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Subvarieties of the Varieties Generated by Aperiodic Commutative Semigroups

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Abstract In this paper, all subvarieties of the varieties $\mathbf{A}_k$ $(k \in N)$ generated by aperiodic commutative semigroups are characterized. Based on this characterization, the structure of lattice of subvarieties of $\mathbf{A}_k$ is investigated.

Keywords aperiodic commutative semigroups; varieties; lattice of varieties.

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1. Introduction

Varieties of commutative semigroups have been extensively studied not only because of their natural continuation of the theory of commutative groups, but also, because of their applications in language and automata theory. There are lots of good results on varieties of commutative semigroups. Schwabauer [1, 2] proved that the lattice of varieties of commutative semigroups, $\mathcal{L}(\mathrm{Com})$, is non-modular. Perkins [3, 4] showed that every variety of commutative semigroups is finitely based, and so the lattice $\mathcal{L}(\mathrm{Com})$ is countable and has no infinite descending chains. Kisielewicz [5] described varieties of commutative semigroups in terms of certain order filters, integer parameters, and the so called remainders. But it is fairly complicated to completely describe the lattice of varieties of commutative semigroups.

In this paper, we pay our main attention to the varieties $\mathbf{A}_k$ determined by the identities set

$$A_k = \{x^{k+1} \approx x^k, xy \approx yx\}$$

for $k \in N$. Clearly, $\mathbf{A}_1 \subset \mathbf{A}_2 \subset \cdots \subset \mathbf{A}_k$ and $\mathbf{A}_k$ is generated by aperiodic commutative semigroups. In particular, $\mathbf{A}_1$ is the variety generated by semilattices. The description of $\mathcal{L}(\mathbf{A}_2)$ is already known [2]. The main aims of this paper are to characterize the subvarieties of $\mathbf{A}_k$ $(k \in N)$ by determining sets of identities and to establish the structure of lattice of subvarieties of $\mathbf{A}_k$ $(k \in N)$.

The reader is referred to [4] and [6] for all notations and terminologies not defined in this paper.

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Let X be a countably infinite alphabet ordered by $<$. Elements of X are referred to as letters. Throughout this paper, x, y, z with or without indices stand for letters and u, v, w with or without indices stand for words over X. We write $u \approx v$ to stand for a semigroup identity and write $u = v$ if u and v are identical words.

Let u be a word over X. The content of u is the set $c(u)$ of letters occurring in u. The multiplicity $m_u(x)$ of x is the number of occurrences of x in u. The length of u is the number $|u|$ of letters in u counting multiplicity.

Let Σ be a set of identities. The variety defined by Σ is denoted by $[\Sigma]$. Two words u, v are called Σ-equivalent if u is derivable from v by invoking the identities in Σ, in this case we write $u \overset{\Sigma}{\approx} v$. If an identity σ is derivable from the identities in Σ, then we write $\Sigma \vdash \sigma$. For a variety $\mathbf{V}$, we write $\mathbf{V} \models \sigma$ if it satisfies σ.

Let $\mathbf{U}$ be a subvariety of a variety $\mathbf{V}$. The set of all varieties $\mathbf{W}$ such that $\mathbf{U} \subseteq \mathbf{W} \subseteq \mathbf{V}$ constitutes a complete lattice and is denoted by $\mathcal{L}(\mathbf{U}, \mathbf{V})$. We write $\mathcal{L}(\mathbf{V}) = \mathcal{L}(\mathbf{T}, \mathbf{V})$, where $\mathbf{T}$ is the variety of trivial semigroups. In this paper, we use the Hasse diagram to represent the lattice of varieties, where a line joining a lower positioned variety to a higher positioned variety indicates containment, denoted by $\subseteq$, and a bolded line indicates containment with covering, denoted by $\prec$.

Recall $\mathbf{A}_1$ is the variety generated by semilattices. It is well known that $\mathbf{A}_1 \models u \approx v$ if and only if $c(u) = c(v)$. A variety is semilattice-free if it does not contain $\mathbf{A}_1$.

Let N denote the set of all positive integers and let $N_0 = N \cup \{0\}$.

In this paper, maximal subvarieties of some varieties in $\mathcal{L}(\mathbf{A}_k)$ $(k \in N)$ will be shown in Section 2. In particular, the unique maximal subvariety of $\mathbf{A}_k$ will be determined. In Section 3, the relations of some subvarieties lattice will be described. The main result of this paper is the structure of $\mathcal{L}(\mathbf{A}_k)$ (Theorem 3.6).

2. Maximal subvarieties of some varieties in $\mathcal{L}(\mathbf{A}_k)$

A word u is said to be in canonical form if

$$u = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n},$$

where $x_1, x_2, \ldots, x_n$ are distinct letters in X with $x_1 < x_2 < \cdots < x_n$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \{1, \ldots, k\}$.

Proposition 2.1 *Each word is A_k-equivalent to a unique word in canonical form.*

Proof This is straightforward.

It is well known that every variety of aperiodic commutative semigroups is finitely based. Thus all subvarieties of $\mathbf{A}_k$ are finitely based. It follows from Proposition 2.1 that each proper subvariety of $\mathbf{A}_k$ possesses a finite basis $A_k \cup \Sigma$, where each identity in Σ is formed by a pair of words in canonical form.

For the rest of this paper, whenever $u \approx v$ is an identity that holds in a subvariety of $\mathbf{A}_k$, we always assume that

$$u = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n} \quad \text{and} \quad v = x_1^{\delta_1} x_2^{\delta_2} \cdots x_n^{\delta_n},$$

where u, v satisfy the following conditions:

$$\begin{aligned} &x_1 < x_2 < \cdots < x_n, \\ &0 \leq \varepsilon_i, \delta_i \leq k \ (i = 1, 2, \ldots, n), \\ &\varepsilon_1 + \cdots + \varepsilon_n > 0, \\ &\delta_1 + \cdots + \delta_n > 0. \end{aligned}$$

We define

$$m(u \approx v) = \min\{\varepsilon_i, \delta_i : \varepsilon_i \neq \delta_i, i = 1, \ldots, n\}.$$

Lemma 2.2 *Let $u \approx v$ be an identity such that $\mathbf{A}_k \nvDash u \approx v$ $(k \geq 2)$ and $\mathbf{A}_1 \vDash u \approx v$. Then $A_k \cup \{u \approx v\} \vdash x^k y^k \approx x^k y^{k-1}$.*

Proof Since $\mathbf{A}_1 \vDash u \approx v$, we have $c(u) = c(v)$. In this case, we may assume that $\varepsilon_i, \delta_i \in \{1, \ldots, k\}$ for $i = 1, 2, \ldots, n$ and there exists a letter $x_i \in c(u) = c(v)$ with $m_u(x_i) \neq m_v(x_i)$. Suppose that $m_u(x_i) = h > m_v(x_i) = l$. Denote by ρ the substitution $t \mapsto x^k$ for all $t \neq x_i$ and $t \mapsto y$ for $t = x_i$. Then

$$x^k y^k \approx x^k \rho(u) y^{k-l-1} \approx x^k \rho(v) y^{k-l-1} \approx x^k y^{k-1}.$$

Hence $A_k \cup \{u \approx v\} \vdash x^k y^k \approx x^k y^{k-1}$. □

Lemma 2.3 *Let $u \approx v$ be an identity such that $\mathbf{A}_k \nvDash u \approx v$ $(k \in N)$ and $\mathbf{A_1} \nvDash u \approx v$. Then $A_k \cup \{u \approx v\} \vdash x^k y^k \approx x^k \vdash x^k y \approx x^k$.*

Proof Without loss of generality, we may assume $y \in c(u) \setminus c(v)$. Denote by ρ the substitution $t \mapsto x^k$ for all $t \neq y$ and $t \mapsto y^k$ for $t = y$. Then $\mathbf{V}$ satisfies

$$x^k \stackrel{A_k}{\approx} x^k \rho(v) \approx x^k \rho(u) \approx x^k \cdots y^k \cdots \stackrel{A_k}{\approx} x^k y^k$$

and so $A_k \cup \{x^k \approx x^k y^k\} \vdash x^k y \stackrel{x^k \approx x^k y^k}{\approx} x^k y^k y \stackrel{A_k}{\approx} x^k y^k \approx x^k$. □

Theorem 2.4 *The identity $x^k y^k \approx x^k y^{k-1}$ is satisfied by every proper subvariety of $\mathbf{A}_k$ $(k \in N)$. In particular, the unique maximal subvariety of $\mathbf{A}_k$ is defined within $\mathbf{A}_k$ by $x^k y^k \approx x^k y^{k-1}$.*

Proof It suffices to show that $x^k y^k \approx x^k y^{k-1}$ is derivable within $\mathbf{A}_k$ from any identity not satisfied by $\mathbf{A}_k$. Let $u \approx v$ be an identity with $\mathbf{A}_k \nvDash u \approx v$. Suppose that $k = 1$. Then by Lemma 2.3, $A_k \cup \{u \approx v\} \vdash x^k y^k \approx x^k y^{k-1}$. Suppose that $k \geq 2$. If $c(u) \neq c(v)$, then $u \approx v$ is not satisfied by any semilattice, and so by Lemma 2.3,

$$A_k \cup \{u \approx v\} \vdash x^k \approx x^k y \vdash x^k y^{k-1} \approx x^k y^k.$$

Therefore assume that $c(u) = c(v)$. By Lemma 2.2, $x^k y^k \approx x^k y^{k-1}$ is derivable within $\mathbf{A}_k$ from any identity not satisfied by $\mathbf{A}_k$. Hence $A_k \cup \{u \approx v\} \vdash x^k y^k \approx x^k y^{k-1}$, and so the variety defined by $A_k \cup \{x^k y^k \approx x^k y^{k-1}\}$ is the unique maximal subvariety of $\mathbf{A}_k$.

Let

$$A_{k,m} = A_k \cup \{x^k y^k \approx x^k y^m\}$$

for $m \in N_0, k \in N$ and $0 \le m \le k-1$ and let $\mathbf{A}_{k,m}$ be the variety defined by $A_{k,m}$. It is obvious that $\mathbf{A}_{k,k-1}$ is the unique maximal subvariety of $\mathbf{A}_k$ and $\mathbf{A}_{k,0}$ is the unique maximal semilattice-free subvariety of $\mathbf{A}_k$.

Theorem 2.5 *$\mathbf{A}_{k,m-1}$ is a maximal subvariety of $\mathbf{A}_{k,m}$ for $1 \le m \le k-1$ and $k \ge 2$.*

Proof Let $\mathbf{V}$ be a proper subvariety of $\mathbf{A}_{k,m}$. Then $\mathbf{V}$ satisfies some identity $u \approx v$ which is not satisfied by $\mathbf{A}_{k,m}$. Suppose that $2 \le m \le k-1$. There are two cases.

Case 1 $c(u) = c(v)$. In this case, we may assume that $\varepsilon_i, \delta_i \in \{1, \ldots, k\}$.

(1.1) $m(u \approx v) < m$. Without loss of generality, we may assume that $m(u \approx v) = \varepsilon_i$ and $\varepsilon_i < \delta_i$ for some $i = 1, \ldots, n$. Denote by ρ the substitution $t \mapsto x^k$ for all $t \ne x_i$ and $t \mapsto y$ for $t = x_i$. Then $x^k y^{\varepsilon_i} \approx x^k \rho(u) \approx x^k \rho(v) \approx x^k y^{\delta_i}$. Hence

$$\begin{aligned} A_{k,m} \cup \{u \approx v\} &\vdash x^k y^{\varepsilon_i} \approx x^k y^{\delta_i} \\ &\vdash x^k y^{\varepsilon_i} \approx x^k y^{\varepsilon_i} y^{\delta_i - \varepsilon_i} \approx x^k y^{\varepsilon_i} y^{2(\delta_i - \varepsilon_i)} \approx \cdots \approx x^k y^k \\ &\vdash x^k y^{m-1} \approx x^k y^k. \end{aligned}$$

(1.2) $m(u \approx v) \ge m$. If $\max\{\varepsilon_i : i = 1, \ldots, n\} = \max\{\delta_i : i = 1, \ldots, n\} = k$, then $u \overset{A_{k,m}}{\approx} v$ and so we may assume that $\max\{\varepsilon_i : i = 1, \ldots, n\} \le k-1$ and $\varepsilon_i < \delta_i$ for some $i = 1, \ldots, n$. Thus

$$\begin{aligned} x_1^{k-1} \cdots x_i^{k-1} \cdots x_n^{k-1} &\approx u x_1^{k-1-\varepsilon_1} \cdots x_i^{k-1-\varepsilon_i} \cdots x_n^{k-1-\varepsilon_n} \\ &\approx v x_1^{k-1-\varepsilon_1} \cdots x_i^{k-1-\varepsilon_i} \cdots x_n^{k-1-\varepsilon_n} \\ &\approx x_1^{k-1-\varepsilon_1+\delta_1} \cdots x_i^{k-1-\varepsilon_i+\delta_i} \cdots x_n^{k-1-\varepsilon_n+\delta_n} \\ &\overset{(a)}{\approx} x_1^k \cdots x_i^k \cdots x_n^k, \end{aligned}$$

where (a) holds by: if $j = i$, then $k-1-\varepsilon_i+\delta_i \ge k$; if $j \ne i$ and $\delta_j \ge \varepsilon_j$, then $k-1-\varepsilon_j+\delta_j \ge k-1 \ge m$; if $j \ne i$ and $\delta_j < \varepsilon_j$, then

$$k-1-\varepsilon_j+\delta_j \ge k-1-(k-1)+m(u \approx v) = m(u \approx v) \ge m$$

by $\delta_j \ge m(u \approx v)$ and $\varepsilon_j \le k-1$. Hence $A_{k,m} \cup \{u \approx v\} \vdash x_1^{k-1} \cdots x_n^{k-1} \approx x_1^k \cdots x_n^k$.

Case 2 $c(u) \ne c(v)$. By Lemma 2.3, $\mathbf{V} \subseteq [A_{k,m}, x^k y^k \approx x^k]$. Clearly, $A_{k,m} \cup \{x^k y^k \approx x^k\} \vdash x^k y^{m-1} \approx x^k y^k y^{m-1} \approx x^k y^k$.

It is obvious that $[A_{k,m}, x^k y^k \approx x^k y^{m-1}]$ and $[A_{k,m}, x_1^k \cdots x_n^k \approx x_1^{k-1} \cdots x_n^{k-1}]$ are incomparable. Hence $\mathbf{A}_{k,m-1}$ is a maximal subvariety of $\mathbf{A}_{k,m}$.

Suppose that $m = 1$. If $\mathbf{A}_1 \subseteq \mathbf{V}$, then $m(u \approx v) \ge 1$. By subcase (1.2), $A_{k,1} \cup \{u \approx v\} \vdash x_1^{k-1} \cdots x_n^{k-1} \approx x_1^k \cdots x_n^k$. If $\mathbf{A}_1 \nsubseteq \mathbf{V}$, then $\mathbf{V} \subseteq [A_{k,1}, x^k y^k \approx x^k]$ by Lemma 2.3. Clearly, $[A_{k,1}, x^k y^k \approx x^k]$ and $[A_{k,1}, x_1^k \cdots x_n^k \approx x_1^{k-1} \cdots x_n^{k-1}]$ are incomparable. Hence $[A_k, x^k y^k \approx x^k] = \mathbf{A}_{k,0}$ is a maximal subvariety of $\mathbf{A}_{k,1}$.

Corollary 2.6 *Let $u \approx v$ be an identity such that $\mathbf{A}_{k,m} \nvDash u \approx v$ and $\max\{m_u(x_i) : x_i \in c(u)\} = \max\{m_v(x_i) : x_i \in c(v)\} = k$. Then $[A_{k,m}, u \approx v] = \mathbf{A}_{k,m(u\approx v)}$.*

Proof It follows from subcase (1.1) of Theorem 2.5 that $A_{k,1} \cup \{u \approx v\} \vdash x^k y^{m(u\approx v)} \approx x^k y^k$. It is easy to show that $A_{k,1} \cup \{x^k y^{m(u\approx v)} \approx x^k y^k\} \vdash u \approx v$, and so $[A_{k,m}, u \approx v] = \mathbf{A}_{k,m(u\approx v)}$.

Corollary 2.7 *Let $u \approx v$ be an identity such that $\mathbf{A}_{k,m-1} \nvDash u \approx v$. Then $[A_{k,m-1}, u \approx v]$ is a maximal subvariety of $[A_{k,m}, u \approx v]$.*

Proof It is similar to the arguments of Theorem 2.5.

3. The structure of lattice $\mathcal{L}(\mathbf{A}_k)$

In order to explore the structure of lattice $\mathcal{L}(\mathbf{A}_k)$, we first define the following identities

$$(k;n) : x_1^k x_2^{k-1} \cdots x_n^{k-1} \approx x_1^{k-1} x_2^{k-1} \cdots x_n^{k-1}$$

for $k, n \in N$ and $k \geq 2$. Let

$$A_{k,m,n} = A_{k,m} \cup \{(k;n)\}$$

and $\mathbf{A}_{k,m,n}$ be the variety defined by $A_{k,m,n}$. Then by Corollary 2.7,

$$\begin{aligned} &\mathbf{A}_{k,m} = \mathbf{A}_{k+1,m,1}, \quad \mathbf{A}_k = \mathbf{A}_{k+1,k,1}, \\ &\mathbf{A}_{k,m,1} \subseteq \mathbf{A}_{k,m,2} \subseteq \cdots \subseteq \mathbf{A}_{k,m,n} \subseteq \cdots, \\ &\mathbf{A}_{k,0,n} \prec \mathbf{A}_{k,1,n} \prec \cdots \prec \mathbf{A}_{k,i,n} \prec \cdots \prec \mathbf{A}_{k,k-1,n} \end{aligned}$$

for $k \geq 2, 0 \leq m \leq k-1$ and $n \in N$.

Lemma 3.1 *Let $\mathbf{V}$ be a variety in the lattice $\mathcal{L}(\mathbf{A}_{k-1,m}, \mathbf{A}_{k,m})$ for $k \geq 2$ and $0 \leq m \leq k-1$. Then $\mathbf{V}$ is defined within $\mathbf{A}_{k,m}$ by finitely many identities of the form either*

$$\sigma_1 : x_1^k x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n} = x_1^{k-1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}$$

with $\varepsilon_2, \ldots, \varepsilon_n \in \{1, \ldots, k-1\}$ or

$$\sigma_2 : x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} = x_1^{\delta_1} \cdots x_n^{\delta_n}$$

with $\max\{\varepsilon_i : i = 1, \ldots, n\} = \max\{\delta_i : i = 1, \ldots, n\} = k-1$ and $m(\sigma_2) \geq m$.

Proof Since every subvariety of $\mathbf{A}_k$ is finitely based, $\mathbf{V}$ is defined within $\mathbf{A}_{k,m}$ by a finite set Σ of identities. It is easy to see that $\mathbf{A}_{k-1,m} \vDash \{\sigma_1, \sigma_2\}$ and $\mathbf{A}_{k,m} \nvDash \{\sigma_1, \sigma_2\}$. It remains to show that any identity $u \approx v \in \Sigma$ in $\mathbf{V}$ which satisfies $\mathbf{A}_{k-1,m} \vDash u \approx v$ and $\mathbf{A}_{k,m} \nvDash u \approx v$ can be transformed into the form either σ_1 or σ_2.

First we note that $m(u \approx v) \geq m$ by the proof of Theorem 2.5. Let

$$\varepsilon = \max\{\varepsilon_i : i = 1, \ldots, n\}, \quad \delta = \max\{\delta_i : i = 1, \ldots, n\}.$$

Without loss of generality, we may assume that $\varepsilon \geq \delta$.

Case 1 $m \geq 1$. In this case $\mathbf{A}_1 \subseteq \mathbf{A}_{k-1,m}$ and so $c(u) = c(v)$.

(1.1) $\varepsilon = \delta = k$. This subcase is impossible since $\mathbf{A}_{k,m} \models u \approx v$.

(1.2) $\varepsilon \leq k-1$. If $\delta = k-1$, then $u \approx v$ is of the form σ_2. If $\delta < k-1$, then it is impossible since $\mathbf{A}_{k-1,m} \not\models u \approx v$.

(1.3) $\varepsilon = k$ and $\delta \leq k-1$. If $\delta < k-1$, then $\mathbf{A}_{k-1,m} \not\models u \approx v$. If $\delta = k-1$. Let $\delta_i = \delta = k-1$ for some $i = 1, \ldots, n$. Then

$$A_{k,m} \cup \{u \approx v\} \vdash u' \approx v,$$

where $u' \stackrel{A_{k,m}}{\approx} u$, $m_{u'}(x_i) = k$ and $m_{u'}(x_j) = m_v(x_j)$ for $i \neq j$. Clearly, $A_{k,m} \cup \{u' \approx v\} \vdash u \approx v$ and so $[\mathbf{A}_{k,m}, u \approx v] = [\mathbf{A}_{k,m}, u' \approx v]$, where $u' \approx v$ is of the form σ_1.

Case 2 $m = 0$. If $c(u) = c(v)$, then by Case 1, $u \approx v$ is of the form either σ_1 or σ_2. If $c(u) \neq c(v)$, say $\{y_1, \ldots, y_l\} \in c(u) \setminus c(v)$ and $\{z_1, \ldots, z_m\} \in c(v) \setminus c(u)$, then, by using the identity $x^k y^k \approx x^k$, we have $u \approx u z_1^k \cdots z_m^k = u'$ and $v \approx v y_1^k \cdots y_l^k = v'$ with $c(u') = c(v')$ and so $[A_{k,m}, u \approx v] = [A_{k,m}, u' \approx v']$. Hence by Case 1, any identity $u \approx v \in \Sigma$ can be transformed into the form either σ_1 or σ_2.

Corollary 3.2 *Let* $\mathbf{V}$ *be a variety in the lattice* $\mathcal{L}(\mathbf{A}_{k,m,l}, \mathbf{A}_{k,m,l+1})$ *for* $k \geq 2, 0 \leq m \leq k-1$ *and* $l \in N$. *Then* $\mathbf{V}$ *is defined within* $\mathbf{A}_{k,m,l+1}$ *by finitely many identities of the form either*

$$\sigma_3 : x_1^k x_2^{k-1} \cdots x_l^{k-1} x_{l+1}^{\varepsilon_1} \cdots x_n^{\varepsilon_{n-l}} = x_1^{k-1} x_2^{k-1} \cdots x_l^{k-1} x_{l+1}^{\varepsilon_1} \cdots x_n^{\varepsilon_{n-l}}$$

with $\varepsilon_1, \ldots, \varepsilon_{n-1} \in \{1, 2, \ldots, k-2\}$ *or*

$$\sigma_4 : x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} = x_1^{\delta_1} \cdots x_n^{\delta_n}$$

with $\max\{\varepsilon_i : i = 1, \ldots, n\} = \max\{\delta_i : i = 1, \ldots, n\} = k-1$, $m(\sigma_4) \geq m$ *and* $|\{\varepsilon_i = k-1 : i = 1, \ldots, n\}| = |\{\delta_i = k-1 : i = 1, \ldots, n\}| = l$.

Proof It is obvious that $\mathcal{L}(\mathbf{A}_{k,m,l}, \mathbf{A}_{k,m,l+1})$ is a sublattice of $\mathcal{L}(\mathbf{A}_{k-1,m}, \mathbf{A}_{k,m})$. Assume that $\mathbf{V}$ is defined within $\mathbf{A}_{k,m,l+1}$ by a finite set Σ of identities, where any identity $u \approx v \in \Sigma$ satisfies $\mathbf{A}_{k,m,l} \models u \approx v$ and $\mathbf{A}_{k,m,l+1} \not\models u \approx v$. Then $u \approx v$ is of the form either σ_1 or σ_2 by Lemma 3.1. Suppose that $u \approx v$ is of the form σ_1. It is easy to show that $u \approx v$ can be reduced to the identity of the form σ_3. Suppose that $u \approx v$ is of the form σ_2. Let $|\{\varepsilon_i = k-1 : i = 1, \ldots, n\}| = a$ and $|\{\delta_i = k-1 : i = 1, \ldots, n\}| = b$.

If $a < l$ or $b < l$, then this case is impossible since $\mathbf{A}_{k,m,l} \not\models u \approx v$.

If $a = l, b > l$ or $a > l, b = l$. By symmetry, it suffices to assume $a = l, b > l$. Then

$$A_{k,m,l+1} \cup \{u \approx v\} \vdash u \approx v',$$

where $v' \stackrel{A_{k,m,l+1}}{\approx} v$ and $\max(m_{v'}(x)) = k$ for $x \in c(v')$. Clearly, $A_{k,m,l} \cup \{u \approx v'\} \vdash u \approx v$ and so $[\mathbf{A}_{k,m,l}, u \approx v] = [\mathbf{A}_{k,m}, u \approx v']$, where $u \approx v'$ is of the form σ_3.

If $a = b = l$, then $u \approx v$ is just the identity of the form σ_4.

Theorem 3.3 *For* $k \geq 2$ *and* $0 \leq j < i \leq k-1$, *the lattice* $\mathcal{L}(\mathbf{A}_{k-1,i}, \mathbf{A}_{k,i})$ *can be embedded into the lattice* $\mathcal{L}(\mathbf{A}_{k-1,j}, \mathbf{A}_{k,j})$.

Proof By Lemma 3.1, any variety $\mathbf{V} \in \mathcal{L}(\mathbf{A}_{k-1,i}, \mathbf{A}_{k,i})$ can be defined by $[A_{k,i}, \Sigma]$, where any identity $u \approx v \in \Sigma$ is of the form either σ_1 or σ_2. Let $\mathbf{V}' = [A_{k,j}, \Sigma]$. Then it is easy to see that $\mathbf{V}' \in \mathcal{L}(\mathbf{A}_{k-1,j}, \mathbf{A}_{k,j})$ and $\mathbf{V}'$ forms a sublattice $\mathcal{L}'$ of $\mathcal{L}(\mathbf{A}_{k-1,j}, \mathbf{A}_{k,j})$.

We define the mapping α from $\mathcal{L}(\mathbf{A}_{k-1,i}, \mathbf{A}_{k,i})$ to $\mathcal{L}'$ by

$$[A_{k,i}, \Sigma] \mapsto [A_{k,j}, \Sigma].$$

If $[A_{k,i}, \Sigma_1] = [A_{k,i}, \Sigma_2]$, then $A_{k,i} \cup \Sigma_1 \vdash \Sigma_2$. Since $A_{k,j} \vdash A_{k,i}$, we have $A_{k,j} \cup \Sigma_1 \vdash \Sigma_2$ and so $[A_{k,j}, \Sigma_1] \subseteq [A_{k,j}, \Sigma_2]$. Similarly, we may show that $[A_{k,j}, \Sigma_2] \subseteq [A_{k,j}, \Sigma_1]$. Hence $[A_{k,j}, \Sigma_1] = [A_{k,j}, \Sigma_2]$ and the definition of α is reasonable. If $[A_{k,j}, \Sigma_1] = [A_{k,j}, \Sigma_2]$, then $A_{k,j} \cup \Sigma_1 \vdash \Sigma_2$ and $A_{k,j} \vdash A_{k,i}$. For any identity $u \approx v \in \Sigma_2$, if $m_u(x) \neq m_v(x)$ for some $x \in c(u) = c(v)$, then $\min\{m_u(x), m_v(x)\} \geq m(u \approx v) \geq i$ and so Σ_2 can be derived only by identities set $A_{k,i} \cup \Sigma_1$. Hence $[A_{k,i}, \Sigma_1] \subseteq [A_{k,i}, \Sigma_2]$. Similarly, we may show that $[A_{k,i}, \Sigma_2] \subseteq [A_{k,i}, \Sigma_1]$. Hence α is injective. It is trivial that α is surjective.

Clearly, $[A_{k,i}, \Sigma_1] \subseteq [A_{k,i}, \Sigma_2]$ if and only if $[A_{k,j}, \Sigma_1] \subseteq [A_{k,j}, \Sigma_2]$. Hence α is an embedding of the lattice $\mathcal{L}(\mathbf{A}_{k-1,i}, \mathbf{A}_{k,i})$ into the lattice $\mathcal{L}(\mathbf{A}_{k-1,j}, \mathbf{A}_{k,j})$. $\square$

Corollary 3.4 *For $k \geq 2$ and $0 \leq j < i \leq k-1$, the lattice $\mathcal{L}(\mathbf{A}_{k,i,n}, \mathbf{A}_{k,i,n+1})$ can be embedded into the lattice $\mathcal{L}(\mathbf{A}_{k,j,n}, \mathbf{A}_{k,j,n+1})$.*

Proof The conclusion follows from Corollary 3.2 and Theorem 3.3.

Theorem 3.5 *For $k \geq 2$ and $0 \leq m \leq k-1$, the lattice $\mathcal{L}(\mathbf{A}_{k-1,m}, \mathbf{A}_{k,m})$ can be embedded into the lattice $\mathcal{L}(\mathbf{A}_{k,m}, \mathbf{A}_{k+1,m})$.*

Proof By Lemma 3.1 any variety $\mathbf{V} \in \mathcal{L}(\mathbf{A}_{k-1,m}, \mathbf{A}_{k,m})$ can be defined by $[A_{k,m}, \Sigma]$, where any identity $u \approx v \in \Sigma$ is of the form either σ_1 or σ_2. Then $c(u) = c(v)$, say $c(u) = c(v) = \{x_1, \dots, x_n\}$. Let $u' = ux_1 \cdots x_n$ and $v' = vx_1 \cdots x_n$ and $u' \approx v' \in \Sigma'$. Let $\mathbf{V}' = [A_{k+1,m}, \Sigma']$. It is easy to see that $\mathbf{V}' \in \mathcal{L}(\mathbf{A}_{k,m}, \mathbf{A}_{k+1,m})$ and $\mathbf{V}'$ forms a sublattice $\mathcal{L}'$ of $\mathcal{L}(\mathbf{A}_{k,m}, \mathbf{A}_{k+1,m})$.

We define the mapping β from $\mathcal{L}(\mathbf{A}_{k-1,m}, \mathbf{A}_{k,m})$ to $\mathcal{L}'$ by

$$[A_{k,m}, \Sigma] \mapsto [A_{k+1,m}, \Sigma'].$$

By an argument similar to that of Theorem 3.3, we may show that β is an embedding of the lattice $\mathcal{L}(\mathbf{A}_{k-1,m}, \mathbf{A}_{k,m})$ into the lattice $\mathcal{L}(\mathbf{A}_{k,m}, \mathbf{A}_{k+1,m})$. $\square$

Now we have the required results for a description of the structure of $\mathcal{L}(\mathbf{A}_k)$.

Theorem 3.6 *The structure of lattice $\mathcal{L}(\mathbf{A}_k)$ is as shown in the following Figure 1.*

Proof This is a consequence of Theorems 2.4, 2.5, 3.3 and 3.5 and Corollaries 2.7 and 3.4. $\square$

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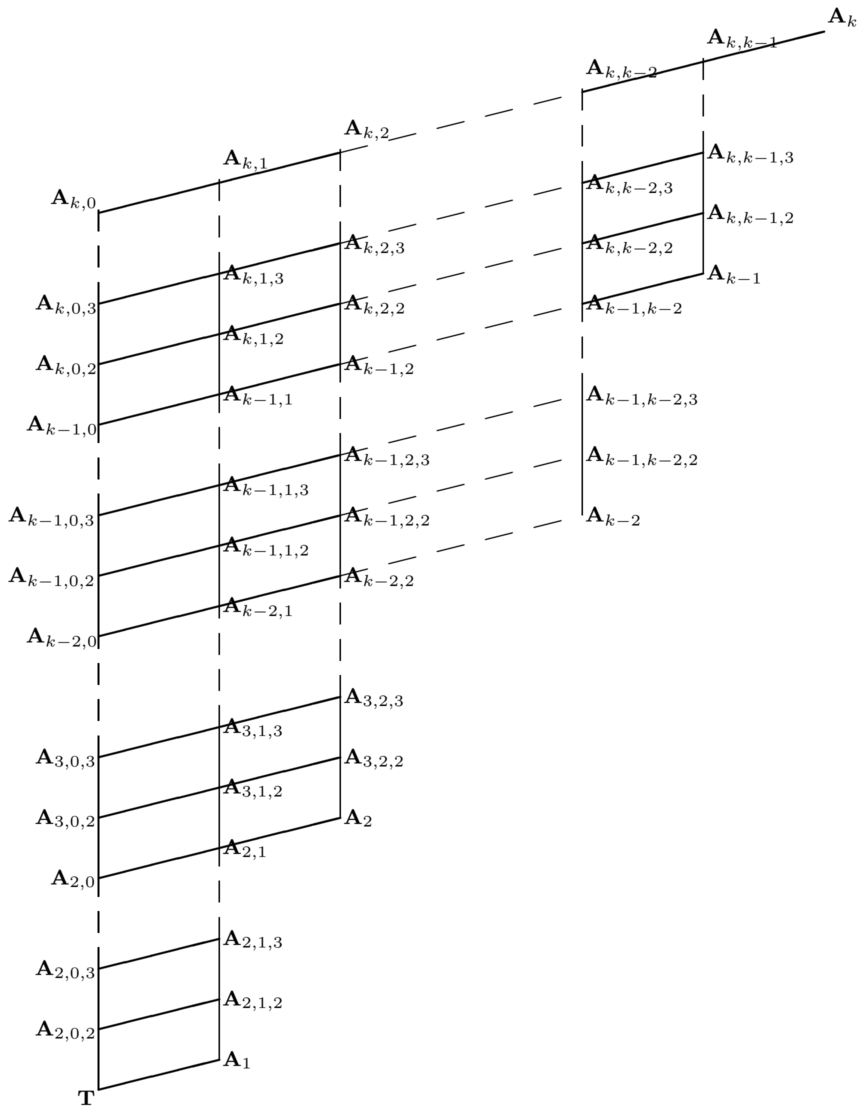

Figure 1 The structure of lattice $\mathcal{L}(\mathbf{A}_k)$

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