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Generation of a Problem about Mean Value Theorem

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Abstract This paper presents a generalized form and its application to a problem, which was proposed by F.P. Callham.

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1. Introduction and main conclusion

Callaham [1] presented a result about Mean Value Theorem.

Suppose $f(x)$ is differentiable on $(-\infty,+\infty)$, and there exist p, q satisfying $p>0$, $q>0$, $p+q=1$ such that for every u, v,

$$\frac{f(v)-f(u)}{v-u}=f'(pu+qv),\quad u\neq v, \tag{1}$$

then $f(x)$ is a linear function or quadratic function.

As the author knows, the conclusion has not been studied further by anyone yet. This result will be generalized in this paper, i.e., we have the following theorem:

Theorem 1 *Suppose $f(x)$ has n-th derivative in $(-\infty,+\infty)$, and there exist $\alpha_i>0\,(i=0,1,2,\ldots,n)$, $\sum_{i=0}^n\alpha_i=1$, such that for any $x_i\,(i=0,1,2,\ldots,n)$ different from each other, the following equality is valid:*

$$\sum_{i=0}^{n}\frac{f(x_i)}{\prod_{\substack{j=0\\ j\neq i}}^{n}(x_i-x_j)}=\frac{1}{n!}f^{(n)}\Big(\sum_{i=0}^{n}\alpha_ix_i\Big), \tag{2}$$

then $f(x)$ is a polynomial of degree not greater than $n+1$.

2. The proof of the main conclusion

From (2) it is known that $f(x)\in C^{\infty}(-\infty,+\infty)$. Because the left side of (2) denotes the n-th difference quotient of function $f(x)$ at points $x_0,x_1,\ldots,x_n$, by using the symbol of difference

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quotient, (2) can be written as

$$f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^{(n)}\Big(\sum_{i=0}^{n} \alpha_i x_i\Big). \tag{2}$$

We shall discuss the following two cases of α_i $(i = 0, 1, \ldots, n)$:

(i) α_i $(i = 0, 1, \ldots, n)$ are not all equal, i.e., at least two of them are not equal to each other. We might as well let $\alpha_0 \neq \alpha_1$;

(ii) $\alpha_0 = \alpha_1 = \cdots = \alpha_n = \frac{1}{n+1}$.

For the Case (i):

Since difference quotient is independent of the order of the points $x_0, x_1, \ldots, x_n$, exchanging the positions of x_0 and x_1 in (2) gives

$$f^{(n)}(\alpha_0 x_0 + \alpha_1 x_1 + \sum_{i=2}^{n} \alpha_i x_i) = f^{(n)}(\alpha_0 x_1 + \alpha_1 x_0 + \sum_{i=2}^{n} \alpha_i x_i). \tag{3}$$

Differentiating both sides of (3) with respect to x_0 and x_1 respectively gives

$$\alpha_0 f^{(n+1)}(\alpha_0 x_0 + \alpha_1 x_1 + \sum_{i=2}^{n} \alpha_i x_i) = \alpha_1 f^{(n+1)}(\alpha_0 x_1 + \alpha_1 x_0 + \sum_{i=2}^{n} \alpha_i x_i), \tag{4}$$

$$\alpha_1 f^{(n+1)}(\alpha_0 x_0 + \alpha_1 x_1 + \sum_{i=2}^{n} \alpha_i x_i) = \alpha_0 f^{(n+1)}(\alpha_0 x_1 + \alpha_1 x_0 + \sum_{i=2}^{n} \alpha_i x_i). \tag{5}$$

It follows from (4) and (5) that

$$(\alpha_0 - \alpha_1)[f^{(n+1)}(\alpha_0 x_0 + \alpha_1 x_1 + \sum_{i=2}^{n} \alpha_i x_i) - f^{(n+1)}(\alpha_0 x_1 + \alpha_1 x_0 + \sum_{i=2}^{n} \alpha_i x_i)] = 0.$$

Since $\alpha_0 \neq \alpha_1$, we have

$$f^{(n+1)}(\alpha_0 x_0 + \alpha_1 x_1 + \sum_{i=2}^{n} \alpha_i x_i) = f^{(n+1)}(\alpha_0 x_1 + \alpha_1 x_0 + \sum_{i=2}^{n} \alpha_i x_i).$$

Then from the arbitrariness of $x_0, x_1, \ldots, x_n$, we know that for any $x \in (-\infty, +\infty)$, $f^{(n+1)}(x) \equiv C$ (C is a constant), which implies that $f(x)$ is a polynomial of degree not greater than $n+1$.

For the Case (ii), the equality (2) gives

$$f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^{(n)}\Big(\frac{1}{n+1} \sum_{i=0}^{n} x_i\Big). \tag{6}$$

Then it will be proved with induction that $f(x)$ is a polynomial of degree not greater than $n+1$ while (6) is valid.

From [1] we know that the proposition is true when $n = 1$. Suppose it is true for n, i.e.,

$$f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^{(n)}\Big(\frac{1}{n+1} \sum_{i=0}^{n} x_i\Big)$$

or

$$f[x_1, x_2, \ldots, x_{n+1}] = \frac{1}{n!} f^{(n)}\Big(\frac{1}{n+1} \sum_{i=1}^{n+1} x_i\Big) \tag{7}$$

is true, $f(x)$ is a polynomial of degree not greater than $n+1$.

For $n+1$,

$$f[x_0, x_1, x_2, \dots, x_{n+1}] = \frac{1}{(n+1)!} f^{(n+1)}\Big(\frac{1}{n+2}\sum_{i=0}^{n+1} x_i\Big). \tag{8}$$

From the definition of difference quotient,

$$f[x_0, x_1, x_2, \dots, x_{n+1}] = \frac{f[x_1, x_2, \dots, x_{n+1}] - f[x_0, x_1, \dots, x_n]}{x_{n+1} - x_0},$$

(6), (7) and (8) it follows

$$\begin{aligned} &\frac{1}{x_{n+1} - x_0}\Big[f^{(n)}\Big(\frac{1}{n+1}\sum_{i=1}^{n+1} x_i\Big) - f^{(n)}\Big(\frac{1}{n+1}\sum_{i=0}^{n} x_i\Big)\Big] \\ &= \frac{1}{n+1} f^{(n+1)}\Big(\frac{1}{n+2}\sum_{i=0}^{n+1} x_i\Big). \end{aligned} \tag{9}$$

From the assumption of induction it is known that

$$f^{(n)}\Big(\frac{1}{n+1}\sum_{i=1}^{n+1} x_i\Big) - f^{(n)}\Big(\frac{1}{n+1}\sum_{i=0}^{n} x_i\Big)$$

is a polynomial of x_{n+1} of degree not greater than 1, and it contains a factor $x_{n+1} - x_0$, so the left side and right side of (9) must be constant with respect to x_{n+1}. From the arbitrariness of x_{n+1}, $f^{(n+1)}(x)$ is constant with respect to x, so $f(x)$ is a polynomial of degree not greater than $n+2$.

By the principle of induction, $f(x)$ is a polynomial of degree not greater than $n+1$ as (6) is valid.

But the conclusion can also be proved without induction.

The validity of (6) gives

$$f[x_0, x_1, \dots, x_{n-1}, x_{n+1}) = \frac{1}{n!} f^{(n)}\Big(\frac{1}{n+1}\sum_{i=0}^{n-1} x_i + \frac{1}{n+1}x_{n+1}\Big), \tag{10}$$

$$f[x_0, x_1, \dots, x_{n-2}, x_n, x_{n+1}) = \frac{1}{n!} f^{(n)}\Big(\frac{1}{n+1}\sum_{i=0}^{n-2} x_i + \frac{1}{n+1}x_n + \frac{1}{n+1}x_{n+1}\Big). \tag{11}$$

Since

$$\begin{aligned} &\frac{f[x_0, x_1, \dots, x_{n-1}, x_{n+1}] - f[x_0, x_1, \dots, x_n]}{x_{n+1} - x_n} \\ &= \frac{f[x_0, x_1, \dots, x_{n-2}, x_n, x_{n+1}] - f[x_0, x_1, \dots, x_{n-1}, x_{n+1}]}{x_n - x_{n-1}}, \end{aligned} \tag{12}$$

using (10), (11) and (12) together with $\sum_{i=0}^{n-2} x_i = \lambda_0$ gives

$$\begin{aligned} &(x_n - x_{n-1})\Big\{f^{(n)}\Big[\frac{1}{n+1}(\lambda_0 + x_{n-1} + x_{n+1})\Big] - f^{(n)}\Big[\frac{1}{n+1}(\lambda_0 + x_{n-1} + x_n)\Big]\Big\} \\ &= (x_{n+1} - x_n)\Big\{f^{(n)}\Big[\frac{1}{n+1}(\lambda_0 + x_n + x_{n+1})\Big] - f^{(n)}\Big[\frac{1}{n+1}(\lambda_0 + x_{n-1} + x_{n+1})\Big]\Big\}. \end{aligned}$$

It can be deduced by differentiating both sides of the equation above in respect to x_n and rearranging that

$$\begin{aligned}&(n+1)f^{(n)}\Big[\frac{1}{n+1}(\lambda_0+x_n+x_{n+1})\Big]\\&=(x_{n+1}-x_n)f^{(n+1)}\Big[\frac{1}{n+1}(\lambda_0+x_n+x_{n+1})\Big]+\\&\quad(x_n-x_{n-1})f^{(n+1)}\Big[\frac{1}{n+1}(\lambda_0+x_{n-1}+x_n)\Big]+\\&\quad(n+1)f^{(n)}\Big[\frac{1}{n+1}(\lambda_0+x_{n-1}+x_n)\Big].\end{aligned}$$

Then differentiating both sides in respect of x_{n+1} and rearranging yields

$$\frac{x_{n+1}-x_n}{n+1}f^{(n+2)}\Big[\frac{1}{n+1}(\lambda_0+x_n+x_{n+1})\Big]=0,$$

so

$$f^{(n+2)}\Big[\frac{1}{n+1}(\lambda_0+x_n+x_{n+1})\Big]=0. \tag{13}$$

From the arbitrariness of $x_0, x_1, \ldots, x_n, x_{n+1}$ we know that, when (6) is valid, $f(x)$ is a polynomial of degree not greater than $n+1$.

Thus, when (2) is valid, $f(x)$ is a polynomial of degree not greater than $n+1$.

Now the proof of Theorem is completed. □

Contrarily, if $f(x)$ is a polynomial of degree not greater than $n+1$, it is easy to verify that (2) is valid now, so we have

Theorem 2 *Function $f(x)$ is a polynomial of degree not greater then $n+1$ if and only if the equality (2) is valid.*

3. An application

About the difference of functions with equidistant knots, we have

Theorem 3 ([2]) *Suppose $f(x) \in C^{m+1}[a,b]$, and $f^{(m+1)}(a) \neq 0$. If $h = \frac{b-a}{m}, x_k = a + kh$ $(k = 0, 1, \ldots, m)$, then η, defined by the relation equation of the m-th difference of $f(x)$ at knots $x_0, x_1, \ldots, x_m$ and its m-th derivative*

$$\Delta^m f(x_0) = h^m f^{(m)}(\eta), \quad \eta \in (a,b) \tag{14}$$

satisfies

$$\lim_{h\to 0^+}\frac{\eta-a}{h}=\frac{m}{2}. \tag{15}$$

As an application of Theorem 2, let us solve a problem contrary to Theorem 3.

Problem Let $\frac{\eta-a}{h}=\frac{m}{2}$, so $\eta = a+\frac{m}{2}h$. By (14), if

$$\Delta^m f(x_0) = h^m f^{(m)}(a+\frac{m}{2}h), \tag{16}$$

then what properties does $f(x)$ possess?

By the relation between difference and difference quotient, (16) can be rewritten as

$$f[x_0, x_1, \ldots, x_m] = \frac{1}{m!} f^{(m)}\Big(\frac{1}{m+1}\sum_{i=0}^{m} x_i\Big). \tag{16'}$$

So from Theorem 2 it can be derived that $f(x)$ is a polynomial of degree not greater then $m+1$. Thus we also have

Theorem 4 *Function $f(x)$ is a polynomial of degree not greater than $m+1$ if and only if the equality (16) is valid.*

References

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