Journal of Mathematical Research & Exposition
Jan., 2010, Vol. 30, No. 1, pp. 167–172
DOI:10.3770/j.issn:1000-341X.2010.01.016
Http://jmre.dlut.edu.cn

Two Results on Square Closed Lie Ideals of Prime Rings

Shu Liang HUANG[1,*], **Shi Tai FU**[2]
1. *Department of Mathematics, Chuzhou University, Anhui* 239012*, P. R. China;*
2. *School of Mathematics and Computer Science, Nanjing Normal University, Jiangsu* 210097*, P. R. China*

Abstract Let R be a 2-torsion free prime ring, d_1 a nonzero derivation, γ a generalized derivation associated with a nonzero derivation d_2, U a square closed Lie ideal of R. In the present paper, we prove that if $[d_1^2(u),u]\in Z(R)$ or γ acts as a homomorphism (or an antihomomorphism) on U, then $U\subseteq Z(R)$.

Keywords prime ring; Lie ideal; derivation; generalized derivation.

Document code A
MR(2000) Subject Classification 16W25; 16N60
Chinese Library Classification O153.3

1. Introduction

Throughout the present paper R will denote an associative ring with center $Z(R)$. For any $x,y\in R$, the symbol $[x,y]$ stands for the commutator $xy-yx$. Given two subsets A and B of R, $[A,B]$ will denote the additive subgroup of R generated by all elements of the form $[a,b]$ where $a\in A$, $b\in B$. For a nonempty subset S of R, we put $C_R(S)=\{x\in R|[x,s]=0 \text{ for all } s\in S\}$. Recall that R is prime if $aRb=0$ implies $a=0$ or $b=0$, and is semiprime if $aRa=0$ implies $a=0$. A ring R is called 2-torsion free, if whenever $2x=0$, with $x\in R$, then $x=0$. An additive mapping $d:R\longrightarrow R$ is called a derivation if $d(xy)=d(x)y+xd(y)$ holds for all $x,y\in R$. An additive mapping $\delta:R\longrightarrow R$ is called a generalized derivation if there exists a derivation $d:R\longrightarrow R$ such that $\delta(xy)=\delta(x)y+xd(y)$ holds for all $x,y\in R$. Let S be a nonempty subset of R and δ a generalized derivation of R. If $\delta(xy)=\delta(x)\delta(y)$ or $\delta(xy)=\delta(y)\delta(x)$ for all $x,y\in S$, then δ is called a generalized derivation which acts as a homomorphism or antihomomorphism on S respectively. A mapping $F:R\longrightarrow R$ is said to be commuting on a subset S of R if $[F(x),x]=0$ for all $x\in S$, and is said to be centralizing on S if $[F(x),x]\in Z(R)$ for all $x\in S$. An additive subgroup U of R is said to be a Lie ideal of R if $[u,r]\in U$ for all $u\in U$ and $r\in R$, a Lie ideal U is called square closed if $u^2\in U$ for all $u\in U$. There is a particular interest in square closed Lie ideals. Such a distinction already appears in several papers involving usual derivations and Lie ideals. By the definition of Lie ideal, we can see that every ideal I in a ring

Received March 17, 2008; Accepted July 7, 2008
Supported by the Natural Science Research Foundation of Anhui Provincial Education Department (Grant Nos. KJ2008B013; KJ2010B200).
* Corresponding author
E-mail address: shulianghuang@sina.com (S. L. HUANG)

R is a Lie ideal of R. However it is not the case on the contrary. Though a square closed Lie ideal seems close to be an ideal, there exist such Lie ideals which are not ideals. For example, let R be any ring and U the additive subgroup of R generated by the idempotents of R. If e is an idempotent in R, and $x \in R$, then it is easy to see that $u = e + ex - exe$ and $v = e + xe - exe$ are idempotents. Hence, $ex - xe = u - v \in U$, namely, U is a square closed Lie ideal of R.

There has been considerable interest in commuting, centralizing, and related mappings in prime and semiprime rings. It is well known that the theory of commuting and centralizing mapping on prime rings was initiated by E.C. Posner, who asserts that the set $\{[d(x), x] \mid x \in R\}$ does not lie in the center of a noncommutative prime ring R, where d is a nonzero derivation of R. In 1982, Chung and Jiang [1] proved that if d is a nonzero derivation of a 2-torsion free prime ring R such that $d^2(x) \in Z(R)$ for all $x \in R$, then R is commutative. Afterward Vukman [2] extended E.C. Posner's theorem by showing that if d is a nonzero derivation of a 2-torsion free prime ring R such that $[[d(x), x], x] = 0$ for all $x \in R$, then R is commutative. Recently, Wong [3] obtained more information about the structure $\{[d(x), x] \mid x \in R\}$ of a noncommutative prime ring R. It is interesting to investigate the identity $d[d(x), x] = [d^2(x), x] \in Z(R)$ if we combine E.C. Posner's with Chung and Jiang's result. Bell and Kappe [4] proved that if d is a derivation of a prime ring R which acts as a homomorphism on a nonzero right ideal I of R, then $d = 0$. Asma, Rehman and Shakir [5] extended this result to Lie ideals. At the same time, Albaş and Argaç [6] extended H.E.Bell's result to generalized derivations. For more other related results of commutative ring, one can see [7]. Motivated by these observations, we extend the above mentioned results to a more general case.

2. Preliminaries

The following results will be used to prove our theorems.

Lemma 2.1 *If U is a square closed Lie ideal of R, then $2uv \in U$ for all $u, v \in U$.*

Proof For all $u, v \in U$, on the one hand

$$uv + vu = (u + v)^2 - u^2 - v^2 \in U.$$

On the other hand,

$$uv - vu \in U.$$

Adding two expressions, we have $2uv \in U$ for all $u, v \in U$. □

Lemma 2.2 *Let R be a 2-torsion free prime ring and $V = \{u \in U \mid d(u) \in U\}$, where U is a Lie ideal of R. If $U \nsubseteq Z(R)$, then $V \nsubseteq Z(R)$.*

Proof Assume that $V \subseteq Z(R)$. It is clear that $[U, U] \subseteq U$ and $d([U, U]) \subseteq U$, so we have $[U, U] \subseteq V \subseteq Z(R)$. Hence $C_R([U, U]) = R$. From [8, Lemma 2], $C_R(U) = Z(R)$. But by [8, Lemma 3], $C_R([U, U]) = C_R(U)$. And hence $R = Z(R)$, a contradiction. □

Lemma 2.3 *Let R be a 2-torsion free prime ring. If d_1 and d_2 are nonzero derivations of R such* *that* $d_1(u)d_2(u) = 0$ *for all* $u \in U$, *where* U *is a square closed Lie ideal of* R, *then* $U \subseteq Z(R)$.

Proof Set $V = \{u \in U |\ d(u) \in U\}$. Suppose on the contrary that $U \not\subseteq Z(R)$. Then $V \not\subseteq Z(R)$ by Lemma 2.2. The linearization of $d_1(u)d_2(u) = 0$ gives $d_1(u)d_2(v)+d_1(v)d_2(u) = 0$ for all $u, v \in U$. In particular, it is true for all $u, v \in V$. Now replace v by $2vd_2(u)$ in the last equation. By Lemma 2.1 and using that R is 2-torsion free, we have the equation $(d_1(u)d_2(v) + d_1(v)d_2(u))d_2(u) + d_1(u)vd_2^2(u) + vd_1d_2(u)d_2(u) = 0$, which reduces to $d_1(u)vd_2^2(u) + vd_1d_2(u)d_2(u) = 0$ for all $u, v \in V$. Replacing v by $2wv$ in the above equation and using that R is 2-torsion free, we have

$$\begin{aligned}
0 &= d_1(u)wvd_2^2(u) + wvd_1d_2(u)d_2(u)\\
&= [d_1(u), w]vd_2^2(u) + wd_1(u)vd_2^2(u) + (wv)d_1d_2(u)d_2(u)\\
&= [d_1(u), w]vd_2^2(u) + w(d_1(u)vd_2^2(u) + vd_1d_2(u)d_2(u))\\
&= [d_1(u), w]vd_2^2(u).
\end{aligned}$$

In other words, we have $[d_1(u), w]Vd_2^2(u) = 0$ and hence either $[d_1(u), w] = 0$ or $d_2^2(u) = 0$ by [8, Lemma 4].

Now let $V_1 = \{u \in V \mid [d_1(u), w] = 0\}$ and $V_2 = \{u \in V \mid d_2^2(u) = 0\}$. Then V_1, V_2 are both additive subgroups of V and $V_1 \bigcup V_2 = V$. But a group cannot be a union of its two proper subgroups, and hence $V_1 = V$ or $V_2 = V$. On the one hand, if $V_1 = V$, then $d_1(V) \subseteq C_R(V) = Z(R)$. We have $V \subseteq Z(R)$ by [8, Lemma 5], a contradiction. On the other hand, if $V_2 = V$, then $d_2^2(V) = 0$ and hence $V \subseteq Z(R)$ by [10, Theorem 4], again a contradiction.

3. The main results

Now we are in a position to prove our main theorems.

Theorem 3.1 *Let* R *be a 2-torsion free prime ring and* d *a nonzero derivation of* R. *Suppose that* U *is a square closed Lie ideal of* R. *If* $[d^2(u), u] \in Z(R)$ *for all* $u \in U$, *then* $U \subseteq Z(R)$.

Proof Set $V = \{u \in U |\ d(u) \in U\}$. Suppose on the contrary that $U \not\subseteq Z(R)$. Then $V \not\subseteq Z(R)$ by Lemma 2.2. First of all, we claim that if $[d^2(u), u] = 0$ for all $u \in U$, then $U \subseteq Z(R)$. In fact, the linearization of $[d^2(u), u] = 0$ gives

$$[d^2(u), v] + [d^2(v), u] = 0. \tag{1}$$

Replace v by $2vu$ in (1). By Lemma 2.1 and using that R is 2-torsion free, we have the equation $([d^2(u), v] + [d^2(v), u])u + [v, u]d^2(u) + 2[d(v)d(u), u] + 2v[d^2(u), u] = 0$. Using $[d^2(u), u] = 0$ and $[d^2(u), v] + [d^2(v), u] = 0$, the above equation reduces to

$$[v, u]d^2(u) + 2[d(v)d(u), u] = 0. \tag{2}$$

Replacing v by $2uv$ in (2) leads to $u([v, u]d^2(u) + 2[d(v)d(u), u]) + 2[d(u)vd(u), u] = 0$ for all $u, v \in U$. In other words, we obtain the relation

$$[d(u)vd(u), u] = 0. \tag{3}$$

Replacing u by $w+u$ and $w-u$ in (3), respectively, we have the following relations

$$[d(u)vd(u)+d(u)vd(w)+d(w)vd(u),w]+[d(u)vd(w)+d(w)vd(u)+d(w)vd(w),u]=0, \quad (4)$$

and

$$[d(u)vd(u)-d(u)vd(w)-d(w)vd(u),w]+[d(u)vd(w)+d(w)vd(u)-d(w)vd(w),u]=0. \quad (5)$$

Combining (4) with (5), we find that

$$[d(u)vd(u),w]+[d(u)vd(w),u]+[d(w)vd(u),u]=0. \quad (6)$$

Replacing w by $2wu$ in (6) and using (3), we can get the following

$$([d(u)vd(u),w]+[d(u)vd(w),u]+[d(w)vd(u),u])u+[[d(w)[vd(u)],u]+[w,u]d(u)vd(u)=0.$$

Making use of (6), we have

$$[[d(w)[vd(u)],u]+[w,u]d(u)vd(u)=0. \quad (7)$$

In particular, the above relation is true for all $u,v,w\in V$. Replacing v by $2d(u)v$ in (7) and making use of (3), we conclude that $[w,u]d(u)d(u)vd(u)=0$ for all $u,v,w\in V$. Then [8, Lemma 4] gives either $[w,u]d(u)d(u)=0$ or $d(u)=0$. In either case we find that

$$[w,u]d(u)d(u)=0. \quad (8)$$

Replacing w by $2wv$ in (8) and using (8), we get $[w,u]vd(u)d(u)=0$ for all $u,v,w\in V$. Therefore, $[w,u]=0$ or $d(u)d(u)=0$ by [8, Lemma 4].

Now let $V_1=\{u\in V\mid [w,u]=0\}$ and $V_2=\{u\in V\mid d(u)d(u)=0\}$. Then V_1,V_2 are both additive subgroups of V and $V_1\bigcup V_2=V$. But a group cannot be a union of its two proper subgroups, and hence $V_1=V$ or $V_2=V$. On the one hand, if $V_1=V$, then $V\subseteq C_R(V)=Z(R)$, a contradiction. On the other hand, if $V_2=V$, then $d(u)d(u)=0$ and hence $V\subseteq Z(R)$ by Lemma 2.3, again a contradiction.

Now we begin with our main result. By hypothesis we have $[d^2(u),u]\in Z(R)$ for all $u\in U$. We have two cases in this situation.

Suppose first that $Z(R)\bigcap U=0$, then $[d^2(u),u]\in Z(R)\bigcap U=0$, and hence $[d^2(u),u]=0$, as required.

Suppose next that $Z(R)\bigcap U\neq 0$.

The linearization of $[d^2(u),u]\in Z(R)$ gives

$$[d^2(u),v]+[d^2(v),u]\in Z(R). \quad (9)$$

Now we can choose $0\neq w\in Z(R)\bigcap U$. Replacing v by $2wv$ in (9) and using (9), we get

$$[v,u]d^2(w)+2[d(v),u]d(w)\in Z(R). \quad (10)$$

Replacing v by $2wv$ in (10) and using (10), we have $[v,u]d(w)d(w)\in Z(R)$ for all $u,v\in U$ and $w\in Z(R)\bigcap U$. Since $d(w)\in Z(R)$, [11, Lemma 4] leads to $d(w)d(w)=0$ or $[v,u]=0$. On the one hand, $d(w)d(w)=0$ implies that $d(w)=0$. Replacing w by $2wv$ and using that $d(w)=0$, we get $wd(v)=0$ for all $v\in U$ and $w\in Z(R)\bigcap U$. This is a contradiction by [8, Lemma 7].

On the other hand, $[v,u]=0$ (i.e., $[U,U]=0$) for all $u,v\in U$. According to [9, Lemma 1], we conclude that $U\subseteq Z(R)$, again a contradiction. The proof of the theorem is completed. $\Box$

Theorem 3.2 *Let R be a 2-torsion free prime ring and U a square closed Lie ideal of R. If γ is a generalized derivation associated with $d\neq 0$, and γ acts as a homomorphism or an antihomomorphism on U, then $U\subseteq Z(R)$.*

Proof Assume that $U\not\subseteq Z(R)$. By hypothesis, for all $x,y\in U$, we have

$$\gamma(xy)=\gamma(x)\gamma(y)=\gamma(x)y+xd(y). \tag{11}$$

Replacing y by $2yz$ according to Lemma 2.1 in (11) and using that R is 2-torsion free, we arrive at $\gamma(x)\gamma(yz)=\gamma(x)yz+xd(yz)$, namely, $\gamma(x)\gamma(y)z+\gamma(x)yd(z)=\gamma(x)yz+xd(y)z+xyd(z)$ for all $x,y,z\in U$. It follows from (11) that $\gamma(x)\gamma(y)z=\gamma(x)yz+xd(y)z$. Hence we get $(\gamma(x)-x)Ud(z)=0$ for all $x,z\in U$. Thus, we have $\gamma(x)-x=0$ or $d(z)=0$ by [8, Lemma 4]. If $\gamma(x)-x=0$, then we have $xy=xy+xd(y)$, namely, $xd(y)=0$. Therefore, [8, Lemma 7] yields that $x=0$ for all $x\in U$, a contradiction. On the other hand, if $d(z)=0$, then we have $U\subseteq Z(R)$ by [8, Lemma 5], again a contradiction. Now assume that γ acts as antihomomorphism on U, so that

$$\gamma(xy)=\gamma(y)\gamma(x)=\gamma(x)y+xd(y). \tag{12}$$

Replacing y by $2xy$ in (12) and using (12), we get $\gamma(y)\gamma(xy)=\gamma(xy)y+xyd(y)$ for all $x,y\in U$, namely, $\gamma(y)\gamma(x)y+\gamma(y)xd(y)=\gamma(x)y^2+xd(y)y+xyd(y)$. It follows from (12) that $\gamma(y)\gamma(x)y=\gamma(x)y^2+xyd(y)y$. Therefore, we obtain that

$$\gamma(y)xd(y)=xyd(y). \tag{13}$$

Again replacing x by $2zx$ in (13) yields

$$\gamma(y)zxd(y)=zxyd(y). \tag{14}$$

On the other hand, multiplying (13) from the left by z, we find that

$$z\gamma(y)xd(y)=zxyd(y). \tag{15}$$

Combining (14) and (15), we have $\gamma(y)zxd(y)=z\gamma(y)xd(y)$ and hence $[\gamma(y),z]xd(y)=0$ for all $x,y,z\in U$. Thus, we get either $[\gamma(y),z]=0$ or $d(y)=0$ by [8, Lemma 4].

Now let $U_1=\{y\in U\mid[\gamma(y),z]=0\}$ and $U_2=\{y\in U\mid d(y)=0\}$. Then U_1,U_2 are both additive subgroups of U and $U_1\bigcup U_2=U$. Thus either $U=U_1$ or $U=U_2$ since a group cannot be a union of two of its proper subgroups. If $U=U_2$, then $d(u)=0$ for all $u\in U$. Thus, by [8, Lemma 5] we get $U\subseteq Z(R)$, a contradiction. On the other hand, if $U=U_1$, then $[\gamma(x),x]=0$ for all $x\in U$. Now the linearization of $[x,\gamma(x)]=0$ gives us

$$[x,y]d(x)+y[x,d(x)]=0. \tag{16}$$

Replacing y by $2yz$ in (16) and using (16), we arrive at $[x,y]zd(x)=0$ (i.e., $[x,y]Ud(x)=0$) for all $x,y,z\in U$. Thus we have either $[x,y]=0$ or $d(x)=0$ by [8, Lemma 4]. Now let $A=\{x\in U\mid[x,y]=0\}$ and $B=\{x\in U\mid d(x)=0\}$. Then A,B are both additive subgroups of

U and $A\bigcup B=U$. Thus either $A=U$ or $B=U$. On the one hand, if $A=U$ (i.e., $[U,U]=0$), then we get $U\subseteq Z(R)$ by [9, Lemma 1], a contradiction. On the other hand, if $B=U$(i.e., $d(U)=0$), then we have $d=0$ by [8, Lemma 5], again a contradiction. The proof of the theorem is completed. $\square$

References

[1] CHUNG L O, LUH J. *Derivations of higher order and commutativity of rings* [J]. Pacific J. Math., 1982, **99**(2): 317–326.

[2] VUKMAN J. *Commuting and centralizing mappings in prime rings* [J]. Proc. Amer. Math. Soc., 1990, **109**(1): 47–52.

[3] WONG T L. *On Lie automorphisms, additive subgroups, and Lie ideals of prime rings* [J]. Comm. Algebra, 2003, **31**(2): 969–979.

[4] BELL H E, KAPPE L C. *Rings in which derivations satisfy certain algebraic conditions* [J]. Acta Math. Hungar., 1989, **53**(3-4): 339–346.

[5] ASMA A, REHMAN N, SHAKIR A. *On Lie ideals with derivations as homomorphisms and anti-homomorphisms* [J]. Acta Math. Hungar., 2003, **101**(1-2): 79–82.

[6] ALBAŞ E, ARGAÇ N. *Generalized derivations of prime rings* [J]. Algebra Colloq., 2004, **11**(3): 399–410.

[7] PINTER-LUCKE J. *Commutativity conditions for rings: 1950–2005* [J]. Expo. Math., 2007, **25**(2): 165–174.

[8] BERGEN J, HERSTEIN I N, KERR J W. *Lie ideals and derivations of prime rings* [J]. J. Algebra, 1981, **71**(1): 259–267.

[9] HERSTEIN I N. *On the Lie structure of an associative ring* [J]. J. Algebra, 1970, **14**: 561–571.

[10] LEE P H, LEE T K. *Lie ideals of prime rings with derivations* [J]. Bull. Inst. Math. Acad. Sinica, 1983, **11**(1): 75–80.

[11] MAYNE J H. *Centralizing mappings of prime rings* [J]. Canad. Math. Bull., 1984, **27**(1): 122–126.