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Commuting Hankel and Toeplitz Operators on the Hardy Space of the Bidisk

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Abstract In this paper, we characterize when the Toeplitz operator T_f and the Hankel operator H_g commute on the Hardy space of the bidisk. For certain types of bounded symbols f and g, we give a necessary and sufficient condition on the symbols to guarantee $T_f H_g = H_g T_f$.

Keywords Hankel operator; Toeplitz operator; Hardy space; bidisk; commuting.

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1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. Its boundary is the unit circle $\mathbb{T}$. The bidisk $\mathbb{D}^2$ and torus $\mathbb{T}^2$ are the subsets of $\mathbb{C}^2$ which are Cartesian products of two copies $\mathbb{D}$ and $\mathbb{T}$, respectively. We write $L^2(\mathbb{T})$ and $H^2(\mathbb{D})$ to denote the usual Lebesgue space on $\mathbb{T}$ and Hardy space on $\mathbb{D}$, respectively. Let $L^2(\mathbb{T}^2) = L^2(\mathbb{T}^2, \mathrm{d}\sigma)$ be the usual Lebesgue space of $\mathbb{T}^2$, where $\mathrm{d}\sigma$ is the normalized Haar measure on $\mathbb{T}^2$, and the Hardy space $H^2(\mathbb{D}^2)$ is the closure of the polynomials in $L^2(\mathbb{T}^2)$. Let P_2 denote the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$, and P denote the orthogonal projection from $L^2(\mathbb{T}^2)$ onto $H^2(\mathbb{D}^2)$.

For a function $f \in L^2(\mathbb{T})$, we define $f^*(w) = \overline{f(\overline{w})}$, $\overline{f}(w) = \overline{f(w)}$ and $\widetilde{f}(w) = f(\overline{w})$, respectively. If $f \in L^2(\mathbb{T}^2)$, we define $f^*(w_1, w_2) = \overline{f(\overline{w}_1, \overline{w}_2)}$, $\overline{f}(w_1, w_2) = \overline{f(w_1, w_2)}$ and $\widetilde{f}(w_1, w_2) = f(\overline{w}_1, \overline{w}_2)$. U is the operator on $L^2(\mathbb{T}^2)$ defined by

$$Uh(w_1, w_2) = \overline{w}_1 \overline{w}_2 \widetilde{h}(w_1, w_2).$$

Clearly, U is a unitary operator on $L^2(\mathbb{T}^2)$.

Definition 1.1 *For $f \in L^\infty(\mathbb{T}^2)$, the Toeplitz operator T_f and Hankel operator H_f with symbol f are defined respectively by*

$$T_f h = P(fh), \quad H_f h = P(U(fh))$$

for functions $h \in H^2(\mathbb{D}^2)$.

Then it is easy to get that both T_f and H_f are bounded linear operators on $H^2(\mathbb{D}^2)$.

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Similarly, we can also define the Toeplitz operator T_f and the Hankel operator H_f on $H^2(\mathbb{D})$.

The general problem that we are interested in is the following: when Toeplitz and Hankel (or two Toeplitz) operators commute, what is the relationship between their symbols? Knowing commutativity of two operators often gives an idea of what these operators look like; conversely, trying to determine commutativity of Toeplitz and Hankel (or two Toeplitz) operators often leads to interesting problems in analysis. In the setting of the classical Hardy space H^2, Brown and Halmos [1] characterized commutativity of Toeplitz operators on $H^2(\mathbb{D})$. Martínez-Avendaño [2] completely solved the problem of when a Hankel operator commutes with a Toeplitz operator, and proved that H_g and T_f commute if and only if one of the following three conditions is satisfied: (i) g is in H^∞; (ii) f and $\widetilde{f}$ are in H^∞; (iii) There exists a nonzero constant λ such that $f+\lambda g, f+\widetilde{f}$ and $f\widetilde{f}$ are in H^∞. Guo and Zheng [3] characterized when a Hankel operator and a Toeplitz operator have a compact commutator.

On the Bergman space of the unit disk, the first complete result was obtained by Axler and Čučković [4] who characterized commuting Toeplitz operators with harmonic symbols. Stroethoff [5] extended their results to essentially commuting Toeplitz operators, and Axler, Čučković and Rao [6] subsequently showed that if two Toeplitz operators commute and the symbol of one of them is analytic and nonconstant, then the other one is also analytic. Čučković and Rao [7] studied Toeplitz operators that commute with Toeplitz operators with monomial symbols.

In several variables, the situation is much more complicated. Gu and Zheng [8] mainly characterized when the semi-commutator $T_fT_g - T_{fg}$ of two Toeplitz operators T_f and T_g on the Hardy space of the bidisk is zero. Zheng [9] made significant contributions in the study of commuting Toeplitz operators on the Bergman space of the unit ball in $\mathbb{C}^n$ with pluriharmonic symbols. Lee [10] studied weighted cases. Lu [11] characterized commuting Toeplitz operators on the Bergman space of the bidisk with $H^\infty(\mathbb{D}^2)+\overline{H^\infty(\mathbb{D}^2)}$ symbols. Choe, Koo and Lee [12] obtained characterization of (essential) commuting Toeplitz operators with pluriharmonic symbols on the Bergman sapce of the polydisk. Recently, on the Hardy space of the bidisk, Lee [13] gave a necessary and sufficient condition for a bounded symbol of a Toeplitz operator that commutes with another Toeplitz operator whose symbol is a certain type of bounded symbol.

Motivated by Martínez-Avendaño [2] and Guo and Zheng [3], it is natural to ask about the relationships between Toeplitz and Hankel operators on the Hardy space of higher dimensional polydisks, but little is known concerning the commutativity of Hankel and Toeplitz operators and many problems still remain open on the polydisk.

In this paper, we investigate the commutativity of Hankel operators and Toeplitz operators on the Hardy space of the bidisk and completely characterize when the Toeplitz operator T_f with a certain type of symbol commutes with the Hankel operator H_g with some special symbol.

To state our main result, we introduce some notations.

Throughout this paper, let $\mathbb{Z}$ denote the set of all integers, $\mathbb{Z}_+$ denote the set of all nonnegative integers, $\mathbb{Z}_-$ denote the set of all negative integers and $\mathbb{N}$ denote the set of all positive integers. As in [14] we can consider multiple Fourier series on the bitorus $\mathbb{T}^2$. The multiple Fourier series on the bitorus $\mathbb{T}^2$ can be viewed as the Fourier transformation on $L^1(\mathbb{T}^2)$. For f in $L^1(\mathbb{T}^2)$, the

Fourier transformation of f on $\mathbb{T}\times\mathbb{T}$ is given by :

$$f_m = f_{m_1,m_2} = (\frac{1}{2\pi})^2 \int_0^{2\pi}\int_0^{2\pi} f(e^{i\theta_1}, e^{i\theta_2})e^{i(m,\theta)}\mathrm{d}\theta_1\mathrm{d}\theta_2,$$

where $m=(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}$, $\theta=(\theta_1,\theta_2)$ and $(m,\theta)=m_1\theta_1+m_2\theta_2$. By Theorem 1.7 in [14], the Fourier transformation is injective, i.e., if $f\in L^1(\mathbb{T}^2)$ and $f_{m_1,m_2}=0$ for all $m\in\mathbb{Z}\times\mathbb{Z}$, then $f\equiv 0$.

Using multiple Fourier series, we have

$$L^2(\mathbb{T}^2) = \Big\{ f : f = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} f_{m_1,m_2}e^{i(m,\theta)} = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} f_{m_1,m_2}z_1^{m_1}z_2^{m_2}, \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} |f_{m_1,m_2}|^2 < +\infty \Big\}$$

$$H^2(\mathbb{D}^2) = \Big\{ h : h = \sum_{(m_1,m_2)\in\mathbb{Z}_+\times\mathbb{Z}_+} f_{m_1,m_2}e^{i(m,\theta)} = \sum_{(m_1,m_2)\in\mathbb{Z}_+\times\mathbb{Z}_+} f_{m_1,m_2}z_1^{m_1}z_2^{m_2}, \sum_{(m_1,m_2)\in\mathbb{Z}_+\times\mathbb{Z}_+} |f_{m_1,m_2}|^2 < +\infty \Big\}$$

and

$$Pf = \sum_{(m_1,m_2)\in\mathbb{Z}_+\times\mathbb{Z}_+} f_{m_1,m_2}z_1^{m_1}z_2^{m_2}, \quad \text{for} \quad f = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} f_{m_1,m_2}z_1^{m_1}z_2^{m_2} \in L^2(\mathbb{T}^2).$$

The multiple Fourier series of $f\in L^2(\mathbb{T}^2)$ can be written as follows

$$f = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} f_{m_1,m_2}z_1^{m_1}z_2^{m_2} = f_{++}(z)+f_{+-}(z)+f_{-+}(z)+f_{--}(z),$$

where

$$f_{++}(z) = \sum_{m\in\mathbb{Z}_+\times\mathbb{Z}_+} f_m z^m, f_{+-}(z) = \sum_{m\in\mathbb{Z}_+\times\mathbb{Z}_-} f_m z^m,$$

$$f_{-+}(z) = \sum_{m\in\mathbb{Z}_-\times\mathbb{Z}_+} f_m z^m, f_{--}(z) = \sum_{m\in\mathbb{Z}_-\times\mathbb{Z}_-} f_m z^m,$$

and for example, $m=(m_1,m_2)\in\mathbb{Z}_+\times\mathbb{Z}_-$ means that $m_1\in\mathbb{Z}_+$ and $m_2\in\mathbb{Z}_-$, and z^m means the product $z_1^{m_1}z_2^{m_2}$.

2. The equation $T_f^* H_g = H_g T_{f^*}$

In this section we will investigate when the equation $T_f^* H_g = H_g T_{f^*}$ holds. Before doing this, we discuss some properties of the Toeplitz and Hankel operators.

Lemma 2.1 *Let $f\in L^\infty(\mathbb{T}^2)$ and suppose $f(z_1,z_2)=\sum_{i=-\infty}^{+\infty} f_i(z_2)z_1^i$ is the Fourier series expansion of f with respect to z_1-variable. Then $f_i(z_2)\in L^\infty(\mathbb{T})$.*

Proof According to the supposition, we know

$$f_j(z_2) = \int_{\mathbb{T}} f(z_1,z_2)\overline{z}_1^j \mathrm{d}\sigma_1(z_1), j\in\mathbb{Z}.$$

Then $|f_j(z_2)| = |\int_{\mathbb{T}}(\sum_{i=-\infty}^{+\infty} f_i(z_2)z_1^i)\overline{z}_1^j \mathrm{d}\sigma_1(z_1)| \le \|f\|_{L^\infty(\mathbb{T}^2)}$, that is, $f_i(z_2) \in L^\infty(\mathbb{T})$.

Lemma 2.2 *If $f \in L^\infty(\mathbb{T}^2)$, then $H_f^* = H_{f^*}$ and $T_f^* = T_{\overline{f}}$.*

Proof Suppose $f = \sum_{(m_1,m_2)\in\mathbb{Z}\times\mathbb{Z}} f_{m_1,m_2} z_1^{m_1} z_2^{m_2}$. For m_1, m_2, n_1 and n_2 in $\mathbb{Z}_+$, we have

$$\begin{aligned}(H_f^*(z_1^{n_1}z_2^{n_2}), z_1^{m_1}z_2^{m_2}) &= (z_1^{n_1}z_2^{n_2}, H_f(z_1^{m_1}z_2^{m_2}))\\ &= (z_1^{n_1}z_2^{n_2}, P(\overline{z_1z_2}\sum_{(j_1,j_2)\in\mathbb{Z}\times\mathbb{Z}} f_{j_1,j_2}\overline{z_1}^{j_1+m_1}\overline{z_2}^{j_2+m_2}))\\ &= (z_1^{n_1}z_2^{n_2}, \sum_{(j_1,j_2)\in\mathbb{Z}\times\mathbb{Z}} f_{j_1,j_2}\overline{z_1}^{j_1+m_1+1}\overline{z_2}^{j_2+m_2+1})\\ &= \overline{f}_{-n_1-m_1-1,-n_2-m_2-1}\end{aligned}$$

and

$$\begin{aligned}(H_{f^*}(z_1^{n_1}z_2^{n_2}), z_1^{m_1}z_2^{m_2}) &= (P(U(f^*z_1^{n_1}z_2^{n_2})), z_1^{m_1}z_2^{m_2})\\ &= \overline{f}_{-n_1-m_1-1,-n_2-m_2-1},\end{aligned}$$

i.e.,

$$(H_f^*(z_1^{n_1}z_2^{n_2}), z_1^{m_1}z_2^{m_2}) = (H_{f^*}(z_1^{n_1}z_2^{n_2}), z_1^{m_1}z_2^{m_2}), \quad \text{for any } \ m_1, m_2, n_1 \text{ and } n_2 \text{ in } Z_+,$$

so we get $H_f^* = H_{f^*}$.

For any $g, h \in H^2(\mathbb{D}^2)$, we have

$$(T_f^* g, h) = (g, T_f h) = \int_{\mathbb{T}^2} g(z)\overline{f(z)h(z)}\mathrm{d}\sigma(z)$$

and

$$(T_{\overline{f}} g, h) = (\overline{f} g, h) = \int_{\mathbb{T}^2} g(z)\overline{f(z)h(z)}\mathrm{d}\sigma(z),$$

so $T_f^* = T_{\overline{f}}$. □

Lemma 2.3 *Suppose $f \in L^\infty(\mathbb{T}^2)$ and*

$$f = \sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} f_{i,j} z_1^i z_2^j \quad \textit{for} \quad z = (z_1, z_2) \in \mathbb{T}^2.$$

Then $H_f \ne 0$ if and only if there exist n_1 and n_2 in $\mathbb{Z}_-$ such that $f_{n_1,n_2} \ne 0$.

Proof Since for m_1, m_2 in $\mathbb{Z}_+$,

$$\begin{aligned}H_f(z_1^{m_1}z_2^{m_2}) &= P(U(fz_1^{m_1}z_2^{m_2}))\\ &= P(\overline{z_1z_2}\sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} f_{i,j}\overline{z}_1^i\overline{z}_2^j\overline{z}_1^{m_1}\overline{z}_2^{m_2})\\ &= P(\sum_{(i,j)\in\mathbb{Z}\times\mathbb{Z}} f_{i,j} z_1^{-i-m_1-1} z_2^{-j-m_2-1})\\ &= \sum_{(i,j)\in\mathbb{Z}_+\times\mathbb{Z}_+} f_{-i-m_1-1,-j-m_2-1} z_1^i z_2^j,\end{aligned}$$

we can conclude that $H_f \ne 0$ if and only if there are n_1 and n_2 in $\mathbb{Z}_-$ such that $f_{n_1,n_2} \ne 0$. □

Now we characterize when the equation $T_f^* H_g = H_g T_{f^*}$ holds.

Theorem 2.1 *Let $f, g \in L^\infty(\mathbb{T}^2)$, and*

$$f = \sum_{i=-\infty}^{+\infty} f_i(z_2) z_1^i, \quad g = \sum_{i=-\infty}^{+\infty} g_i(z_2) z_1^i.$$

Then $T_f^* H_g = H_g T_{f^*}$ if and only if for all $k, l \in \mathbb{Z}_+$,

$$\sum_{j=0}^{+\infty} T_{f_{j-l}}^* H_{g_{-j-k-1}} = \sum_{j=0}^{+\infty} H_{g_{-l-j-1}} T_{f_{j-k}^*},$$

where T_{f_i} and H_{g_j} are Toeplitz and Hankel operators on the $H^2(\mathbb{T})$, respectively.

Proof Since for k, l, α and β in $\mathbb{Z}_+$,

$$\begin{aligned}
(T_f^* H_g z_1^k z_2^\alpha, z_1^l z_2^\beta) &= (H_g z_1^k z_2^\alpha, T_f z_1^l z_2^\beta) \\
&= (P(U(\sum_{i=-\infty}^{+\infty} g_i(z_2) z_1^{k+i} z_2^\alpha)), P(\sum_{j=-\infty}^{+\infty} f_j(z_2) z_1^{l+j} z_2^\beta)) \\
&= (\sum_{i=-\infty}^{+\infty} g_i(\overline{z}_2) \overline{z}_1^{k+i+1} \overline{z}_2^{\alpha+1}, P(\sum_{j=-\infty}^{+\infty} f_j(z_2) z_1^{l+j} z_2^\beta)) \\
&= (\sum_{i=-\infty}^{+\infty} g_{-l-i-k-1}(\overline{z}_2) z_1^{i+l} z_2^{-\alpha-1}, P(\sum_{j=-\infty}^{+\infty} f_j(z_2) z_1^{l+j} z_2^\beta)) \\
&= \sum_{i=-l}^{+\infty} (g_{-l-i-k-1}(\overline{z}_2) z_2^{-\alpha-1}, P_2(f_i(z_2) z_2^\beta)) \\
&= ((\sum_{i=0}^{+\infty} T_{f_{i-l}}^* H_{g_{-i-k-1}}) z_2^\alpha, z_2^\beta)
\end{aligned}$$

and

$$(H_g T_{f^*} z_1^k z_2^\alpha, z_1^l z_2^\beta) = (T_{f^*} z_1^k z_2^\alpha, H_g^* z_1^l z_2^\beta) = ((\sum_{i=0}^{+\infty} H_{g_{-l-i-1}} T_{f_{i-k}^*}) z_2^\alpha, z_2^\beta),$$

we have

$$(T_f^* H_g z_1^k z_2^\alpha, z_1^l z_2^\beta) = (H_g T_{f^*} z_1^k z_2^\alpha, z_1^l z_2^\beta)$$

if and only if

$$((\sum_{j=0}^{+\infty} T_{f_{j-l}}^* H_{g_{-j-k-1}}) z_2^\alpha, z_2^\beta) = ((\sum_{j=0}^{+\infty} H_{g_{-l-j-1}} T_{f_{j-k}^*}) z_2^\alpha, z_2^\beta).$$

Hence we can conclude that

$$T_f^* H_g = H_g T_{f^*},$$

if and only if for all k, $l \in \mathbb{Z}_+$,

$$\sum_{j=0}^{+\infty} T_{f_{j-l}}^* H_{g_{-j-k-1}} = \sum_{j=0}^{+\infty} H_{g_{-l-j-1}} T_{f_{j-k}^*}. \quad \square$$

In the light of Theorem 2.1, we can get the following two results.

Corollary 2.1 *Let $f, g \in L^\infty(\mathbb{T}^2)$ as in Theorem 2.1. If $T_f^* H_g = H_g T_{f^*}$, then we have*

$$T^*_{f_{-(l+1)}} H_{g_{-k}} = -H_{g_{-(l+1)}} T_{f^*_{-k}}$$

for k in $\mathbb{N}$ and l in $\mathbb{Z}_+$.

Proof If $T_f^* H_g = H_g T_{f^*}$, then

$$(T_f^* H_g z_1^{k-1} z_2^\alpha, z_1^{l+1} z_2^\beta) = (H_g T_{f^*} z_1^{k-1} z_2^\alpha, z_1^{l+1} z_2^\beta), \tag{1}$$

$$(T_f^* H_g z_1^k z_2^\alpha, z_1^l z_2^\beta) = (H_g T_{f^*} z_1^k z_2^\alpha, z_1^l z_2^\beta) \tag{2}$$

for k in $\mathbb{N}$ and l, α, β in $\mathbb{Z}_+$.

From (1) and (2), we get

$$\sum_{i=-(l+1)}^{+\infty} (g_{-l-i-k-1}(\overline{z}_2) z_2^{-\alpha-1}, P_2(f_i(z_2) z_2^\beta)) = \sum_{i=-(k-1)}^{+\infty} (P_2(\overline{f}_i(\overline{z}_2) z_2^\alpha), \overline{g}_{-l-i-k-1}(z_2) z_2^{-\beta-1}), \tag{3}$$

$$\sum_{i=-l}^{+\infty} (g_{-l-i-k-1}(\overline{z}_2) z_2^{-\alpha-1}, P_2(f_i(z_2) z_2^\beta)) = \sum_{i=-k}^{+\infty} (P_2(\overline{f}_i(\overline{z}_2) z_2^\alpha), \overline{g}_{-l-i-k-1}(z_2) z_2^{-\beta-1}). \tag{4}$$

It is trivial to get

$$(g_{-k}(\overline{z}_2) z_2^{-\alpha-1}, P_2(f_{-(l+1)}(z_2) z_2^\beta)) = -(P_2(\overline{f}_{-k}(\overline{z}_2) z_2^\alpha), \overline{g}_{-l-1}(z_2) z_2^{-\beta-1}),$$

i.e.,

$$(T^*_{f_{-(l+1)}} H_{g_{-k}} z_2^\alpha, z_2^\beta) = (-H_{g_{-(l+1)}} T_{f^*_{-k}} z_2^\alpha, z_2^\beta). \tag{5}$$

Since the equation (5) holds for all $\alpha, \beta \in \mathbb{Z}_+$, we obtain the desired conclusion. □

Corollary 2.2 *Let f, $g \in L^\infty(\mathbb{T}^2)$, $f = \sum_{i=-\infty}^{+\infty} f_i(z_2) z_1^i$ and $g = g_j(z_2) z_1^j$, where $j \in \mathbb{Z}_-$. Suppose $f = f_{++} + f_{--}$ and $H_g \neq 0$. Then $T_f^* H_g = H_g T_{f^*}$ if and only if $f_{--} = 0$.*

Proof Since for k, l, α and β in $\mathbb{Z}_+$,

$$(T_f^* H_g z_1^k z_2^\alpha, z_1^l z_2^\beta) = (H_g z_1^k z_2^\alpha, T_f z_1^l z_2^\beta) = (g_j(\overline{z}_2) \overline{z}_2^{\alpha+1} \overline{z}_1^{j+k+1}, \sum_{i=-l}^{+\infty} P_2(f_i(z_2) z_2^\beta) z_1^{l+i}),$$

we get

(a) $(T_f^* H_g z_1^k z_2^\alpha, z_1^l z_2^\beta) = 0$, if $k > -j-1$;

(b) $(T_f^* H_g z_1^k z_2^\alpha, z_1^l z_2^\beta) = (\overline{z}_2^{\alpha+1} g_j(\overline{z}_2),\ P_2(f_{-j-k-l-1}(z_2) z_2^\beta))$, if $0 \le k \le -j-1$.

Similarly, we get

(c) $(H_g T_{f^*} z_1^k z_2^\alpha, z_1^l z_2^\beta) = 0$, if $l > -j-1$;

(d) $(H_g T_{f^*} z_1^k z_2^\alpha, z_1^l z_2^\beta) = (P_2(\overline{f}_{-j-k-l-1}(\overline{z}_2) z_2^\alpha),\ \overline{z}_2^{\beta+1} \overline{g}_j(z_2))$, if $0 \le l \le -j-1$.

If $T_f^* H_g = H_g T_{f^*}$, then we obtain

(i) If $0 \le k \le -j-1$ and $l > -j-1$,

$$(\overline{z}_2^{\alpha+1} g_j(\overline{z}_2), P_2(f_{-j-k-l-1}(z_2) z_2^\beta)) = 0;$$

(ii) If $0 \le l \le -j-1$ and $k > -j-1$,

$$(P_2(\overline{f}_{-j-k-l-1}(\overline{z}_2)z_2^{\alpha}), \overline{z}_2^{\beta+1}\overline{g}_j(z_2)) = 0;$$

(iii) If $0 \le k \le -j-1$ and $0 \le l \le -j-1$,

$$(\overline{z}_2^{\alpha+1}g_j(\overline{z}_2), P_2(f_{-j-k-l-1}(z_2)z_2^{\beta})) = (P_2(\overline{f}_{-j-k-l-1}(\overline{z}_2)z_2^{\alpha}), \overline{z}_2^{\beta+1}\overline{g}_j(z_2)).$$

Since the equations (1), (2) and (3) hold for all $\alpha, \beta \in \mathbb{Z}_+$, we have $T^*_{f_0}H_{g_j} = H_{g_j}T_{f_0^*}$ and $T^*_{f_i}H_{g_j} = H_{g_j}T_{f_i^*} = 0$ where $i \in \mathbb{Z}_-$.

Since $f = \sum_{i=-\infty}^{+\infty} f_i(z_2)z_1^i$, $f = f_{++} + f_{--}$ and $H_g \ne 0$, we have

(iv) $\overline{f_i} \in H^\infty(\mathbb{D})$ for all $i \in \mathbb{Z}_-$,

(v) $H_{g_j} \ne 0$ and there exists $\alpha_0 \in \mathbb{Z}_+$ such that $H_{g_j}z_2^{\alpha_0} \ne 0$.

By Lemma 2.2 and the fact that $T^*_{f_i}H_{g_j} = 0$, we get $\overline{f_i} \cdot (H_{g_j}z_2^{\alpha_0}) = 0$. Since $\overline{f_i} \in H^\infty(\mathbb{D})$ and $H_{g_j}z_2^{\alpha_0} \ne 0$, using the fact in [1] that a non-zero analytic function cannot vanish on a set of positive measure, we obtain $\overline{f_i} = 0$ for all $i \in \mathbb{Z}_-$, that is, $f_i = 0$ for all $i \in \mathbb{Z}_-$. So we have $f_{--} = 0$.

Suppose $f_{--} = 0$. Since $f = f_{++} + f_{--}$, we have $f \in H^\infty(\mathbb{D}^2)$. Since for $n_1, n_2, m_1, m_2 \in \mathbb{Z}_+$,

$$\begin{aligned}(T^*_f H_g z_1^{n_1}z_2^{n_2}, z_1^{m_1}z_2^{m_2}) &= (H_g z_1^{n_1}z_2^{n_2}, T_f z_1^{m_1}z_2^{m_2}) = (\overline{z}_1\overline{z}_2\widetilde{g}\overline{z}_1^{n_1}\overline{z}_2^{n_2}, fz_1^{m_1}z_2^{m_2}) \\ &= (\overline{z}_1^{n_1+1}\overline{z}_2^{n_2+1}\widetilde{g}\overline{f}, z_1^{m_1}z_2^{m_2})\end{aligned}$$

and

$$(H_g T_{f^*} z_1^{n_1}z_2^{n_2}, z_1^{m_1}z_2^{m_2}) = (\overline{z}_1^{n_1+1}\overline{z}_2^{n_2+1}\widetilde{g}\overline{f}, z_1^{m_1}z_2^{m_2}),$$

we obtain

$$(T^*_f H_g z_1^{n_1}z_2^{n_2}, z_1^{m_1}z_2^{m_2}) = (H_g T_{f^*} z_1^{n_1}z_2^{n_2}, z_1^{m_1}z_2^{m_2})$$

for all $n_1, n_2, m_1, m_2 \in \mathbb{Z}_+$, that is,

$$T^*_f H_g = H_g T_{f^*}. \quad \square$$

3. Commutativity of Toeplitz and Hankel operators

In this section we characterize when a Toeplitz operator T_f commutes with a Hankel operator H_g. we are not able to obtain a characterization when two symbols are all arbitrary bounded functions. In the course of the proof of the main Theorem 3.4, we will make use of some known results obtained in [2] for commutativity of Toeplitz and Hankel operators on the Hardy space of the unit disk.

Theorem 3.1 *Let $f, g \in L^\infty(\mathbb{T}^2)$, $f = f_{++} + f_{--}$, $f = \widetilde{f}$, $g = g_j(z_2)z_1^j$ and $H_g \ne 0$ where j is in $\mathbb{Z}_-$. If $H_gT_f = T_fH_g$, then f is a constant.*

Proof Using $f = \widetilde{f}$, we have $(\overline{f})^* = f$. By Lemma 2.2, $T_f = T^*_{\overline{f}}$. Combining these two facts together, we see that $H_gT_f = T_fH_g$ is equivalent to $T^*_{\overline{f}}H_g = H_gT_{\overline{f}^*}$. By Corollary 2.2, we obtain f_{++} is a constant. Applying $f = \widetilde{f}$ again, it follows that f is a constant. $\square$

Theorem 3.2 *Let $f, g \in L^\infty(\mathbb{T}^2)$. Then $T_f H_g = H_g T_f$ if and only if $T_{\widetilde{f}} H_g = H_g T_{\widetilde{f}}$.*

Proof Let us define the anti-unitary involution V on H^2 by $Vf = f^*$. It is easy to check that $VT_fV = T_{f^*}$ for $f \in L^\infty(\mathbb{T}^2)$, and $VH_gV = H_g^*$ for any Hankel operator H_g. Clearly, $V^2 = I$. Thus $T_f H_g = H_g T_f$ implies that $VT_fVVH_gV = VH_gVVT_fV$, which in turn implies that $T_{f^*}H_g^* = H_g^* T_{f^*}$. Taking adjoints, we get $H_g T_{\widetilde{f}} = T_{\widetilde{f}} H_g$. Applying this to $\widetilde{f}$, it follows that $T_{\widetilde{f}} H_g = H_g T_{\widetilde{f}}$ implies $T_f H_g = H_g T_f$. □

In view of the last two theorems, we can get the following useful corollary.

Corollary 3.1 *Let $f, g \in L^\infty(\mathbb{T}^2)$, $f = f_{++} + f_{--}$, $g = g_j(z_2) z_1^j$ and $H_g \neq 0$ where j is in $\mathbb{Z}_-$. If $T_f H_g = H_g T_f$, then $f + \widetilde{f}$ is a constant.*

Proof Since $T_f H_g = H_g T_f$, we get $T_{\widetilde{f}} H_g = H_g T_{\widetilde{f}}$ with the help of Theorem 3.2. Therefore $T_{f+\widetilde{f}} H_g = H_g T_{f+\widetilde{f}}$. Using Theorem 3.1, we obtain $f + \widetilde{f}$ is a constant function. □

Next, we will discuss the commutativity of a Toeplitz operator T_f and a Hankel operator H_g.

Theorem 3.3 *Let $f, g \in L^\infty(\mathbb{T}^2)$, where $f = \sum_{i=-\infty}^{+\infty} f_i(z_2) z_1^i$ and $g = \sum_{i=-\infty}^{+\infty} g_i(z_2) z_1^i$. Then $T_f H_g = H_g T_f$ if and only if*

$$\sum_{j=0}^{+\infty} H_{g_{-j-n_1-1}} T_{f_{j-m_1}} = \sum_{j=0}^{+\infty} T_{f_{n_1-j}} H_{g_{-j-m_1-1}}$$

for all $m_1, n_1 \in \mathbb{Z}_+$.

Proof For m_1, m_2, n_1 and n_2 in $\mathbb{Z}_+$, we define

$$\begin{aligned} C_{m_1,m_2,n_1,n_2} &= (H_g T_f z_1^{m_1} z_2^{m_2}, z_1^{n_1} z_2^{n_2}) = (T_f z_1^{m_1} z_2^{m_2}, H_g^* z_1^{n_1} z_2^{n_2}) \\ &= (P(\sum_{j=-\infty}^{+\infty} f_j(z_2) z_1^{m_1+j} z_2^{m_2}), \overline{z}_1 \overline{z}_2 (\sum_{i=-\infty}^{+\infty} \overline{g}_i(z_2) \overline{z}_1^{i+n_1} \overline{z}_2^{n_2})) \\ &= (\sum_{j=0}^{+\infty} P_2(f_{j-m_1}(z_2) z_2^{m_2}) z_1^j, \sum_{i=-\infty}^{+\infty} \overline{z}_2^{n_2+1} \overline{g}_{-i-n_1-1}(z_2) z_1^j) \\ &= \sum_{j=0}^{+\infty} (P_2(f_{j-m_1}(z_2) z_2^{m_2}), \overline{z}_2^{n_2+1} \overline{g}_{-j-n_1-1}(z_2)) \\ &= ((\sum_{j=0}^{+\infty} H_{g_{-j-n_1-1}} T_{f_{j-m_1}}) z_2^{m_2}, z_2^{n_2}) \end{aligned}$$

and

$$D_{m_1,m_2,n_1,n_2} = (T_f H_g z_1^{m_1} z_2^{m_2}, z_1^{n_1} z_2^{n_2}) = ((\sum_{j=0}^{+\infty} T_{f_{n_1-j}} H_{g_{-j-m_1-1}}) z_2^{m_2}, z_2^{n_2}).$$

Then we get for all $m_1, m_2, n_1, n_2 \in \mathbb{Z}_+$,

$$C_{m_1,m_2,n_1,n_2} = D_{m_1,m_2,n_1,n_2}$$

if and only if for all $m_1, m_2, n_1, n_2 \in \mathbb{Z}_+$,

$$((\sum_{j=0}^{+\infty} H_{g_{-j-n_1-1}} T_{f_{j-m_1}}) z_2^{m_2}, z_2^{n_2}) = ((\sum_{j=0}^{+\infty} T_{f_{n_1-j}} H_{g_{-j-m_1-1}}) z_2^{m_2}, z_2^{n_2}).$$

Hence we obtain

$$T_f H_g = H_g T_f$$

if and only if for all $n_1, m_1 \in \mathbb{Z}_+$,

$$\sum_{j=0}^{+\infty} H_{g_{-j-n_1-1}} T_{f_{j-m_1}} = \sum_{j=0}^{+\infty} T_{f_{n_1-j}} H_{g_{-j-m_1-1}}. \quad \square$$

By means of Theorem 3.3, we have the following discussions.

Corollary 3.2 *Let $f, g \in L^\infty(\mathbb{T}^2)$, where $f = \sum_{i=-\infty}^{+\infty} f_i(z_2) z_1^i$ and $g = \sum_{i=-\infty}^{+\infty} g_i(z_2) z_1^i$. If $T_f H_g = H_g T_f$, then we have*

$$H_{g_{-n-1}} T_{f_{-m}} + T_{f_{n+1}} H_{g_{-m}} = 0$$

for m in $\mathbb{N}$ and n in $\mathbb{Z}_+$.

Proof For m in $\mathbb{N}$ and n in $\mathbb{Z}_+$, we define

$$C_{n,m} = \sum_{j=0}^{+\infty} H_{g_{-j-n-1}} T_{f_{j-m}}$$

and

$$D_{n,m} = \sum_{j=0}^{+\infty} T_{f_{n-j}} H_{g_{-j-m-1}}.$$

Since $T_f H_g = H_g T_f$, by Theorem 3.3 it follows that $C_{n,m} = D_{n,m}$. In fact, we have the following expressions for $C_{n,m}$ and $D_{n,m}$.

$$\begin{aligned} C_{n,m} &= H_{g_{-n-1}} T_{f_{-m}} + \sum_{j=1}^{+\infty} H_{g_{-j-n-1}} T_{f_{j-m}} = H_{g_{-n-1}} T_{f_{-m}} + \sum_{s=0}^{+\infty} H_{g_{-(s+1)-n-1}} T_{f_{(s+1)-m}} \\ &= H_{g_{-n-1}} T_{f_{-m}} + \sum_{s=0}^{+\infty} H_{g_{-s-(n+1)-1}} T_{f_{s-(m-1)}} = H_{g_{-n-1}} T_{f_{-m}} + C_{n+1,m-1} \end{aligned}$$

and

$$\begin{aligned} D_{n+1,m-1} &= \sum_{j=0}^{+\infty} T_{f_{(n+1)-j}} H_{g_{-j-(m-1)-1}} = T_{f_{(n+1)}} H_{g_{-m}} + \sum_{j=1}^{+\infty} T_{f_{(n+1)-j}} H_{g_{-j-(m-1)-1}} \\ &= T_{f_{(n+1)}} H_{g_{-m}} + \sum_{s=0}^{+\infty} T_{f_{(n+1)-(s+1)}} H_{g_{-(s+1)-(m-1)-1}} \\ &= T_{f_{(n+1)}} H_{g_{-m}} + \sum_{s=0}^{+\infty} T_{f_{n-s}} H_{g_{-s-m-1}} = T_{f_{(n+1)}} H_{g_{-m}} + D_{n,m}. \end{aligned}$$

Since $C_{n,m} = D_{n,m}$, it follows that

$$H_{g_{-n-1}} T_{f_{-m}} + T_{f_{n+1}} H_{g_{-m}} = 0,$$

for m in $\mathbb{N}$ and n in $\mathbb{Z}_+$. $\Box$

Corollary 3.3 *Let $f, g \in L^\infty(\mathbb{T}^2)$, $f = \sum_{i=-\infty}^{+\infty} f_i(z_2)z_1^i$ and $g = g_j(z_2)z_1^j$, where j is in $\mathbb{Z}_-$. Then $T_f H_g = H_g T_f$ if and only if*

(i) $H_{g_j} T_{f_{-i}} = T_{f_i} H_{g_j} = 0$, $i > 0$;

(ii) $H_{g_j} T_{f_{-i}} = T_{f_i} H_{g_j}$, $j + 1 \le i \le 0$.

Proof Define

$$\begin{aligned} C_{m_1,m_2,n_1,n_2} &= (H_g T_f z_1^{m_1} z_2^{m_2}, z_1^{n_1} z_2^{n_2}) \\ &= (P(\sum_{i=-\infty}^{+\infty} f_i(z_2) z_1^{i+m_1} z_2^{m_2}), \overline{z}_1 \overline{z}_2 (\overline{g}_j(z_2) \overline{z}_1^j \overline{z}_1^{n_1} \overline{z}_2^{n_2})) \\ &= (\sum_{i=-m_1}^{+\infty} P_2(f_i(z_2) z_2^{m_2}) z_1^{i+m_1}, \overline{g}_j(z_2) \overline{z}_2^{n_2+1} \overline{z}_1^{j+n_1+1}) \end{aligned}$$

for m_1, m_2, n_1, n_2 in $\mathbb{Z}_+$. Then we have

(a) $C_{m_1,m_2,n_1,n_2} = 0$, if $n_1 > -j-1$.

(b) $C_{m_1,m_2,n_1,n_2} = (P_2(f_{-j-m_1-n_1-1}(z_2) z_2^{m_2}), \overline{g}_j(z_2) \overline{z}_2^{n_2+1})$, if $0 \le n_1 \le -j-1$.

Similarly define

$$D_{m_1,m_2,n_1,n_2} = (T_f H_g z_1^{m_1} z_2^{m_2}, z_1^{n_1} z_2^{n_2}) = (g_j(\overline{z}_2) \overline{z}_2^{m_2+1} \overline{z}_1^{j+m_1+1}, \sum_{i=n_1}^{-\infty} P_2(\overline{f}_i(z_2) z_2^{n_2}) z_1^{n_1-i}),$$

and we have

(c) $D_{m_1,m_2,n_1,n_2} = 0$, if $m_1 > -j-1$,

(d) $D_{m_1,m_2,n_1,n_2} = (g_j(\overline{z}_2) \overline{z}_2^{m_2+1}, P_2(\overline{f}_{j+m_1+n_1+1}(z_2) z_2^{n_2}))$, if $0 \le m_1 \le -j-1$.

Suppose $T_f H_g = H_g T_f$. Since $T_f H_g = H_g T_f$ if and only if for all $m_1, m_2, n_1, n_2 \in \mathbb{Z}_+$, $C_{m_1,m_2,n_1,n_2} = D_{m_1,m_2,n_1,n_2}$, it follows that

(i) $H_{g_j} T_{f_{-j-m_1-n_1-1}} = 0$, if $0 \le n_1 \le -j-1$ and $m_1 \ge -j-1$,

(ii) $T_{f_{j+m_1+n_1+1}} H_{g_j} = 0$, if $0 \le m_1 \le -j-1$ and $n_1 \ge -j-1$,

(iii) $T_{f_{j+m_1+n_1+1}} H_{g_j} = H_{g_j} T_{f_{-j-m_1-n_1-1}}$, if $0 \le m_1 \le -j-1$ and $0 \le n_1 \le -j-1$.

Combining all these facts, we have

$$H_{g_j} T_{f_{-i}} = T_{f_i} H_{g_j} = 0$$

for $i \in \mathbb{N}$, and

$$H_{g_j} T_{f_{-i}} = T_{f_i} H_{g_j}$$

for $j + 1 \le i \le 0$.

It is easy to get that the converse is true. $\Box$

Finally, we will discuss the relationship between f and g under the condition of $g = g_j(z_2)z_1^j$.

Theorem 3.4 *Let $f, g \in L^\infty(\mathbb{T}^2)$, $f = \sum_{i=-\infty}^{+\infty} f_i(z_2)z_1^i$, $g = g_j(z_2)z_1^j$ and $f = f_{++} + f_{--}$, where $j \in \mathbb{Z}_-$. Then $T_f H_g = H_g T_f$ if and only if one of the following conditions is satisfied:*

(i) $g_j(z_2)$ is in $H^\infty(\mathbb{D})$.

(ii) f *and* $\widetilde{f}$ *are in* $H^\infty(\mathbb{D}^2)$.

(iii) $f_i(z_2) = 0$ *for all nonzero integers* i *and there exists a nonzero constant* λ *such that* $f_0 + \lambda g_j$, $f_0 + \widetilde{f_0}$ *and* $f_0 \cdot \widetilde{f_0}$ *are in* $H^\infty(\mathbb{D})$.

Proof Suppose $T_f H_g = H_g T_f$. Then we have

(i) $g_j(z_2)$ is in $H^\infty(\mathbb{D})$ provided that $H_g = 0$.

(ii) If $T_f = \mu I$, where μ is a constant, then we have f is a constant function, i.e., f and $\widetilde{f}$ are in $H^\infty(\mathbb{D}^2)$.

(iii) If $H_g \neq 0$ and $T_f \neq \mu I$, then by Corollaries 3.1 and 3.3, we have

(a) $f + \widetilde{f} = C$, where C is a constant,

(b) $H_{g_j} T_{f_{-i}} = T_{f_i} H_{g_j} = 0$, for all $i \in \mathbb{N}$,

(c) $T_{f_0} H_{g_j} = H_{g_j} T_{f_0}$.

Let $g_j(z_2) = \sum_{k=-\infty}^{+\infty} b_k z_2^k$ and $f_i(z_2) = \sum_{k=-\infty}^{+\infty} a_{i,k} z_2^k$ for all $i \in \mathbb{Z}$. Since $f = \sum_{i=-\infty}^{+\infty} f_i(z_2) z_1^i$, $f = f_{++} + f_{--}$ and $f + \widetilde{f} = C$, we have $a_{i,k} = -a_{-i,-k}$ for $i \in \mathbb{N}, k \in \mathbb{Z}_+$ and $a_{i,k} = 0$ for $i, k \in \mathbb{Z}$ where the product of i and k is negative. Thus $f_i(z_2) = \sum_{k=0}^{+\infty} a_{i,k} z_2^k$ and $f_{-i}(z_2) = -\sum_{k=0}^{\infty} a_{i,k} z_2^{-k}$ for $i \in \mathbb{N}$.

For m, n in $\mathbb{Z}_+$ and i in $\mathbb{N}$, using the fact (b), we have

$$\begin{aligned}(H_{g_j} T_{f_{-i}} z_2^m, z_2^n) &= (T_{f_{-i}} z_2^m, H_{g_j}^* z_2^n) = -(P_2(\sum_{k=0}^{+\infty} a_{i,k} z_2^{m-k}), \overline{z}_2^{n+1} \overline{g}_j(z_2)) \\ &= -(P_2(\sum_{k=0}^{+\infty} a_{i,k} z_2^{m-k}), \sum_{k=-\infty}^{+\infty} \overline{b}_k \overline{z}_2^{k+n+1}) = -\sum_{k=0}^{m} a_{i,k} b_{-(m+n+1)+k} \\ &= 0,\end{aligned}$$

that is,

$$\sum_{k=0}^{m} a_{i,k} b_{-(m+n+1)+k} = 0, \quad \text{for all} \ \ m, n \in \mathbb{Z}_+, \quad i \in \mathbb{N}. \tag{6}$$

Now we claim that for any $m \in \mathbb{Z}_+$, $a_{i,m} = 0$, $i \in \mathbb{N}$.

Since $H_g \neq 0$, there exists $l \in \mathbb{N}$ such that $b_{-l} \neq 0$. Let $n + 1 = l$. Then the equation (6) can be written as

$$\sum_{k=0}^{m} a_{i,k} b_{-(m+l)+k} = 0, \quad \text{for all} \ \ m \in \mathbb{Z}_+, \quad i \in \mathbb{N}. \tag{7}$$

If $m = 0$, by Equation (7) and the fact $b_{-l} \neq 0$, we have $a_{i,0} b_{-l} = 0$ and $a_{i,0} = 0$, $i \in \mathbb{N}$.

If $m = 1$, by Equation (7), we have $a_{i,0} b_{-l-1} + a_{i,1} b_{-l} = 0$. Since $a_{i,0} = 0$ and $b_{-l} \neq 0$, we get $a_{i,1} = 0$, $i \in \mathbb{N}$.

Now suppose the conclusion holds when $0 \le m \le N$, that is,

$$a_{i,m} = 0, \quad \text{for} \ \ 0 \le m \le N, \quad i \in \mathbb{N}. \tag{8}$$

If $m = N + 1$, by Equation (7), we get

$$\sum_{k=0}^{N} a_{i,k} b_{-(N+1+l)+k} + a_{i,N+1} b_{-l} = 0.$$

Since the Equation (8) holds and $b_{-l} \neq 0$, we have $a_{i,(N+1)} = 0$.

Hence by the induction we obtain

$$a_{i,m} = 0 \quad \text{for all} \;\; m \in \mathbb{Z}_+, \quad i \in \mathbb{N},$$

i.e.,

$$f_i = 0, \quad \text{for all} \;\; i \in \mathbb{N}.$$

Since $a_{-i,-k} = -a_{i,k}$ for $i \in \mathbb{N}, k \in \mathbb{Z}_+$, we get $f_i = 0$ for all nonzero integers i.

By the fact (c) and [2], we get that there exists a nonzero constant λ such that $f_0 + \lambda g_j$, $f_0 + \widetilde{f_0}$ and $f_0 \cdot \widetilde{f_0}$ are in $H^\infty(\mathbb{D})$.

By [2] and Corollary 3.3, it is easy to obtain that the converse holds.

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