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Complete Convergence for Weighted Sums of Arrays of Rowwise Negatively Associated Random Variables

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Abstract In this paper we obtain theorems of complete convergence for weighted sums of arrays of rowwise negatively associated (NA) random variables. These results improve and extend the corresponding results obtained by Sung (2007), Wang et al. (1998) and Li et al. (1995) in independent sequence case.

Keywords complete convergence; negatively associated random variable; weighted sums; slowly varying function.

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1. Introduction

The concept of complete convergence was introduced by Hsu and Robbins in [14] as follows: A sequence of random variables $\{X_n, n \geq 1\}$ is said to converge completely to a constant C if $\sum_{n=1}^{\infty} P(|X_n - C| > \varepsilon) < \infty$ for all $\varepsilon > 0$. From then on, many authors devote their study to complete convergence [7, 8, 10–13].

Recently, Sung [10] proved the following result:

Theorem A *Let $\{X_n, n \geq 1\}$ be a sequence of zero-mean independent random variables which is stochastically dominated by a random variable X, i.e., $P(|X_n| > x) \leq CP(|X| > x)$ for all $x > 0$ and all $n \geq 1$, where C is a positive constant. Assume that $E|X|^{\gamma} < \infty$, where $\gamma = p(t + \beta + 1) > 0$ and $p > 0$. Let $\{a_{ni}, i \geq 1, n \geq 1\}$ be an array of real numbers satisfying*

$$|a_{ni}| = O(1), \ \sum_{i=1}^{\infty} |a_{ni}|^{\alpha} = O(n^{\beta}) \ \text{ for some } \ \alpha < \gamma. \tag{1}$$

Assume that $\sum_{i=1}^{\infty} a_{ni} X_i$ is finite a.s.,

(i) If $1 \leq \gamma < 2$, then

$$\sum_{n=1}^{\infty} n^t P\Big(n^{-1/p} |\sum_{i=1}^{\infty} a_{ni} X_i| > \varepsilon\Big) < \infty \ \text{ for all } \varepsilon > 0. \tag{2}$$

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(ii) If $\gamma \geq 2$, and

$$\sum_{i=1}^{\infty} a_{ni}^2 = O(n^q) \quad \text{for some} \quad q < 2/p, \tag{3}$$

then (2) holds.

Let Z be the set of integers and let

$$N(n, m+1) = \sharp\{k \in Z : |a_{nk}| \geq (m+1)^{-1/p}\}, \quad p \geq 2, \; n \geq 1, \; m \geq 1.$$

Wang et al. [13] proved the following result:

Theorem B *Let $r > 1$, $p > 2$. Let $\{X, X_i, i \in Z\}$ be a sequence of i.i.d. random variables and let $\{a_{ni}, i \in Z\}$ for each $n \geq 1$ be a constant sequence with*

$$N(n, m+1) \approx m^{q(r-1)/p}, \quad n \geq 1, \; m \geq 1, \; 2 \leq q < p, \tag{4}$$

$$EX = 0, \quad \text{when} \quad 1 \leq q(r-1), \tag{5}$$

$$\sum_{i \in Z} a_{ni}^2 = O(n^\delta) \quad \text{as} \quad n \to \infty, \; \text{when } 2 \leq q(r-1), \quad \text{where } 0 < \delta < 2/p. \tag{6}$$

Then the following statements are equivalent:

(i) $E|X|^{p(r-1)} < \infty;$ (7)

(ii) $\displaystyle\sum_{i \in Z} 2^{i(r-1)} \max_{2^{i-1} \leq n < 2^i} P\Big(n^{1/p} \Big| \sum_{k \in N} a_{nk} X_k \Big| > \varepsilon\Big) < \infty, \quad \forall \varepsilon > 0.$ (8)

When $p = 2$, taking $q = 2$, and taking

$$\sum_{i \in Z} |a_{ni}|^{2(r-1)} = O(1) \quad \text{as} \quad n \to \infty, \tag{9}$$

and

$$E|X|^{2(r-1)} \log(1 + |X|) < \infty, \tag{10}$$

instead of (6) and (7), respectively, then the above results still hold.

A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets A and B of $\{1, 2, \ldots, n\}$,

$$\mathrm{Cov}(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0.$$

Whenever f_1 and f_2 are coordinatewise increasing and such that the covariance exists. An infinite family of random variables $\{X_i, i \geq 1\}$ is NA if for every positive integer $n \geq 2$, $\{X_i, 1 \leq i \leq n\}$ is NA. An array $\{X_{ni}, i \geq 1, n \geq 1\}$ is rowwise NA if for every positive integer n, the sequence of random variables $\{X_{ni}, i \geq 1\}$ is NA. This definition was introduced by Alam and Saxena [1] and carefully studied by Joag-Dev and Proschan [2] and Block et al. [3]. NA sequences have many good properties and extensive applications in multivariate statistical analysis and reliability theory, and the notion of NA random variables has received more and more attention in recent years. We refer to Joag-Dev and Proschan [2] for fundamental properties, Matula [4] for the three series theorem, Su et al. [5] for a moment inequality, a weak invariance principle and an example to show that there exists infinite family of non-degenerate non-independent strictly stationary NA random variables, Shao [6] for the Rosenthal type maximal inequality and the Kolmogorov exponential inequality, Su and Qin [7] and Qiu and Gan [8] for complete convergence, Qiu and Yang [9] for strong laws of large numbers.

The main purpose of this paper is to extend and improve Theorem A and the sufficient part of Theorem B to arrays of rowwise NA random variables $\{X_{ni}, i \geq 1, n \geq 1\}$ which are stochastically dominated by a random variable X. That is,

$$P(|X_{ni}| > x) \leq CP(|X| > x) \text{ for all } x > 0 \text{ and for all } i \geq 1 \text{ and } n \geq 1,$$

where C is a positive constant.

Throughout this paper, we assume that $\sum_{i=1}^{\infty} a_{ni}X_{ni}$ is finite a.s., C always stands for a positive constant which may differ from one place to another.

2. Preliminaries

In order to prove our main result, we need the following lemmas.

Lemma 1 ([6]) *Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables with $EX_n = 0$ and $E|X_n|^2 < \infty,\ n \geq 1$. Let $S_k = \sum_{i=1}^{k} X_i$, $B_n = \sum_{i=1}^{n} EX_i^2$. Then for all $x > 0, b > 0$*

$$P(\max_{1\leq k\leq n} |S_k| \geq x) \leq 2P(\max_{1\leq k\leq n} |X_k| \geq b) + 4\exp(-\frac{x^2}{8B_n}) + 4(\frac{B_n}{4(xb + B_n)})^{x/(12b)}$$

Lemma 2 ([6]) *Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables with $EX_n = 0$ and $E|X_n|^p < \infty,\ n \geq 1$, where $1 \leq p \leq 2$. Then*

$$E|\sum_{i=1}^{n} X_i|^p \leq 2^{3-p}\sum_{i=1}^{n} E|X_i|^P, \quad \forall n \geq 2.$$

Lemma 3 *Let $\{X_{ni}, i \geq 1, n \geq 1\}$ be an array of random variables which is stochastically dominated by a random variable X. Then for any $q > 0$ and $x > 0$*

(i) $E|X_{ni}|^q I(|X_{ni}| \leq x) \leq C\,[E|X|^q I(|X| \leq x) + x^q P(|X| > x)]$,

(ii) $E|X_{ni}|^q I(|X_{ni}| > x) \leq CE|X|^q I(|X| > x)$.

Lemma 4 *Let $h(x) > 0$ be a slowly varying function as $x \to +\infty$ and X be a random variable with $E|X|^\gamma h(|X|^p) < \infty$ for some $\gamma > 0$ and some $p > 0$. Then*

(i) $\sum_{i=0}^{\infty} 2^{-i\delta/p}h(2^i)E|X|^{\gamma+\delta}I(|X| \leq 2^{(i+1)/p}) \leq C + CE|X|^\gamma h(|X|^p)$ *for any* $\delta > 0$,

(ii) $\sum_{i=0}^{\infty} 2^{i\delta/p}h(2^i)E|X|^{\gamma-\delta}I(|X| > 2^{i/p}) \leq C + CE|X|^\gamma h(|X|^p)$ *for some* $\delta > 0$ *such that* $\gamma - \delta > 0$,

(iii) $\sum_{i=0}^{\infty} 2^{i\gamma/p}h(2^i)P(|X| > 2^{i/p}) \leq C + CE|X|^\gamma h(|X|^p)$.

Proof First of all, we mention that the proofs of (ii) and (iii) are similar to that of (i), so we only prove (i). By the property of slowly varying function [11], we have

$$\sum_{i=0}^{\infty} 2^{-i\delta/p}h(2^i)E|X|^{\gamma+\delta}I(|X| \leq 2^{(i+1)/p})$$
$$\leq \sum_{i=0}^{\infty} 2^{-i\delta/p}h(2^i) + \sum_{i=0}^{\infty} 2^{-i\delta/p}h(2^i)\sum_{j=0}^{i} E|X|^{\gamma+\delta}I(2^{j/p} < |X| \leq 2^{(j+1)/p})$$

$$\leq C+\sum_{j=0}^{\infty}E|X|^{\gamma+\delta}I(2^{j/p}<|X|\leq 2^{(j+1)/p})\sum_{i=j}^{\infty}2^{-i\delta/p}h(2^i)$$
$$\leq C+C\sum_{j=0}^{\infty}h(2^j)E|X|^{\gamma}I(2^{j/p}<|X|\leq 2^{(j+1)/p})$$
$$\leq C+CE|X|^{\gamma}h(|X|^p).$$

Lemma 5 *Let $h(x)>0$ be a slowly varying function as $x\to+\infty$ and $\{X_{ni}, i\geq 1, n\geq 1\}$ be an array of random variables which is stochastically dominated by a random variable X satisfying $E|X|^{\gamma}h(|X|^p)<\infty$, where $\gamma=p(t+\beta+1)>0$ and $p>0$. Let $\{a_{ni}, i\geq 1, n\geq 1\}$ be an array of real numbers satisfying (1). Then we have*

(i) $\sum_{j=0}^{\infty}2^{j(t+1)}h(2^j)\max_{2^j\leq n<2^{j+1}}\sum_{i=1}^{\infty}E|n^{-1/p}a_{ni}X_{ni}I(|X_{ni}|\leq n^{1/p})|^{\gamma+\delta}\leq C+CE|X|^{\gamma}h(|X|^p)$ *for any $\delta>0$.*

(ii) $\sum_{j=0}^{\infty}2^{j(t+1)}h(2^j)\max_{2^j\leq n<2^{j+1}}\sum_{i=1}^{\infty}E|n^{-\frac{1}{p}}a_{ni}X_{ni}I(|X_{ni}|>n^{\frac{1}{p}})|^{\gamma-\delta}\leq C+CE|X|^{\gamma}h(|X|^p)$ *for some $\delta>0$ such that $\gamma-\delta\geq\alpha$ and $\gamma-\delta>0$.*

Proof First we note that (1) implies

$$\sum_{i=1}^{\infty}|a_{ni}|^{\alpha+r}=O(n^{\beta})\ \text{ for any }\ r\geq 0.$$

Thus by C_r-inequality, Lemmas 3 and 4, we can get

$$\sum_{j=0}^{\infty}2^{j(t+1)}h(2^j)\max_{2^j\leq n<2^{j+1}}\sum_{i=1}^{\infty}E|n^{-1/p}a_{ni}X_{ni}I(|X_{ni}|\leq n^{1/p})|^{\gamma+\delta}$$
$$\leq C\sum_{j=0}^{\infty}2^{-j(\beta+\frac{\delta}{p})}h(2^j)\max_{2^j\leq n<2^{j+1}}\sum_{i=1}^{\infty}|a_{ni}|^{\gamma+\delta}\left[E|X|^{\gamma+\delta}I(|X|\leq n^{\frac{1}{p}})+n^{\frac{\gamma+\delta}{p}}P(|X|>n^{\frac{1}{p}})\right]$$
$$\leq C\sum_{j=0}^{\infty}2^{-\delta j/p}h(2^j)E|X|^{\gamma+\delta}I(|X|\leq 2^{(j+1)/p})+C\sum_{j=0}^{\infty}2^{j\gamma/p}h(2^j)P(|X|>2^{j/p})$$
$$\leq C+CE|X|^{\gamma}h(|X|^p).$$

Therefore, (i) is proved. By Lemmas 3 and 4, we also get

$$\sum_{j=0}^{\infty}2^{j(t+1)}h(2^j)\max_{2^j\leq n<2^{j+1}}\sum_{i=1}^{\infty}E|n^{-1/p}a_{ni}X_{ni}I(|X_{ni}|>n^{1/p})|^{\gamma-\delta}$$
$$\leq C\sum_{j=0}^{\infty}2^{j(t+1)}h(2^j)\max_{2^j\leq n<2^{j+1}}n^{-(\gamma-\delta)/p}\sum_{i=1}^{\infty}|a_{ni}|^{\gamma-\delta}E|X|^{\gamma-\delta}I(|X|>n^{1/p})$$
$$\leq C\sum_{j=0}^{\infty}2^{\delta j/p}h(2^j)E|X|^{\gamma-\delta}I(|X|>2^{j/p})$$
$$\leq C+CE|X|^{\gamma}h(|X|^p).$$

Thus (ii) is proved. □

3. Main results and proofs

Theorem 1 *Let $h(x)>0$ be a slowly varying function as $x\to+\infty$ and $\{X_{ni}, i\geq 1, n\geq 1\}$*

be an array of zero-mean rowwise NA random variables which is stochastically dominated by a random variable X *satisfying* $E|X|^\gamma h(|X|^p) < \infty$*, where* $\gamma = p(t+\beta+1) > 0$ *and* $p > 0$*. Let* $\{a_{ni}, i \geq 1, n \geq 1\}$ *be an array of real numbers satisfying (1) for some* $\alpha < \gamma$*.*

(i) If $\gamma \geq 2$*, and* $\{a_{ni}, i \geq 1, n \geq 1\}$ *satisfies (3). Moreover, we assume that* $E|X|^2 < \infty$ *when* $\gamma = 2$*. Then*

$$\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} P\Big(n^{-1/p}|\sum_{i=1}^{\infty} a_{ni}X_{ni}| > \varepsilon\Big) < \infty \text{ for all } \varepsilon > 0, \tag{11}$$

and

$$\sum_{n=1}^{\infty} n^t h(n) P\Big(n^{-1/p}|\sum_{i=1}^{\infty} a_{ni}X_{ni}| > \varepsilon\Big) < \infty \text{ for all } \varepsilon > 0. \tag{12}$$

(ii) If $1 < \gamma < 2$*, then (11) and (12) hold.*

(iii) If $\gamma = 1$*, and* $E|X| < \infty$*, then (11) and (12) hold.*

Proof If (11) holds, then

$$\sum_{n=1}^{\infty} n^t h(n) P\Big(n^{-1/p}|\sum_{i=1}^{\infty} a_{ni}X_{ni}| > \varepsilon\Big) = \sum_{j=0}^{\infty} \sum_{2^j \leq n < 2^{j+1}} n^t h(n) P\Big(n^{-1/p}|\sum_{i=1}^{\infty} a_{ni}X_{ni}| > \varepsilon\Big)$$
$$\leq C + C\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} P\Big(n^{-1/p}|\sum_{i=1}^{\infty} a_{ni}X_{ni}| > \varepsilon\Big) < \infty,$$

(12) holds. So, it suffices to show that (11) holds.

If $t < -1$, by the property of slowly varying function [11], (11) holds and so we assume that $t \geq -1$. Define $U_{ni} = X_{ni}I(|X_{ni}| \leq n^{1/p}) + n^{1/p}I(X_{ni} > n^{1/p}) - n^{1/p}I(X_{ni} < -n^{1/p}), V_{ni} = X_{ni}I(|X_{ni}| > n^{1/p}) - n^{1/p}I(X_{ni} > n^{1/p}) + n^{1/p}I(X_{ni} < -n^{1/p})$, for any $i \geq 1, n \geq 1$. Without loss of generality, we may assume that $a_{ni} > 0$ for $i \geq 1, n \geq 1$. Then by Joag-Dev and Proschan [2], $\{a_{ni}n^{-1/p}U_{ni}, i \geq 1, n \geq 1\}$ and $\{a_{ni}n^{-1/p}V_{ni}, i \geq 1, n \geq 1\}$ are arrays of rowwise NA random variables. Since we assume that $\sum_{i=1}^{\infty} a_{ni}X_{ni}$ is finite a.s., there exists positive integer k_n such that $P(n^{-1/p}|\sum_{i=k_n+1}^{\infty} a_{ni}X_{ni}| > \varepsilon/2) < 1/n^{(t+2)}, \forall\, n \geq 1$. Thus in order to prove (11), we only need to show that

$$\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} P(n^{-1/p}|\sum_{i=1}^{k_n} a_{ni}X_{ni}| > \varepsilon/2) < \infty. \tag{13}$$

Since $EX_{ni} = 0$ for any $i \geq 1, n \geq 1$, we can get

$$\begin{aligned}
&\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} P(n^{-1/p}|\sum_{i=1}^{k_n} a_{ni}X_{ni}| > \varepsilon/2) \\
&\leq \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} P(n^{-1/p}|\sum_{i=1}^{k_n} a_{ni}(U_{ni} - EU_{ni})| > \varepsilon/4) + \\
&\quad \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} P(n^{-1/p}|\sum_{i=1}^{k_n} a_{ni}(V_{ni} - EV_{ni})| > \varepsilon/4) \\
&\overset{\text{Def}}{=} A + B.
\end{aligned} \tag{14}$$

(i) When $\gamma \geq 2$, by the condition (i) of Theorem 1, we have $E|X|^2 < \infty$. For B, we get by C_r-inequality and (3) that

$$D_{n1} = \sum_{i=1}^{k_n} \mathrm{Var}(n^{-1/p} a_{ni} V_{ni}) \leq \sum_{i=1}^{\infty} a_{ni}^2 n^{-2/p} \Big[EX_{ni}^2 I(|X_{ni}| > n^{1/p}) + n^{2/p} P(|X_{ni}| > n^{1/p}) \Big]$$
$$\leq Cn^{(q-2/p)} + Cn^q P(|X| > n^{1/p}) \leq Cn^{(q-2/p)}.$$

Since $q < \frac{2}{p}$, we can take $b > 0$ such that $(\frac{2}{p} - q)\frac{\varepsilon}{48b} > t+1$. So we have

$$\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} \exp\Big(- \frac{(\varepsilon/4)^2}{8D_{n1}} \Big) \leq \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \exp\Big(- C2^{j(2/p-q)} \Big) < \infty, \quad (15)$$

and

$$\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} \Big(\frac{D_{n1}}{4(b\varepsilon/4 + D_{n1})} \Big)^{\varepsilon/(48b)} \leq C \sum_{j=0}^{\infty} 2^{j[(q-\frac{2}{p})\frac{\varepsilon}{48b} + t + 1]} h(2^j) < \infty. \quad (16)$$

Take $\delta > 0$ such that $\gamma - \delta \geq \alpha$ and $\gamma - \delta \geq 1$. By Markov's inequality, C_r-inequality, Jensen's inequality, (1), Lemmas 4 and 5, we get

$$\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} P(\max_{1 \leq i \leq k_n} |n^{-1/p} a_{ni}(V_{ni} - EV_{ni})| \geq b)$$
$$\leq \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} \sum_{i=1}^{\infty} P(|n^{-1/p} a_{ni}(V_{ni} - EV_{ni})| \geq b)$$
$$\leq C \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} \sum_{i=1}^{\infty} \Big[E|n^{-\frac{1}{p}} a_{ni} X_{ni} I(|X_{ni}| > n^{\frac{1}{p}})|^{\gamma-\delta} + a_{ni}^{\gamma-\delta} P(|X_{ni}| > n^{\frac{1}{p}}) \Big]$$
$$\leq C + C \sum_{j=0}^{\infty} 2^{j\gamma/p} h(2^j) P(|X| > 2^{j/p}) < \infty. \quad (17)$$

By (15)–(17) and Lemma 1 we have $B < \infty$. For A, we have

$$D_{n2} = \sum_{i=1}^{k_n} Var(n^{-1/p} a_{ni} U_{ni}) \leq \sum_{i=1}^{\infty} a_{ni}^2 n^{-2/p} \Big[EX_{ni}^2 I(|X_{ni}| \leq n^{1/p}) + n^{2/p} P(|X_{ni}| > n^{1/p}) \Big]$$
$$\leq Cn^{(q-2/p)} + Cn^q P(|X| > n^{1/p}) \leq Cn^{(q-2/p)}.$$

Taking $b > 0$ such that $(\frac{2}{p} - q)\frac{\varepsilon}{48b} > t+1$, similar to the proof of (15) and (16), we have

$$\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} \exp\Big(- \frac{(\varepsilon/4)^2}{8D_{n2}} \Big) < \infty, \quad (18)$$

and

$$\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} \Big(\frac{D_{n2}}{4(b\varepsilon/4 + D_{n2})} \Big)^{\varepsilon/(48b)} < \infty. \quad (19)$$

Taking $\delta > 0$, by Markov's inequality, C_r-inequality, Jensen's inequality, (1), Lemmas 4 and 5, we get

$$\sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} P(\max_{1 \leq i \leq k_n} |n^{-1/p} a_{ni}(U_{ni} - EU_{ni})| \geq b)$$

$$\leq \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} \sum_{i=1}^{\infty} P(|n^{-1/p} a_{ni}(U_{ni} - EU_{ni})| \geq b)$$

$$\leq C \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} \sum_{i=1}^{\infty} \Big[E|n^{-\frac{1}{p}} a_{ni} X_{ni} I(|X_{ni}| \leq n^{\frac{1}{p}})|^{\gamma+\delta} + a_{ni}^{\gamma+\delta} P(|X_{ni}| > n^{\frac{1}{p}}) \Big]$$

$$\leq C + C \sum_{j=0}^{\infty} 2^{j\gamma/p} h(2^j) P(|X| > 2^{j/p}) < \infty. \tag{20}$$

By (18)–(20) and Lemma 1 we have $A < \infty$. So (11) holds.

(ii) Taking $\delta > 0$ such that $\gamma + \delta \leq 2$, we get by Markov's inequality, C_r-inequality, Jensen's inequality and Lemma 2–Lemma 5 that

$$A = (4/\varepsilon)^{\gamma+\delta} \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} E|\sum_{i=1}^{k_n} n^{-1/p} a_{ni}(U_{ni} - EU_{ni})|^{\gamma+\delta}$$

$$\leq C \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} \sum_{i=1}^{\infty} \Big[E|n^{-\frac{1}{p}} a_{ni} X_{ni} I(|X_{ni}| \leq n^{\frac{1}{p}})|^{\gamma+\delta} + a_{ni}^{\gamma+\delta} P(|X_{ni}| > n^{\frac{1}{p}}) \Big]$$

$$< \infty. \tag{21}$$

We now take $\delta > 0$ such that $\gamma - \delta \geq \alpha$ and $\gamma - \delta \geq 1$. Then we also get that

$$B = (4/\varepsilon)^{\gamma-\delta} \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} E|\sum_{i=1}^{k_n} n^{-1/p} a_{ni}(V_{ni} - EV_{ni})|^{\gamma-\delta}$$

$$\leq C \sum_{j=0}^{\infty} 2^{j(t+1)} h(2^j) \max_{2^j \leq n < 2^{j+1}} \sum_{i=1}^{\infty} \Big[E|n^{-\frac{1}{p}} a_{ni} X_{ni} I(|X_{ni}| > n^{\frac{1}{p}})|^{\gamma-\delta} + a_{ni}^{\gamma-\delta} P(|X_{ni}| > n^{\frac{1}{p}}) \Big]$$

$$< \infty. \tag{22}$$

Thus (11) holds by (14), (21) and (22) when $1 < \gamma < 2$.

(iii) Next, we consider the case $\gamma = 1$. Similarly to the proof of (21), we have

$$A < \infty. \tag{23}$$

Now, we prove that $B < \infty$. By Lemma 3, there exists a positive constant $C_1 > 0$ such that

$$E|X_{ni}|I(|X_{ni}| > x) \leq C_1 E|X|I(|X| > x) \ \text{ for all } \ x > 0 \text{ and all } n \geq 1. \tag{24}$$

Condition (1) implies that there exists a positive constant $C_2 > 0$ such that

$$\sum_{i=1}^{\infty} |a_{ni}| \leq C_2 n^{\beta} \ \text{ for all } \ n \geq 1. \tag{25}$$

Since $E|X| = E|X|^{\gamma} < \infty$, there exists a positive constant $M > 0$, when $n^{1/p} \geq M$, we have that

$$E|X|I(|X| > n^{1/p}) \leq \frac{\varepsilon}{16 C_1 C_2}. \tag{26}$$

Since $p(t + \beta + 1) = 1$ and $t \geq -1$, we can get that $\beta - 1/p = -(t+1) \leq 0$. For $n^{1/p} \geq M$, by (24)–(26) and Lemma 3, we have

$$\sum_{i=1}^{k_n} n^{-1/p} |a_{ni}| E|X_{ni}| I(|X_{ni}| > n^{1/p}) \leq C_1 n^{-1/p} E|X|I(|X| > n^{1/p}) \sum_{i=1}^{\infty} |a_{ni}|$$

$$\le C_1C_2n^{-1/p}n^{\beta}E|X|I(|X|>n^{1/p})$$
$$\le n^{\beta-1/p}\frac{\varepsilon}{16}\le\frac{\varepsilon}{16}. \tag{27}$$

By (27), we have

$$\sum_{i=1}^{k_n}|a_{ni}|P(|X_{ni}|>n^{1/p})\le\frac{\varepsilon}{16}. \tag{28}$$

Taking $\delta>0$ such that $\gamma-\delta=1-\delta\ge\alpha$ and $\gamma-\delta>0$. Let $2^{j_0/p}\ge M$. By Markov's inequality, C_r-inequality, Lemmas 4 and 5, (27) and (28), we have

$$\begin{aligned}B\le&C+\sum_{j=j_0}^{\infty}2^{j(t+1)}h(2^j)\cdot\\&\max_{2^j\le n<2^{j+1}}P\Big(\sum_{i=1}^{k_n}n^{-1/p}a_{ni}\Big[|X_{ni}|I(|X_{ni}|>n^{1/p})+n^{1/p}I(|X_{ni}|>n^{1/p})\Big]>\frac{\varepsilon}{8}\Big)\\\le&C+C\sum_{j=j_0}^{\infty}2^{j(t+1)}h(2^j)\cdot\\&\max_{2^j\le n<2^{j+1}}E\Big|\sum_{i=1}^{k_n}n^{-1/p}a_{ni}[|X_{ni}|I(|X_{ni}|>n^{1/p})+n^{1/p}I(|X_{ni}|>n^{1/p})]\Big|^{1-\delta}\\\le&C+C\sum_{j=j_0}^{\infty}2^{j(t+1)}h(2^j)\cdot\\&\max_{2^j\le n<2^{j+1}}\sum_{i=1}^{\infty}\Big[E|n^{-1/p}a_{ni}X_{ni}I(|X_{ni}|>n^{1/p})|^{1-\delta}+a_{ni}^{1-\delta}P(|X_{ni}|>n^{1/p})\Big]\\\le&C+CE|X|h(|X|^p)<\infty.\end{aligned} \tag{29}$$

Thus (11) holds by (14), (23) and (29) when $\gamma=1$.

Remark 1 (i) If there exists a positive constant $M>0$ such that $h(x)\ge M$ for sufficiently large x, then we can get $E|X|^{p(t+\beta+1)}<\infty$ from $E|X|^{p(t+\beta+1)}h(|X|^p)<\infty$.

(ii) Put $h(x)\equiv1$, $X_{ni}=X_i$, $\forall i\ge1$, $n\ge1$. Let $\{X_i,i\ge1\}$ be a sequence of independent random variables. Then Theorem A is obtained from Theorem 1, since independent random variables are a special case of NA random variables.

(iii) Let $\beta=0$, $t=r-2$, $h(x)\equiv1$. If condition (4) holds, then (1) holds according to (2.11) of [13], where $\alpha=\tilde q(r-1)$, $\gamma=p(r-1)$, $2\le q<\tilde q<p$. When $0<q(r-1)<2$, by (2.11) of [13], we have $\sum_{i\in Z}a_{ni}^2=O(1)$. Therefore, if (4) and (6) hold, we have $\sum_{i\in Z}a_{ni}^2=O(n^{\delta})$, where $0<\delta<2/p$. Thus Theorem 1 extends and improves the sufficient part of Theorem B in i.i.d. case to NA random variables when $p>2$.

If condition (1) on the weights is replaced by a weaker condition

$$|a_{ni}|=O(1),\ \sum_{i=1}^{\infty}|a_{ni}|^{p(t+\beta+1)}=O(n^{\beta}), \tag{30}$$

we have the following theorem.

Theorem 2 *Let $\{X_{ni},i\ge1,\,n\ge1\}$ be an array of zero-mean rowwise NA random variables* *which is stochastically dominated by a random variable X satisfying $E|X|^\gamma \log(1+|X|) < \infty$, where $\gamma = p(t+\beta+1) > 0$ and $p > 0$. Let $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of real numbers satisfying (30).*

(i) If $\gamma \ge 2$, and $\{a_{ni}, i \ge 1, n \ge 1\}$ satisfies (3), then

$$\sum_{j=0}^{\infty} 2^{j(t+1)} \max_{2^j \le n < 2^{j+1}} P\Big(n^{-1/p}|\sum_{i=1}^{\infty} a_{ni}X_{ni}| > \varepsilon\Big) < \infty \quad \text{for all} \quad \varepsilon > 0, \tag{31}$$

moreover

$$\sum_{n=1}^{\infty} n^t P\Big(n^{-1/p}|\sum_{i=1}^{\infty} a_{ni}X_{ni}| > \varepsilon\Big) < \infty \quad \text{for all} \quad \varepsilon > 0. \tag{32}$$

(ii) If $1 \le \gamma < 2$, then (31) and (32) hold.

Proof The proof is similar to that of Theorem 1, so it is omitted here. □

Remark 2 (i) Obviously, Theorem 2 extends and improves the sufficient part of Theorem B in i.i.d. case to NA random variables when $p = 2$.

(ii) If $0 < \gamma = p(t+\beta+1) < 1$. Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of arbitrary random variables, under conditions of Theorem 1 or Theorem 2, the results still hold according to the proof of [10].

Corollary 1 *Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of zero-mean rowwise NA random variables which is stochastically dominated by a random variable X satisfying $E|X|^p < \infty$ for some $p > 2$. Let $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of real numbers satisfying (3) and*

$$\sum_{i=1}^{\infty} |a_{ni}|^\theta = O(1) \text{ for some } 2 \le \theta < p.$$

Then

$$\sum_{n=1}^{\infty} P\Big(n^{-1/p}|\sum_{i=1}^{\infty} a_{ni}X_{ni}| > \varepsilon\Big) < \infty \text{ for all } \varepsilon > 0.$$

Proof Let $t = 0$ and $\beta = 0$. Clearly, $|a_{ni}| = O(1)$. Thus the result follows from Theorem 1(i). □

Corollary 2 *Let $\{X_{ni}, i \ge 1, n \ge 1\}$ be an array of zero-mean rowwise NA random variables which is stochastically dominated by a random variable X satisfying $E|X|^2 \log|X| < \infty$. Let $\{a_{ni}, i \ge 1, n \ge 1\}$ be an array of real numbers satisfying*

$$\sum_{i=1}^{\infty} |a_{ni}|^2 = O(1).$$

Then

$$\sum_{n=1}^{\infty} P\Big(n^{-1/2}|\sum_{i=1}^{\infty} a_{ni}X_{ni}| > \varepsilon\Big) < \infty \text{ for all } \varepsilon > 0.$$

Proof Let $t = 0$, $\beta = 0$ and $p = 2$. Clearly $|a_{ni}| = O(1)$. Thus the result follows from Theorem 2(i).

Remark 3 When $X_{ni} = X_i$, $\forall n \ge 1$, $i \ge 1$, let $\{X_i, i \ge 1\}$ be a sequence of i.i.d. random variables. Corollarys 1 and 2 were proved by Li et al. [12]. Hence Corollaries 1 and 2 extend the results of Li et al. [12].

From Theorem 1, we can obtain a result on the rate of convergence of moving average processes.

Corollary 3 *Let* $\{X_{ni}, -\infty < i < \infty, -\infty < n < \infty\}$ *be an array of zero-mean rowwise NA random variables which is stochastically dominated by a random variable* X *satisfying* $E|X|^{p(t+2)} < \infty$ *for some* $0 < p < 2$ *and* $p(t+2) > 1$*. Let* $\{a_n, -\infty < n < \infty\}$ *be a sequence of real numbers such that* $\sum_{n=-\infty}^{\infty} |a_n| < \infty$*. Set* $a_{ni} = \sum_{j=i+1}^{i+n} a_j$ *for each* i *and* n*. Then*

$$\sum_{n=1}^{\infty} n^t P\Big(|\sum_{i=-\infty}^{\infty} a_{ni}X_{ni}|/n^{1/p} > \varepsilon\Big) < \infty \quad \text{for all} \quad \varepsilon > 0.$$

Proof The proof is similar to that of Sung [10] and is omitted here.

Remark 4 Corollary 3 extends the Corollary 3 of Sung [10] on independent random variables to arrays of rowwise NA random variables.

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