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Positive Solutions of a Weak Semipositone Third-Order Three-Point Boundary Value Problem

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Abstract The positive solutions are studied for the nonlinear third-order three-point boundary value problem

$$u'''(t) = f(t, u(t)),\ \text{a.e. } t \in [0, 1], \quad u(0) = u'(\eta) = u''(1) = 0,$$

where the nonlinear term $f(t, u)$ is a Carathéodory function and there exists a nonnegative function $h \in L^1[0, 1]$ such that $f(t, u) \geq -h(t)$. The existence of n positive solutions is proved by considering the integrations of "height functions" and applying the Krasnosel'skii fixed point theorem on cone.

Keywords singular ordinary differential equation; multi-point boundary value problem; positive solution; existence; multiplicity.

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1. Introduction

Let $\frac{1}{2} < \eta < 1$. The purpose of this paper is to consider the positive solutions of following nonlinear third-order three-point boundary value problem

$$\text{(P)} \quad u'''(t) = f(t, u(t)),\ \text{a.e. } t \in [0, 1], \quad u(0) = u'(\eta) = u''(1) = 0.$$

Here the function u^* is called positive solution of (P), if u^* is a solution of (P) and $u^*(t) > 0$, $0 < t \leq 1$.

Throughout this paper, we assume that $f : [0, 1]\times(-\infty, +\infty) \to (-\infty, +\infty)$ is a Carathéodory function and there exists a nonnegative function $h \in L^1[0, 1]$ such that $f(t, u) \geq -h(t),\ (t, u) \in [0, 1] \times (-\infty, +\infty)$. Here, the function $f(t, u)$ is called Carathéodory function, if

(C1) For a.e. $t \in [0, 1]$, $f(t, \cdot) : (-\infty, +\infty) \to (-\infty, +\infty)$ is continuous.

(C2) For any $u \in (-\infty, +\infty)$, $f(\cdot, u) : [0, 1] \to (-\infty, +\infty)$ is measurable.

(C3) For each $r > 0$, there is a nonnegative function $j_r \in L^1[0, 1]$ such that $|f(t, u)| \leq j_r(t),\ (t, u) \in [0, 1] \times (-\infty, r]$.

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It is well known that the problem (P) is called positive if $h(t) \equiv 0$, and semipositone if $h(t) \equiv M \geq 0$. In this paper, we require only that $h(t)$ is a nonnegative integrable function. Therefore we allow that the nonlinear term $f(t, u)$ does not have the lower bound on its domain. In this sense the problem (P) is called weak semipositone.

The positive solutions of positive and semipositone problem (P) have been studied by many authors [1–6]. Some weak semipositone problems with other boundary conditions have been followed with interest [7–9]. In this paper we will use new technique to investigate the weak semipositone problem (P). We will introduce the "height functions" to describe the feature of the nonlinear term on bounded sets and apply the integrations of "height functions" to dominate the growth of nonlinear term. After that, we will establish the local existence of n positive solutions for the problem (P) by the Krasnosel'skii fixed point theorem of cone expansion-compression type, where n is an arbitrary positive integer. The idea of this paper comes from papers [10–13].

2. Symbols

Let $C[0,1]$ be the Banach space with the norm $\|u\| = \max_{0\leq t\leq 1} |u(t)|$ and $0 < \alpha < \eta \leq \beta \leq 1$.

Let $G(t, s)$ be the Green function of the homogeneous linear problem $u'''(t) = 0$, $u(0) = u'(\eta) = u''(1) = 0$. From [1], $G(t, s)$ has exact expression

$$G(t,s) = \begin{cases} ts - \frac{1}{2}t^2, & 0 \leq s \leq \eta,\ 0 \leq t \leq s, \\ \frac{1}{2}s^2, & 0 \leq s \leq \eta,\ 0 \leq s \leq t, \\ \eta t - \frac{1}{2}t^2, & \eta \leq s \leq 1,\ 0 \leq t \leq s, \\ \frac{1}{2}s^2 - ts + \eta t, & \eta \leq s \leq 1,\ 0 \leq s \leq t. \end{cases}$$

Obviously, $G(t, s) > 0,\ 0 < t, s < 1$. Further, we have

$$\frac{\partial}{\partial t}G(t,s) = \begin{cases} s - t, \\ 0, \\ \eta - t, \\ \eta - s, \end{cases} \quad \frac{\partial^2}{\partial t^2}G(t,s) = \begin{cases} -1, \\ 0, \\ -1, \\ 0, \end{cases} \quad \begin{cases} 0 \leq s \leq \eta, & 0 \leq t \leq s, \\ 0 \leq s \leq \eta, & 0 \leq s \leq t, \\ \eta \leq s \leq 1, & 0 \leq t \leq s, \\ \eta \leq s \leq 1, & 0 \leq s \leq t. \end{cases}$$

So $G(0, s) = \frac{\partial}{\partial t}G(\eta, s) = \frac{\partial^2}{\partial t^2}G(1, s) = 0$, $0 \leq s \leq 1$.

Let $q(t) = \min\{\eta t,\ 2\eta t - t^2\}$, $H(s) = \max_{0\leq t\leq 1} G(t, s)$. By Lemma 2.1 in [5], $H(s) = \frac{1}{2}s^2$ if $0 \leq s \leq \eta$, and $H(s) = \frac{1}{2}\eta^2$ if $\eta \leq s \leq 1$.

Let $\tau = \min\{\alpha\eta, 2\beta\eta - \beta^2\}$. It is easy to see $\tau = \min_{\alpha\leq t\leq\beta} q(t)$.

Let $K = \{u \in C[0,1] : u(t) \geq \|u\|q(t),\ 0 \leq t \leq 1\}$. Obviously, K is a cone of nonnegative functions in $C[0,1]$.

Let $u_0(t) = \int_0^1 G(t,s)h(s)\mathrm{d}s$. Then $u_0 \in C[0,1]$ is a nonnegative function.

Define the operator T as follows

$$(Tu)(t) = \int_0^1 G(t,s)[f(s, u(s) - u_0(s)) + h(s)]\mathrm{d}s, \quad 0 \leq t \leq 1,\ u \in K.$$

We introduce the following "height functions" to describe the growths of nonlinear term on bounded set $[0,1]\times[0,r]$:

$$\varphi(t,r)=\max\{f(t,u-u_0(t)):0\le u\le r\}+h(t),$$
$$\psi(t,r)=\min\{f(t,u-u_0(t)):\tau r\le u\le r\}+h(t).$$

Since $f(t,u)$ is a Carathéodory function, we see that, for any $r>0$, the function $\varphi(t,r)$ and $\psi(t,r)$ are well defined almost everywhere in $[0,1]$. Since $\psi(t,r)\le\varphi(t,r)\le j_{r+\|u_0\|}(t)+h(t)$, we assert that $\varphi(t,r)$ and $\psi(t,r)$ are integrable function on $[0,1]$. If $f(t,u)$ and $h(t)$ are given, then $\varphi(t,r)$ and $\psi(t,r)$ are computable.

We will use the control constants $A=\frac{2}{\eta^2},\ B=\frac{2}{\alpha^2\min\{\alpha\eta,2\beta\eta-\beta^2\}}$.

3. Lemmas

Lemma 3.1 *We have* $q(t)H(s)\le G(t,s)\le H(s)$, $0\le t,s\le 1$.

Proof See Lemma 2.2 in [5]. □

Lemma 3.2 *We have* $\max_{0\le t,s\le1}G(t,s)\le A^{-1}$, $\min_{\sigma\le t,s\le 1-\sigma}G(t,s)\ge B^{-1}$.

Proof By Lemma 3.1, we have

$$\min_{\alpha\le t,s\le\beta}G(t,s)\ge\min_{\alpha\le t\le\beta}q(t)\min_{\alpha\le s\le\beta}H(s)=\frac{1}{2}\alpha^2\min\{\alpha\eta,2\beta\eta-\beta^2\}=B^{-1},$$
$$\max_{0\le t,s\le 1}G(t,s)\le\max_{0\le s\le1}H(s)=\frac{1}{2}\eta^2=A^{-1}.\ \square$$

Lemma 3.3 *There exists $M>0$ such that* $\max_{0\le t,s\le1}|\frac{\partial^i}{\partial t^i}G(t,s)|\le M,\ 0\le i\le 3$.

Proof By the expressions of $G(t,s)$ and its partial derivatives, $G(t,s)$ and $\frac{\partial}{\partial t}G(t,s)$ are continuous functions on $[0,1]\times[0,1]$, and $\frac{\partial^2}{\partial t^2}G(t,s)$ is a bounded function on $[0,1]\times[0,1]$. From this, the proof is completed. □

Lemma 3.4 $T:K\to K$ *is completely continuous.*

Proof We have the decomposition of operator $T=S\circ J$, where, for $0\le t\le1$,

$$(Ju)(t)=f(t,u(t)-u_0(t))+h(t),\quad (Su)(t)=\int_0^1G(t,s)u(s)\mathrm{d}s.$$

Since $f(t,u)$ is a Carathéodory function, we can prove that $J:K\to L[0,1]$ is continuous by [14, Lemmas 1.2, 1.3 and Theorem 1.1]. Applying the Arzela-Ascoli theorem, we can prove that $S:L[0,1]\to C[0,1]$ is a linear and completely continuous operator. Hence $T:K\to C[0,1]$ is completely continuous. Moreover, by Lemma 3.1 we have, for any $0\le t\le1$,

$$\begin{aligned}(Tu)(t)&=\int_0^1G(t,s)[f(s,u(s)-u_0(s))+h(s)]\mathrm{d}s\\&\ge q(t)\int_0^1H(s)[f(s,u(s)-u_0(s))+h(s)]\mathrm{d}s\\&\ge q(t)\max_{0\le t\le1}\int_0^1G(t,s)[f(s,u(s)-u_0(s))+h(s)]\mathrm{d}s=\|Tu\|q(t).\end{aligned}$$

Therefore $T : K \to K$. $\Box$

Lemma 3.5 is the Krasnosel'skii fixed point theorem of cone expansion-compression type.

Lemma 3.5 *Let X be a Banach space and K be a cone in X. Assume that Ω_1, Ω_2 are bounded open subsets of K with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and $F : K \to K$ is a completely continuous operator such that one of the following conditions is satisfied:*

(1) $\|F(x)\| \le \|x\|,\ x \in \partial\Omega_1$,and $\|F(x)\| \ge \|x\|,\ x \in \partial\Omega_2$;

(2) $\|F(x)\| \ge \|x\|,\ x \in \partial\Omega_1$, and $\|F(x)\| \le \|x\|,\ x \in \partial\Omega_2$.

Then F has a fixed point in $\overline{\Omega}_2 \backslash \Omega_1$.

Lemma 3.6 *If $\tilde{u} \in K$ is a fixed point of the operator T, then $u^* = \tilde{u} - u_0$ is a solution of the problem (P).*

Proof Since $\tilde{u}$ is a fixed point of T, we have

$$\tilde{u}(t) = (T\tilde{u})(t) = \int_0^1 G(t,s)[f(s,\tilde{u}(s) - u_0(s)) + h(s)]\mathrm{d}s, \quad 0 \le t \le 1.$$

Since $\tilde{u} = u^* + u_0$ and $u_0(t) = \int_0^1 G(t,s)h(s)\mathrm{d}s$, we have

$$u^*(t) = \int_0^1 G(t,s)f(s,u^*(s))\mathrm{d}s, \quad 0 \le t \le 1.$$

Applying Lemma 3.3 and using the mean value theorem if $i = 0, 1$, and the generalized mean value theorem [15, Theorem 2.3.7] if $i = 2$, we get that, for $0 \le i \le 3$,

$$\left|\frac{\partial^i}{\partial t^i}G(t+\triangle t,s) - \frac{\partial^i}{\partial t^i}G(t,s)\right| \le M|\triangle t|, \quad 0 \le t+\triangle t, t, s \le 1.$$

Since the condition (C3) guarantees that $f(s,u^*(s))$ is integrable, we get that

$$(u^*)^{(i)}(t) = \int_0^1 \frac{\partial^i}{\partial t^i}G(t,s)f(s,u^*(s))\mathrm{d}s, \quad 0 \le t \le 1,\ 0 \le i \le 2,$$

by the rule of computing partial derivative [16, p129, Theorem 6]. By making use of the expression of $\frac{\partial^2}{\partial t^2}G(t,s)$, we have

$$(u^*)''(t) = \begin{cases} -\int_\eta^t f(s,u^*(s))\mathrm{d}s, & 0 \le t \le \eta, \\ -\int_t^1 f(s,u^*(s))\mathrm{d}s, & \eta \le t \le 1. \end{cases}$$

Applying the properties of infinite integration [16, p162, Theorem 3], we obtain

$$(u^*)'''(t) = f(t,u^*(t)), \quad \text{a.e. } t \in [0,1].$$

In addition, we see $u^*(0) = (u^*)'(\eta) = (u^*)''(1) = 0$ from the expressions of $\frac{\partial^i}{\partial t^i}G(t,s)$. Therefore, u^* is a solution of the problem (P). $\Box$

4. Results

Let $\bar{h} = \frac{1}{2}\int_0^1 h(t)\mathrm{d}t$. We obtain the following local existence results.

Theorem 4.1 *Assume that there exist two positive numbers $a < b$ such that one of the following* *conditions is satisfied:*

(a1) $\int_0^1 \varphi(t,a)\mathrm{d}t \le aA$, $\int_\alpha^\beta \psi(t,b)\mathrm{d}t \ge bB$;

(a2) $\int_\alpha^\beta \psi(t,a)\mathrm{d}t \ge aB$, $\int_0^1 \varphi(t,b)\mathrm{d}t \le bA$.

Then problem (P) has at least one solution $u^* \in C[0,1]$ *such that* $u^* + u_0 \in K$ *and* $a \le \|u^* + u_0\| \le b$. *In addition,* u^* *is a positive solution if* $a > \bar{h}$.

Proof We prove only the case (a1). Let $\Omega_a = \{u \in K : \|u\| < a\}$, $\Omega_b = \{u \in K : \|u\| < b\}$.

If $u \in \partial\Omega_a$, then $\|u\| = a$. Thus $0 \le u(t) \le a$, $0 \le t \le 1$. By the definition of $\varphi(t,a)$, we see

$$f(t, u(t) - u_0(t)) + h(t) \le \varphi(t,a), \quad 0 \le t \le 1.$$

Applying the assumption (a1) and Lemma 3.2, we get

$$\begin{aligned}\|Tu\| &= \max_{0\le t\le 1}\int_0^1 G(t,s)[f(s,u(s)-u_0(s))+h(s)]\mathrm{d}s\\ &\le \max_{0\le t,s\le 1} G(t,s)\int_0^1 [f(s,u(s)-u_0(s))+h(s)]\mathrm{d}s\\ &\le A^{-1}\int_0^1 \varphi(s,a)\mathrm{d}s \le A^{-1}aA = a = \|u\|.\end{aligned}$$

If $u \in \partial\Omega_b$, then $\|u\| = b$. Thus

$$\tau b = b \min_{\alpha\le t\le\beta} q(t) \le \|u\|\, q(t) \le u(t) \le b, \quad \alpha \le t \le \beta.$$

By the definition of $\psi(t,b)$, we have

$$f(t, u(t) - u_0(t)) + h(t) \ge \psi(t,b), \quad \alpha \le t \le \beta.$$

From this we get

$$\begin{aligned}\|Tu\| &\ge \max_{\alpha\le t\le \beta}\int_\alpha^\beta G(t,s)[f(s,u(s)-u_0(s))+h(s)]\mathrm{d}s\\ &\ge \min_{\alpha\le t,s\le \beta} G(t,s)\int_\alpha^\beta [f(s,u(s)-u_0(s))+h(s)]\mathrm{d}s\\ &\ge B^{-1}\int_\alpha^\beta \psi(s,b)\mathrm{d}s \ge B^{-1}bB = b = \|u\|.\end{aligned}$$

By Lemmas 3.4 and 3.5, the operator T has a fixed point $\tilde{u} \in K$ and $a \le \|\tilde{u}\| \le b$. Let $u^* = \tilde{u} - u_0$. Then u^* is a solution of (P) by Lemma 3.6. Moreover, $u^* + u_0 \in K$ and $a \le \|u^* + u_0\| \le b$.

By the definition of $G(t,s)$, if $0 \le t \le \eta$,

$$\begin{aligned}\int_0^\eta G(t,s)h(s)\mathrm{d}s &= \int_0^t \frac{1}{2}s^2 h(s)\mathrm{d}s + \int_t^\eta (ts - \frac{1}{2}t^2)h(s)\mathrm{d}s\\ &\le \frac{1}{2}t^2\int_0^t h(s)\mathrm{d}s + \frac{1}{2}t^2\int_t^\eta h(s)\mathrm{d}s \le \frac{1}{2}\eta t\int_0^\eta h(s)\mathrm{d}s;\end{aligned}$$

if $\eta \le t \le 1$,

$$\int_\eta^1 G(t,s)h(s)\mathrm{d}s = \int_\eta^t (\frac{1}{2}s^2 - ts + \eta t)h(s)\mathrm{d}s + \int_t^1 (\eta t - \frac{1}{2}t^2)h(s)\mathrm{d}s$$

$$\le (\eta t - \frac{1}{2}t^2)\int_\eta^t h(s)\mathrm{d}s + (\eta t - \frac{1}{2}t^2)\int_t^1 h(s)\mathrm{d}s$$
$$= \frac{1}{2}(2\eta t - t^2)\int_\eta^1 h(s)\mathrm{d}s.$$

It follows

$$u_0(t) = \int_0^1 G(t,s)h(s)\mathrm{d}s \le \frac{1}{2}q(t)\int_0^1 h(s)\mathrm{d}s = \bar{h}q(t), \quad 0 \le t \le 1.$$

If $a > \bar{h}$, then

$$u^*(t) = \tilde{u}(t) - u_0(t) \ge \|\tilde{u}\|q(t) - \bar{h}q(t) \ge (a - \bar{h})q(t) > 0, \quad 0 < t < 1.$$

It shows that u^* is a positive solution of (P). □

Modeling the proof of Theorem 3.1, we obtain the following results concerned with multiple positive solutions where $[c]$ is the integer part of c.

Theorem 4.2 *Assume that there exist $n+1$ positive numbers $a_1 < a_2 < \cdots < a_{n+1}$ such that one of the following conditions is satisfied:*

(b1) $\int_0^1 \varphi(t, a_{2k-1})\mathrm{d}t < a_{2k-1}A$, $k = 1, 2, \ldots, [\frac{n+2}{2}]$, *and*

$$\int_\alpha^\beta \psi(t, a_{2k})\mathrm{d}t > a_{2k}B, \quad k = 1, 2, \ldots, [\frac{n+1}{2}];$$

(b2) $\int_\alpha^\beta \psi(t, a_{2k-1})\mathrm{d}t > a_{2k-1}B$, $k = 1, 2, \ldots, [\frac{n+2}{2}]$, *and*

$$\int_0^1 \varphi(t, a_{2k})\mathrm{d}t < a_{2k}A, \quad k = 1, 2, \ldots, [\frac{n+1}{2}].$$

Then problem (P) has at least n solutions $u_k^* \in C[0,1]$, $k = 1, 2, \ldots, n$, such that $u_k^* + u_0 \in K$ and $a_k < \|u_k^* + u_0\| < a_{k+1}$. In addition, $u_k^*, u_{k+1}^*, \ldots, u_n^*$ are positive solutions if $a_k > \bar{h}$.

Applying Theorem 4.1, we can prove Theorem 4.3. Similar results arise in papers [7–9] for the other boundary value problems.

Theorem 4.3 *Assume there exists a set $e \subset [\alpha, \beta]$ with zero measure such that*

$$\lim_{u\to+\infty} \inf_{t\in[\alpha,\beta]\backslash e} f(t,u)/u = +\infty.$$

Then problem

$$u'''(t) = \lambda f(t, u(t)), \text{ a.e. } t \in [0,1], \quad u(0) = u'(\eta) = u''(1) = 0$$

has a positive solution $u^* \in C[0,1]$ for sufficient small $\lambda > 0$.

Proof Now, the nonlinear term is $\lambda f(t,u)$. The corresponding control functions are $\lambda\varphi(t,r)$ and $\lambda\psi(t,r)$.

Since $\lim_{u\to+\infty} \inf_{t\in[\alpha,\beta]\backslash e} f(t,u)/u = +\infty$, there exists $\tilde{r} > \max\{\frac{1}{4}\bar{h}, \|u_0\|\} \ge 0$ such that $\inf_{t\in[\alpha,\beta]\backslash e} f(t,u)/u \ge 1$, $u \ge \tilde{r}$. So,

$$f(t,u) \ge u \ge \tilde{r}, \quad t \in [\alpha,\beta]\backslash e,\ \tilde{r} \le u < +\infty.$$

It follows

$$f(t,u-u_0(t)) \geq \tilde{r}, \quad t\in[\alpha,\beta]\backslash e,\ 2\tilde{r}\leq u<+\infty.$$

Let $a=4\tilde{r}$. Then, for $t\in[\alpha,\beta]\backslash e$,

$$\begin{aligned}\varphi(t,a) &= \max\{f(t,u-u_0(t)) : 0\leq u\leq a\}+h(t)\\ &\geq \max\{f(t,u-u_0(t)) : 2\tilde{r}\leq u\leq 4\tilde{r}\}\geq \tilde{r}.\end{aligned}$$

Hence

$$\int_0^1 \varphi(t,a)\mathrm{d}t \geq \int_\alpha^\beta \varphi(t,a)\mathrm{d}t \geq \tilde{r}(\beta-\alpha)>0.$$

Let $\lambda^*=aA[\int_0^1\varphi(t,a)\mathrm{d}t]^{-1}$ and $0<\lambda\leq\lambda^*$.

By the definition of λ^*, we have

$$\int_0^1 \lambda\varphi(t,a)\mathrm{d}t \leq \lambda^*\int_0^1 \varphi(t,a)\mathrm{d}t = aA.$$

On the other hand, since $\lim_{u\to+\infty}\inf_{t\in[\alpha,\beta]\backslash e} f(t,u)/u=+\infty$, there exists $b>0$ such that $\frac{1}{2}\tau b>\tilde{r}$ and

$$f(t,u)\geq u\geq \frac{bB}{\lambda(\beta-\alpha)}, \quad t\in[\alpha,\beta]\backslash e,\ \frac{1}{2}\tau b\leq u<+\infty.$$

Since $\|u_0\|<\frac{1}{2}\tau b$, we have

$$f(t,u-u_0(t))\geq u\geq \frac{bB}{\lambda(\beta-\alpha)}, \quad t\in[\alpha,\beta]\backslash e,\ \tau b\leq u<+\infty.$$

It follows, for any $t\in[\alpha,\beta]\backslash e$,

$$\lambda\psi(t,b)=\min\{\lambda f(t,u-u_0(t) : \tau b\leq u\leq b\}+\lambda h(t)\geq \frac{bB}{\beta-\alpha}.$$

Hence

$$\int_\alpha^\beta \lambda\psi(t,b)\mathrm{d}t \geq \frac{bB}{\beta-\alpha}\cdot(\beta-\alpha)=bB.$$

Since $a>\bar{h}$, by Theorem 4.1, the problem has a positive solution $u^*\in C[0,1]$. $\square$

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