there exist sequences $\{u_n\}$, $\{f_n\}$ in $L^2(\Omega)$ such that $u_n \to u$, $f_n \to f$ in $L^s(\Omega)$ and $f_n = Au_n$.

Now from Lemma 3.11, we get that

$$-\langle \vartheta, \alpha(\mathrm{grad}\, u_n)\rangle = \beta_x(u_n(x))$$

a.e. on Γ. Then the continuity of both α and β_x implies that $-\langle \vartheta, \alpha(\mathrm{grad}\, u)\rangle = \beta_x(u(x))$ a.e. on Γ. □

Lemma 3.13 *If $\frac{2N}{N+1} < p \leq 2 \leq s$, or $s \geq p \geq 2$, and $f, u \in L^s(\Omega)$ satisfy $f \in A_s u$. Then, the results of Lemma 3.11 are still true.*

Proof Similarly to the proofs of Proposition 2.2 in [4] and Propostion 2.2 in [7], the result is valid.

Lemmas 3.12 and 3.13 imply the following result:

Proposition 3.6 *Let $f \in L^s(\Omega)$, $u \in L^s(\Omega)$ be such that $f \in A_s u$. Then, the results of Lemma 3.11 are true.*

Theorem 3.1 *Let $f \in L^s(\Omega)$ satisfy*

$$\int_\Gamma \beta_-(x)\mathrm{d}\Gamma(x) + \int_\Omega g_-(x)\mathrm{d}x < \int_\Omega f(x)\mathrm{d}x < \int_\Gamma \beta_+(x)\mathrm{d}\Gamma(x) + \int_\Omega g_+(x)\mathrm{d}x.$$

Then, Equation (1.1) has a solution in $L^s(\Omega)$.

Proof Let A_s be the m-accretive mapping as in Definition 3.1 and $C_i : L^s(\Omega) \to L^s(\Omega)$ be as in Proposition 3.3, i.e., $(C_i u)(x) = g_i(x, u(x))$ for $x \in \Omega$, and $i = 1, 2$. We need to prove that $A_s + C_1$ is boundedly-inversely-compact. In fact, we only need to show that if $w \in A_s u + C_1 u$ with $\{w\}$ and $\{u\}$ being bounded in $L^s(\Omega)$, then $\{u\}$ is relatively compact in $L^s(\Omega)$. For this, we need to discuss the following two cases:

(i) If $\frac{2N}{N+1} < p \leq s < 2$, or $s \geq 2$ and $\frac{2N}{N+1} < p \leq 2$, for $N \geq 1$, then the relative compactness of $\{u\}$ in $L^s(\Omega)$ follows from Proposition 3.2.

(ii) If $2 \leq p \leq s$, define a function $\chi_k : R \to R$ by $\chi_k(t) = |(t \wedge k) \vee (-k)|^{s-p+1}\mathrm{sgn}\, t$. Then, we have

$$\begin{aligned}
\mathrm{const} &\geq \|w\|_s \|u\|_s^{s-p+1} \geq \|w\|_s \|u\|_{(s-p+1)s'}^{s-p+1} \\
&\geq (|u|^{s-p+1}\mathrm{sgn}\, u, w) \geq (|u|^{s-p+1}\mathrm{sgn}\, u, B_p u) + \lim_{k \to +\infty}(\chi_k(u), \partial\Phi_p(u)) \\
&\geq (|u|^{s-p+1}\mathrm{sgn} u, B_p u) \geq \mathrm{const} \int_\Omega |\mathrm{grad}(|u|^{1+\frac{s-p}{p}}\mathrm{sgn}\, u)|^p \mathrm{d}x,
\end{aligned}$$

where $\frac{1}{s}+\frac{1}{s'}=1$. Moreover,

$$\int_\Omega \Big||u|^{1+\frac{s-p}{p}}\operatorname{sgn} u\Big|^p \mathrm{d}x = \|u\|_s^s.$$

Therefore, $\{|u|^{1+\frac{s-p}{p}}\operatorname{sgn} u\}$ is bounded in $W^{1,p}(\Omega)$, and hence $\{|u|^{1+\frac{s-p}{p}}\operatorname{sgn} u\}$ is relatively compact in $L^p(\Omega)$. This implies that $\{u\}$ is relatively compact in $L^s(\Omega)$ since the Nemytskii mapping $u\in L^p(\Omega)\to |u|^{\frac{p}{s}}\operatorname{sgn} u\in L^s(\Omega)$ is continuous.

Now by using methods similar to those employed in [1–7], it is easy to show that all the conditions of Theorem 2.1 are satisfied. Further, from Propositions 3.4 and 3.5, we have $f\in \operatorname{int}[R(A_s)+R(C_1)]$. Then Theorem 2.1 implies that $f\in \operatorname{int} R(A_s+C_1+C_2)$. Therefore, Proposition 3.6 implies that the Theorem 3.1 holds. □

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