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A Note on Chromatic Uniqueness of Completely Tripartite Graphs

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Abstract Let $P(G,\lambda)$ be the chromatic polynomial of a simple graph G. A graph G is chromatically unique if for any simple graph H, $P(H,\lambda)=P(G,\lambda)$ implies that H is isomorphic to G. Many sufficient conditions guaranteeing that some certain complete tripartite graphs are chromatically unique were obtained by many scholars. Especially, in 2003, Zou Hui-wen showed that if $n>\frac{1}{3}m^2+\frac{1}{3}k^2+\frac{1}{3}mk+\frac{1}{3}m-\frac{1}{3}k+\frac{2}{3}\sqrt{m^2+k^2+mk}$, where n,k and m are non-negative integers, then the complete tripartite graph $K(n-m,n,n+k)$ is chromatically unique (or simply χ–unique). In this paper, we prove that for any non-negative integers n,m and k, where $m\geq 2$ and $k\geq 0$, if $n\geq \frac{1}{3}m^2+\frac{1}{3}k^2+\frac{1}{3}mk+\frac{1}{3}m-\frac{1}{3}k+\frac{4}{3}$, then the complete tripartite graph $K(n-m,n,n+k)$ is χ–unique, which is an improvement on Zou Hui-wen's result in the case $m\geq 2$ and $k\geq 0$. Furthermore, we present a related conjecture.

Keywords complete tripartite graph; chromatic polynomial; chromatic uniqueness; color partition.

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1. Introduction

We consider only finite, undirected and simple graphs. Notation and terminology that are not defined here may be found in [1, 2].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, order $p(G)$ and size $q(G)$. Denote by $\overline{G}$ the complement of G. Let $O_n=\overline{K_n}$, where K_n denotes the complete graph with n vertices. For disjoint graphs G and H, $G\vee H$ denotes the graphs whose vertex-set is $V(G)\cup V(H)$ and whose edge-set is $\{wv\in V(G)|w\in V(G),v\in V(H)\}\cup E(G)\cup E(H)$. $G\vee H$ is called the join of G and H. We denote by $K(n_1,n_2,n_3)$ the complete tripartite graph with three parts of n_1,n_2,n_3 vertices, respectively. Let S be a set of s edges of G. We denote by $G-S$ the graph by deleting all edges in S from G. Let $N_3(G)$ denote the number of triangles in G, and $\lceil\theta\rceil$ denote the smallest integer greater than or equal to θ.

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Let $P(G,\lambda)$ be the chromatic polynomial of G and $m_r(G)$ denote the number of distinct partitions of $V(G)$ into r color classes. Let $\lambda_{(r)} = \lambda(\lambda-1)\cdots(\lambda-r+1)$. Then we have $P(G,\lambda)=\sum_{r=1}^{p} m_r(G)\lambda_{(r)}$ (see [1]). The notion of chromatic uniqueness was first introduced and studied by Chao and Whitehead Jr. in 1978 (see [3]). Koh and Teo in their expository paper (see [4, 5]), gave a survey of most of the work before 1997. Two graphs H and G are said to be chromatically equivalent (in notation: $H \sim G$) if $P(H,\lambda)=P(G,\lambda)$. Let $\langle G\rangle = \{H|H\sim G\}$. A graph G is chromatically unique if $\langle G\rangle = \{G\}$. The polynomial $\sigma(G,\chi)=\sum_{r=1}^{p} m_r(G)\chi^r$ is called the σ–polynomial of G (see [6]). Clearly, $P(H,\chi)=P(G,\chi)$ iff $\sigma(G,\chi)=\sigma(H,\chi)$.

The chromatic uniqueness of certain complete tripartite graphs have been studied by many authors. It has been shown in [7]–[11] that the following complete tripartite graphs are χ–unique:

$K(n_1,n_2,n_3)$ for $|n_i-n_j|\le 1$ and $1\le i,j\le 3$ (see [7]);

$K(n,n,n+k)$ for $n\ge 2$ and $0\le k\le 3$, $K(n-k,n,n+k)$ for $n\ge 5$ and $0\le k\le 2$ (see [8]);

$K(n-k,n,n)$ for $n>\frac{1}{3}k^2+k$ (see [9,10]);

$K(n,n,n+k)$ for $n>\frac{1}{3}(k^2+k)$ (see [9]);

$K(n-k,n,n+k)$ for $n>k^2+\frac{2\sqrt{3}}{3}k$ (see [9]);

$K(n-k,n,n)$ for $n\ge k+2\ge 4$ (see [11]).

Especially, Zou Hui-wen obtained the following result in 2003.

Theorem 1.1 ([12]) *Let $G=K(n_1,n_2,n_3), n_1\le n_2\le n_3$ and $a=\{2[(n_1-n_2)^2+(n_1-n_3)^2+(n_2-n_3)^2]\}^{\frac{1}{2}}$. If $n_1+n_2+n_3>\frac{1}{4}a^2+a$, then G is χ-unique.*

We may also formulate Theorem 1.1 in another way as follows.

Theorem 1.2 ([12]) *Let $K(n_1,n_2,n_3)=K(n-m,n,n+k)$, where m and k are non-negative integers. If $n>\frac{1}{3}m^2+\frac{1}{3}k^2+\frac{1}{3}mk+\frac{1}{3}m-\frac{1}{3}k+\frac{2}{3}\sqrt{m^2+k^2+mk}$, then $K(n_1,n_2,n_3)$ is χ-unique.*

In this paper, we show that for any non-negative integers n,m and k, where $m\ge 2$ and $k\ge 0$, if $n\ge\frac{1}{3}m^2+\frac{1}{3}k^2+\frac{1}{3}mk+\frac{1}{3}m-\frac{1}{3}k+\frac{4}{3}$, then the complete tripartite graph $K(n-m,n,n+k)$ is χ-unique, which is an improvement of Theorem 1.2 in the case $m\ge 2$ and $k\ge 0$. Note that when $(m,k)\in\{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2)\}$, the conclusion of this result is trivial. Furthermore, we present a related conjecture.

Conjecture 1.3 For any non-negative integers m, k and n, where $(m,k)\in\{(m,k)|m=0$ and $k\ge 4$, or $m=1$ and $k\ge 3\}$, let $G=K(n-m,n,n+k)$. If $n\ge\frac{1}{3}m^2+\frac{1}{3}k^2+\frac{1}{3}mk+\frac{1}{3}m-\frac{1}{3}k+\frac{4}{3}$, then G is χ-unique.

2. Preliminaries

Lemma 2.1 ([13]) *Let G and H be two graphs with $G\sim H$. Then $p(G)=p(H)$, $q(G)=q(H)$, $N_3(G)=N_3(H)$ and $m_r(G)=m_r(H)$ for $r=1,2,\ldots,p(G)$.*

Lemma 2.2 ([13]) *Let $n_0\ge m_0\ge 2$. Then $K(n_0,m_0)$ is χ-unique.*

Lemma 2.3 ([6]) *Let G and H be two disjoint graphs. Then*

$$\sigma(G \vee H, \tau) = \sigma(G, \tau)\sigma(H, \tau).$$

In particular, $\sigma(K(n_1, n_2, \ldots, n_t), \tau) = \prod_{i=1}^{t} \sigma(O_{n_i}, \tau)$.

Lemma 2.4 ([10]) *Let $G = K(n_1, n_2, n_3)$. Then*

(i) $m_4(G) = \sum_{i=1}^{3} 2^{n_i - 1} - 3$;

(ii) If $H \in \langle G \rangle$, there exists a completely tripartite graph $F = K(m_1, m_2, m_3)$ such that $H = F - S$ and $m_1 + m_2 + m_3 = n_1 + n_2 + n_3$, where S is a set of s edges of F and $s = q(F) - q(G)$.

Lemma 2.5 ([10]) *Let $G = K(n_1, n_2, n_3)$ with $n_3 \geq n_2 \geq n_1 \geq 2$ and let $H = G - S$ for a set S of s edges of G. If $n_1 \geq s + 1$, then $s \leq m_4(H) - m_4(G) \leq 2^s - 1$.*

Lemma 2.6 ([11]) *For any integers $n_3 \geq n_2 \geq n_1 \geq 2$, we have*

$$\begin{aligned}\langle K(n_1, n_2, n_3) \rangle \subseteq \{K(x, y, z) - S | 1 \leq x \leq y \leq z, n_2 \leq z \leq n_3,\\ x + y + z = n_1 + n_2 + n_3, S \subset E(K(x, y, z)),\\ |S| = xy + xz + yz - n_1 n_2 - n_1 n_3 - n_2 n_3 \geq 0\}.\end{aligned}$$

In particular, if $z = n_3$, then $K(n_1, n_2, n_3)$ is isomorphic to $K(x, y, z)$.

Lemma 2.7 ([11]) *For any integers n and m with $n \geq m + 2 \geq 4$, $K(n - m, n, n)$ is χ-unique.*

Lemma 2.8 *For any integers n and m with $m \geq 2$, if $n \geq \frac{1}{3}m^2 + \frac{1}{3}m + \frac{4}{3}$, then $K(n - m, n, n)$ is χ-unique.*

Proof From $m \geq 2$, we have $\lceil \frac{1}{3}m^2 + \frac{1}{3}m + \frac{4}{3} \rceil \geq m + 2 \geq 4$. Thus, by Lemma 2.7, this lemma is true.

3. Main results

Theorem 3.1 *For any non-negative integers m, k and n, where $m \geq 2$ and $k \geq 0$, let $G = K(n - m, n, n + k)$, if $n \geq \frac{1}{3}m^2 + \frac{1}{3}k^2 + \frac{1}{3}mk + \frac{1}{3}m - \frac{1}{3}k + \frac{4}{3}$, then G is χ-unique.*

Proof If $k = 0$, then Theorem 3.1 is true by Lemma 2.8. We shall consider the case $k \geq 1$ in the following.

Suppose $H \in \langle G \rangle$. Since $m \geq 2$ and $k \geq 1$, by calculation, we have

$$n - m \geq \frac{1}{3}m^2 + \frac{1}{3}k^2 + \frac{1}{3}mk - \frac{2}{3}m - \frac{1}{3}k + \frac{4}{3} \geq 2.$$

Consequently, we have $n + k \geq n \geq n - m \geq 2$. Then by Lemma 2.6, we have

$$\begin{aligned}H \in \{K(x, y, z) - S | 1 \leq x \leq y \leq z, n \leq z \leq n + k, x + y + z = 3n + k - m,\\ |S| = s = xy + yz + xz - n(n - m) - (n - m)(n + k) - n(n + k) \geq 0\}.\end{aligned}$$

Next, there are 4 cases to be considered. When $k = 1$, we just need to consider Cases 1 and 2; When $k = 2$, we just need to consider Cases 1, 2 and 3; When $k \geq 3$, we have to consider all the 4 cases.

Case 1 $z = n + k$.

By Lemma 2.6, we conclude that H is isomorphic to G.

Case 2 $z = n$.

We distinguish the following two subcases.

Subcase 2.1 $x \leq y = z = n$.

We set $\beta(H) = m_4(H) - m_4(F)$ in the following proof. Let $H = F - S = K(n+k-m, n, n) - S$. According to $n + k - m \leq n$, we have $k \leq m$. By Lemma 2.4, we deduce that

$$s = q(F) - q(G) = km > 0.$$

By the conditions of the theorem, we have

$$n \geq \frac{1}{3}m^2 + \frac{1}{3}k^2 + \frac{1}{3}mk + \frac{1}{3}m - \frac{1}{3}k + \frac{4}{3} \geq mk + m - k + 1.$$

So

$$s + 1 = mk + 1 \leq n + k - m.$$

Obviously, we have $n + k - m \geq mk + 1 \geq 2$. Consequently, by Lemma 2.5, we have

$$km \leq \beta(H) \leq 2^{km} - 1.$$

Using Lemma 2.4, we have

$$\begin{aligned} m_4(G) - m_4(H) &= (2^{n-m-1} + 2^{n-1} + 2^{n+k-1} - 3) - (2^{n+k-m-1} + 2^{n-1} + 2^{n-1} - 3 + \beta(H)) \\ &\geq 2^{n-m-1} + 2^{n+k-1} - 2^{n+k-m-1} - 2^{n-1} - 2^{km} + 1 \\ &\geq 2^{n-m-1} + 2^{n+k-1} - 2^{n+k-m} - 2^{n-1} + 1. \end{aligned}$$

Since $m \in \{(m, k) | m \geq 2, k \geq 1\}$, we have

$$\frac{1}{2} + 2^k 2^{m-1} - 2^k - 2^{m-1} > 0, \quad \text{i.e., } (\frac{1}{2} + 2^k 2^{m-1} - 2^k - 2^{m-1})2^{n-m} > 0.$$

Hence

$$2^{n-m-1} + 2^{n+k-1} - 2^{n+k-m} - 2^{n-1} > 0, \quad \text{i.e., } m_4(G) - m_4(H) > 1.$$

This contradicts that $m_4(G) = m_4(H)$.

Subcase 2.2 $z = n$ and $x \leq y \leq n - 1$.

Let $H = F - S = K(x, y, n) - S$. Let V_1, V_2, V_3 be the unique 3-independent partition of F such that $|V_1| = x$, $|V_2| = y$, $|V_3| = n$. By Lemma 2.1, $x + y = 2n + k - m$, $N_3(G) = N_3(H)$. Hence, we shall consider the number of triangles in G and H. Without loss of generality, let $S = \{e_1, e_2, \ldots, e_s\} \subset E(F)$. It is not hard to see that $N_3(e_i) \leq n$. Then

$$N_3(H) \geq N_3(F) - ns \tag{1}$$

and the equality holds only if $N_3(e_i) = n$ for all $e_i \in S$.

Let $\eta = N_3(F) - N_3(G)$. It is obvious that $N_3(F) = xyn$, $N_3(G) = n(n-m)(n+k)$ and $\eta = xyn - n(n-m)(n+k)$. So, we have

$$N_3(G) = N_3(F) - \eta. \tag{2}$$

Since $N_3(G) = N_3(H)$, from (1) and (2) it follows that

$$\eta \le sn.$$

Assume that $f(z) = \eta - sz$. Recalling that $s = xy + xn + yn - n(n-m) - (n-m)(n+k) - n(n+k)$, we have

$$f(n) = \eta - sn = n^2[2n + k - m - (x+y)] = 0, \text{ i.e., } \eta = sn.$$

From (1) and (2), we have $N_3(G) = N_3(H) = N_3(F) - sn$ and $N_3(e_i) = n$ for all $e_i \in S$. Thus for every edge one end-vertex belongs to V_1, whereas the other end-vertex belongs to V_2. Hence $\overline{H}$ contains K_n as its component. Set $\overline{H} = \overline{H_1} \bigcup K_n$. Then $H = H_1 \vee O_n$. From Lemma 2.3 and $\sigma(H,\tau) = \sigma(K(n-m, n, n+k), \tau)$, we have

$$\sigma(H_1 \vee O_n, \tau) = \sigma(O_{n-m} \vee O_n \vee O_{n+k}, \tau).$$

So

$$\sigma(H_1, \tau) = \sigma(O_{n-m} \vee O_{n+k}, \tau) = \sigma(K(n-m, n+k), \tau).$$

Hence, from Lemma 2.2 and the conditions of the theorem, we have $H_1 = K(n-m, n+k)$. So $y = n+k$, which contradicts $y \le n-1$.

Case 3 $z = n+k-1$ $(k \ge 2)$.

Let $H = F - S = K(n-k-m+u+1, n+k-u, n+k-1) - S$, where u is a positive integer. According to $n-k-m+u+1 \le n+k-u \le n+k-1$, we have

$$1 \le u \le \frac{1}{2}(m+2k-1).$$

By Lemma 2.4, we deduce that

$$\begin{aligned} s &= q(F) - q(G) = -u^2 + (m+2k-1)u - k^2 - km + m + 2k - 1 \\ &= -[u - \frac{1}{2}(m+2k-1-\sqrt{m^2+2m+4k-3})][u - \frac{1}{2}(m+2k-1+\sqrt{m^2+2m+4k-3})]. \end{aligned}$$

From $s \ge 0$, we get

$$\frac{1}{2}(m+2k-1-\sqrt{m^2+2m+4k-3}) \le u \le \frac{1}{2}(m+2k-1+\sqrt{m^2+2m+4k-3}).$$

Set $g(u) = n-k-m+u+1-(s+1) = u^2 + (2-m-2k)u + n + km + k^2 - 3k - 2m + 1$. We shall consider the domain of u. There are two cases to be considered.

(i) If $\frac{1}{2}(m+2k-1-\sqrt{m^2+2m+4k-3}) < 1$, then we have $1 \le u \le \frac{1}{2}(m+2k-1)$.

(ii) If $\frac{1}{2}(m+2k-1-\sqrt{m^2+2m+4k-3}) \ge 1$, then we get

$$\frac{1}{2}(m+2k-1-\sqrt{m^2+2m+4k-3}) \le u \le \frac{1}{2}(m+2k-1).$$

By calculation, we have

$$g(u) \ge \min\{g(u)\} = g[\frac{1}{2}(m+2k-2)] = n - (\frac{1}{4}m^2 + m + k).$$

By the conditions of the theorem, it follows that

$$n \geq \frac{1}{3}m^2 + \frac{1}{3}k^2 + \frac{1}{3}mk + \frac{1}{3}m - \frac{1}{3}k + \frac{4}{3} \geq \frac{1}{4}m^2 + m + k.$$

So

$$g(u) \geq 0, \text{ i.e., } s + 1 \leq n - k - m + u + 1.$$

From $n \geq \frac{1}{4}m^2 + m + k$, we have $n - k - m + u + 1 \geq 2$. Consequently, by Lemma 2.5, we have

$$s \leq \beta(H) \leq 2^s - 1 \leq 2^{n-k-m+u} - 1.$$

Using Lemma 2.4, we have

$$\begin{aligned}
&m_4(G) - m_4(H)\\
&\quad = (2^{n-m-1} + 2^{n-1} + 2^{n+k-1} - 3) - (2^{n-k-m+u} + 2^{n+k-u-1} + 2^{n+k-2} - 3 + \beta(H))\\
&\quad \geq 2^{n-m-1} + 2^{n-1} + 2^{n+k-1} - 2^{n-k-m+u} - 2^{n+k-u-1} - 2^{n+k-2} - 2^{n-k-m+u} + 1\\
&\quad = 2^{n-m-1} + 2^{n-1} + 2^{n+k-2} - 2^{n+k-u-1} - 2^{n-k-m+u+1} + 1.
\end{aligned}$$

Set

$$\Gamma(n, m, k, u) = 2^{n-m-1} + 2^{n-1} + 2^{n+k-2} - 2^{n+k-u-1} - 2^{n-k-m+u+1} + 1,$$

where

$$(m, k) \in \{(m, k)|m \geq 2, k \geq 2\}, \ u \in \{u|1 \leq u \leq \frac{1}{2}(m + 2k - 1)\}$$

or

$$u \in \{u|\frac{1}{2}(m + 2k - 1 - \sqrt{m^2 + 2m + 4k - 3}) \leq u \leq \frac{1}{2}(m + 2k - 1)\}.$$

There are two cases to consider.

(i) If $u \leq \frac{1}{2}(m + 2k - 2)$, by the convexity and the monotone increasing property of the function 2^x, we have $m + 2k \geq 2u + 2$ and $2^{m+2k+u-1} - 2^{m+2k} \geq 0$. So, $2^k + 2^{k+m} \geq 2^{u+2}$, i.e., $2^{k+u} + 2^{k+m+u} - 2^{2u+2} \geq 0$. Therefore, we get

$$2^{k+u} + 2^{m+k+u} + 2^{m+2k+u-1} - 2^{m+2k} - 2^{2u+2} \geq 0.$$

This leads to $\Gamma(n, m, k, u) = 2^{n-u-k-m-1}(2^{k+u} + 2^{m+k+u} + 2^{m+2k+u-1} - 2^{m+2k} - 2^{2u+2}) + 1 \geq 1$.

(ii) If $u = \frac{1}{2}(m+2k-1)$, then $\Gamma(n, m, k, u) = \Gamma(n, m, k, \frac{m}{2} + k - \frac{1}{2}) = 2^{n-1}(1 + 2^{-m} + 2^{k-1} - 2^{\frac{1}{2}-\frac{m}{2}} - 2^{\frac{3}{2}-\frac{m}{2}}) + 1 > 0$.

From (i) and (ii) it follows that $m_4(G) - m_4(H) > 0$, this is impossible.

Cases 4 $z = n + k - t$ ($k \geq 3$ and $2 \leq t \leq k - 1$).

Let $H = F - S = K(n - k - m + u + t, n + k - u, n + k - t) - S$, where u is a positive integer. According to $n - k - m + u + t \leq n + k - u \leq n + k - t$, we can easily obtain that

$$t \leq u \leq \frac{1}{2}(m + 2k - t).$$

By Lemma 2.4, we deduce that

$$s = q(F) - q(G) = -u^2 + u(m + 2k - t) + 2kt + mt - km - k^2 - t^2.$$

Because of $2 \le t \le k-1$ and $(m,k) \in \{(m,k)|m \ge 2, k \ge 3\}$, we have $m^2 - 3t^2 + 4kt + 2mt > 0$. So

$$s = -[u - \frac{1}{2}(m+2k-t-\sqrt{m^2-3t^2+4kt+2mt})][u - \frac{1}{2}(m+2k-t+\sqrt{m^2-3t^2+4kt+2mt})].$$

From $s \ge 0$, we get

$$\frac{1}{2}(m+2k-t-\sqrt{m^2-3t^2+4kt+2mt}) \le u \le \frac{1}{2}(m+2k-t+\sqrt{m^2-3t^2+4kt+2mt}).$$

Now we consider the domain of u. There are two cases to be considered.

(i) If $\frac{1}{2}(m+2k-t-\sqrt{m^2-3t^2+4kt+2mt}) < t$, then we have $t \le u \le \frac{1}{2}(m+2k-t)$.

(ii) If $\frac{1}{2}(m+2k-t-\sqrt{m^2-3t^2+4kt+2mt}) \ge t$, then we have

$$\frac{1}{2}(m+2k-t-\sqrt{m^2-3t^2+4kt+2mt}) \le u \le \frac{1}{2}(m+2k-t).$$

Set $h(u) = n-k-m+u+t-(s+1) = u^2+u(t-m-2k+1)+t^2+k^2+km-2kt-mt+n-k-m+t-1$. By calculation, we have, respectively,

$$h(u) \ge \min\{h(u)\} = h[\frac{1}{2}(m+2k-t-1)] = \frac{1}{4}(3t^2-m^2+2t-2mt-4kt+4n-2m-5)$$

and

$$\min\{3t^2+(2-2m-4k)t\} = -\frac{1}{3}m^2 - \frac{4}{3}k^2 - \frac{4}{3}mk + \frac{2}{3}m + \frac{4}{3}k - \frac{1}{3}.$$

So

$$\min\{h(u)\} \ge n - (\frac{1}{3}m^2 + \frac{1}{3}k^2 + \frac{1}{3}mk + \frac{1}{3}m - \frac{1}{3}k + \frac{4}{3}).$$

By the conditions of the theorem, it follows that

$$h(u) \ge \min\{h(u)\} \ge 0.$$

Hence

$$s + 1 \le n - k - m + u + t.$$

From $n \ge \frac{1}{3}m^2 + \frac{1}{3}k^2 + \frac{1}{3}mk + \frac{1}{3}m - \frac{1}{3}k + \frac{4}{3}$, we have $n-k-m+u+t \ge 2$. Consequently, by Lemma 2.5, we have

$$s \le \beta(H) \le 2^s - 1 \le 2^{n-k-m+u+t-1} - 1.$$

Using Lemma 2.4, we have

$$\begin{aligned}
&m_4(G) - m_4(H) \\
&= (2^{n-m-1} + 2^{n-1} + 2^{n+k-1} - 3) - (2^{n-k-m+u+t-1} + 2^{n+k-u-1} + 2^{n+k-t-1} - 3 + \beta(H)) \\
&\ge 2^{n-m-1} + 2^{n-1} + 2^{n+k-1} - 2^{n-k-m+u+t-1} - 2^{n+k-u-1} - 2^{n+k-t-1} - 2^{n-k-m+u+t-1} + 1 \\
&= 2^{n-m-1} + 2^{n-1} + 2^{n+k-1} - 2^{n-k-m+u+t} - 2^{n+k-u-1} - 2^{n+k-t-1} + 1 \\
&\ge 2^{n-m-1} + 2^{n-1} + 2^{n+k-1} - 2^{n+k-u-1} - 2^{n+k-t-1} - 2^{n+k-u} + 1 \\
&\ge 2^{n-m-1} + 2^{n-1} + 2^{n+k-1} - 2^{n+k-t} - 2^{n+k-u} + 1 \\
&\ge 2^{n-m-1} + 2^{n-1} + 2^{n+k-1} - 2^{n+k-t+1} + 1.
\end{aligned}$$

Since $n+k-1 \ge n+k-t+1$, it follows that $m_4(G) - m_4(H) \ge 1$, which is impossible. The proof is completed. □

Remark 3.2 We shall discuss the improvement of Theorem 3.1 with respect to Theorem 1.2 in the following cases. Note that the judgement condition in the brackets was obtained from Theorem 1.2.

(i) If $k = 0, m = 2$, then for $n \geq 4$ ($n \geq 4$), $K(n-2, n, n)$ is χ–unique. Theorem 1.2 has not been improved in this case.

(ii) If $k = 0, m = 3$, then for $n \geq 6$ ($n \geq 7$), $K(n-3, n, n)$ is χ–unique. Theorem 1.2 has been improved in this case.

(iii) If $k = 1, m = 2$, then for $n \geq 4$ ($n \geq 5$), $K(n-2, n, n+1)$ is χ–unique. Theorem 1.2 has been improved in this case.

(iv) If $k = 2, m = 2$, then for $n \geq 6$ ($n \geq 7$), $K(n-2, n, n+2)$ is χ–unique. Theorem 1.2 has been improved in this case.

(v) For the other cases, we have $\frac{2}{3}\sqrt{m^2 + k^2 + mk} > \frac{7}{3}$. Theorem 1.2 has been improved largely in these cases. For example, when $n \geq 102$ ($n \geq 112$), $K(n-10, n, n+10)$ is χ–unique; When $n \geq 10002$ ($n \geq 10116$), $K(n-100, n, n+100)$ is χ–unique; When $n \geq 1000002$ ($n \geq 1001155$), $K(n-1000, n, n+1000)$ is χ–unique.

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