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A Trick Formula to Illustrate the Period Three Bifurcation Diagram of the Logistic Map

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Abstract In this paper we develop an algebraic formula in analyzing the bifurcation of the logistic map. Based on this formula, we give a new, short and rigorous proof for the period-3 bifurcation diagram of the logistic map. Therefore, we simplify the work in previous references and improve some insufficiently rigorous slight defects in them.

Keywords positive definite formula; period-3 bifurcation; logistic map.

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1. Introduction

Since Li and Yorke [1] published their famous paper "Period three implies chaos", finding period solutions and their existence conditions for various one dimensional maps has become highly interesting. The logistic map,

$$x_{n+1} = \mu x_n(1 - x_n) = f_\mu(x_n) \tag{1}$$

is a standard and familiar example of nonlinear dynamic system. May [2] reported its complicated behaviors and many researchers investigated its period-3 solutions by using various methods [3–6]. They proved that $\mu_0 = 1 + \sqrt{8}$ is the smallest positive value for which $f_\mu(f_\mu(f_\mu(x))) = x$ has a non-trivial solution. Special features of the logistic map such as tangent bifurcation condition or the existence condition of the multiple roots of an algebraic equation have been employed to show the results. Furthermore, that the bifurcation function is positive definite before the bifurcation occurs has rarely been checked. In this paper, we develop an important formula and give the bifurcation diagram of the logistic map a new, short and rigorous proof. Only simple algebraic calculations are involved.

2. A formula for logistic map bifurcation

The period-3 solutions of the logistic map are determined by the equation

$$f_\mu(f_\mu(f_\mu x))) - x = 0. \tag{2}$$

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To exclude the fixed points of the logistic map and only focus on the period-3 solutions, we consider the equation $g(x) = 0$, where

$$g(x) = \frac{f_\mu(f_\mu(f_\mu(x)))) - x}{f_\mu(x) - x} = 0. \tag{3}$$

Let $z = -\mu x$ in (3). We have

$$\begin{aligned} g(z) =& g_\mu(-\frac{z}{\mu}) = z^6 + (3\mu+1)z^5 + (3\mu^2+4\mu+1)z^4 + (\mu^3+5\mu^2+3\mu+1)z^3 + \\ & (2\mu^3+3\mu^2+3\mu+1)z^2 + (\mu^3+2\mu^2+2\mu+1)z + \mu^2+\mu+1. \end{aligned} \tag{4}$$

Clearly, the logistic map has period-3 orbits if and only if the equation $g(z) = 0$ has real roots. We introduce a new parameter $\lambda = 7 + 2\mu - \mu^2$, then it is easy to see that

$$\lambda > 0 \text{ if } 0 \le \mu < 1 + \sqrt{8};$$

$$\lambda = 0 \text{ if } \mu = 1 + \sqrt{8};$$

$$\lambda < 0 \text{ if } \mu > 1 + \sqrt{8}.$$

By direct computation, we have the following square sum formula which is the main new result in this paper. It is not hard to verify the formula, but it is not so easy to find the formula:

$$g(z) = (z^3 + \frac{3\mu+1}{2}z^2 + (2\mu+3-\frac{\lambda}{2})z + \frac{\mu+5}{2} - \frac{\lambda}{2})^2 + \frac{1}{4}\lambda(z+1)^2(z+\mu)^2. \tag{5}$$

3. Local bifurcation of the logistic map

It has been shown that the period-3 bifurcation of the logistic map has the diagram as follows:

Theorem 1 (Local Bifurcation Theorem) *(i) If $0 \le \mu < 1 + \sqrt{8}$, then $g(z)$ is positive definite. Hence the equation $g(z) = 0$ does not have any real root and the logistic map does not have any period-3 orbit;*

(ii) If $\mu = 1 + \sqrt{8}$, then $g(z) = 0$ has three different roots and each root has multiplicity 2. *These three roots give one period-3 orbit of the logistic map;*

(iii) If $\mu > 1 + \sqrt{8}$ and $|\mu - (1 + \sqrt{8})|$ is sufficiently small, then the equation $g(z) = 0$ has six simple roots which give two period-3 orbits of the logistic map.

In other words, there is a local period-3 bifurcation for the logistic map when $\mu = 1 + \sqrt{8}$.

These results were proved [3–6] with the help of the tangent bifurcation condition or the existence condition of the multiple roots of an algebraic equation. In these references, the author only determines the value $1 + \sqrt{8}$, but does not give a rigorous proof for the positive definite in the case of $0 \le \mu < 1 + \sqrt{8}$. Here we give a new and straightforward proof based on our formula (5) by using simple algebra.

Proof If $0 \le \mu < 1 + \sqrt{8}$, then $\lambda > 0$. It follows from (5) that $g(z)$ is positive definite (see Figure 1(a)).

If $\mu = 1 + \sqrt{8}$, then $\lambda = 0$,

$$g(z) = (z^3 + (2 + 3\sqrt{2})z^2 + (5 + 4\sqrt{2})z + 3 + \sqrt{2})^2. \tag{6}$$

Hence three roots of $g(z) = 0$ (each with multiplicity two) are:

$$z_k = \frac{2\sqrt{7}}{3}\cos(\frac{1}{3}\arccos(-\frac{1}{2\sqrt{7}} + \frac{2k\pi}{3})) - \frac{2+3\sqrt{2}}{3}, \quad k = 0, 1, 2. \tag{7}$$

They generate a period-3 orbit of the logistic map. The graph of $g(z)$ is given in Figure 1(b).

If $\mu > 1 + \sqrt{8}$ and $|\mu - (1 + \sqrt{8})|$ is sufficiently small, then $\lambda < 0$ and $|\lambda|$ is sufficiently small. We can easily see that equation $g(z) = 0$ has six simple roots by factoring $g(z)$ in (5). The graph of $g(z)$ is shown in Figure 1(c). The rigorous proof can be obtained from small parameter method or as a special example of Global bifurcation of the logistic map (See next section).

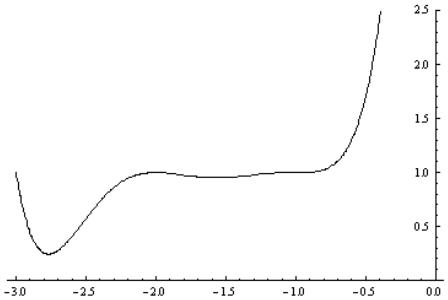

Figure 1(a) $0 \leq \mu < 1 + \sqrt{8}$, or $\lambda > 0$

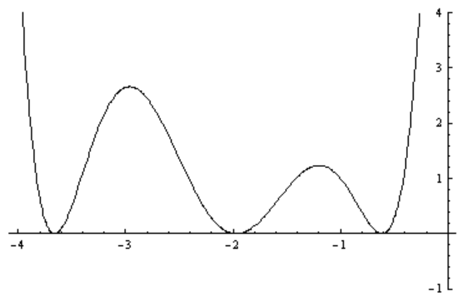

Figure 1(b) $\mu = 1 + \sqrt{8}$, or $\lambda = 0$

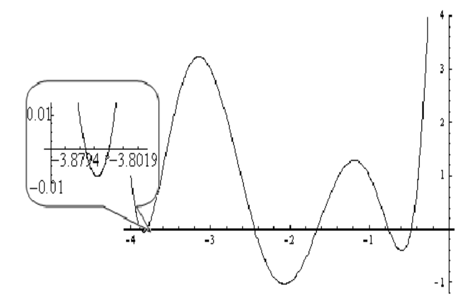

Figure 1(c) $\mu > 1 + \sqrt{8}$ or $\lambda < 0$

4. Global bifurcation of the logistic map

In fact, the condition that $\mu - (1 + \sqrt{8})$ is sufficiently small in part 3 of local bifurcation is not necessary. The bifurcation behavior is global so that we have the following diagram:

Theorem 1 (Global Bifurcation Theorem) *(i) If* $0 \leq \mu < 1 + \sqrt{8}$*, then* $g(z)$ *is positive definite. Hence the equation* $g(z) = 0$ *does not have any real root and the logistic map does not have any period-3 orbit;*

(ii) If $\mu = 1 + \sqrt{8}$*, then* $g(z) = 0$ *has three different roots and each root has multiplicity* 2*. These three roots give one period-3 orbit of the logistic map;*

(iii) If $\mu > 1 + \sqrt{8}$*, then the equation* $g(z) = 0$ *always has six simple roots which give two*

period-3 orbits of the logistic map.

Proof We only need to show part 3. If $\mu > 1 + \sqrt{8}$, we have the factorization

$$g(z) = g_1(z) g_2(z), \tag{8}$$

where

$$g_1(z) = z^3 + (\frac{3\mu+1}{2} - \frac{\sqrt{-\lambda}}{2})z^2 + (2\mu + 3 - \frac{\lambda}{2} - \frac{\mu+1}{2}\sqrt{-\lambda})z + \frac{\mu+5}{2} - \frac{\lambda}{2} - \frac{1}{2}\mu\sqrt{-\lambda} \tag{9}$$

and

$$g_2(z) = z^3 + (\frac{3\mu+1}{2} + \frac{\sqrt{-\lambda}}{2})z^2 + (2\mu + 3 - \frac{\lambda}{2} + \frac{\mu+1}{2}\sqrt{-\lambda})z + \frac{\mu+5}{2} - \frac{\lambda}{2} + \frac{1}{2}\mu\sqrt{-\lambda}. \tag{10}$$

It suffices to show that $g_1(z)$ and $g_2(z)$ have three roots respectively and they are all different from each other.

It is easy to check that

$$g_1(-\mu) = g_1(-1) = -1, \tag{11}$$

$$g_1(-\mu+1) = \frac{1}{2}\mu(\mu-3) + \frac{1}{2}(\mu-2)\sqrt{-\lambda} > 0 \tag{12}$$

and

$$g_1(0) = \frac{1}{2}(\mu^2 - \mu - 2) - \frac{1}{2}\mu\sqrt{\mu^2 - \mu - 7} > 0. \tag{13}$$

Therefore $g_1(z) = 0$ has three different roots located in three open intervals $(-\mu, -\mu+1)$, $(-\mu+1, -1)$ and $(-1, 0)$.

Similarly for $g_2(z)$, we have

$$g_2(-\mu) = g_2(-1) = -1, \tag{14}$$

$$g_2(0) = \frac{1}{2}(\mu^2 - \mu - 2) + \frac{1}{2}\mu\sqrt{\mu^2 - \mu - 7} > 0 \tag{15}$$

and

$$g_2(-\frac{3\mu + 1 + \sqrt{-\lambda} + \sqrt{\Delta}}{6}) = \frac{D}{216}, \tag{16}$$

where

$$\sqrt{-\lambda} = \sqrt{\mu^2 - 2\mu - 7}, \tag{17}$$

$$\sqrt{\Delta} = 2\sqrt{\mu^2 - 2\mu - \sqrt{-\lambda}} \tag{18}$$

and

$$\begin{aligned} D =& 12\mu^2\sqrt{\Delta} - 24\mu\sqrt{\Delta} - 12\sqrt{-\lambda}\sqrt{\Delta} - 16(\sqrt{-\lambda})^3 - (\sqrt{\Delta})^3 + \\ & 24\mu^2 - 48\mu - 120\sqrt{-\lambda} - 112 \\ =& 12\sqrt{\Delta}(\mu^2 - 2\mu - \sqrt{-\lambda}) - 16(\sqrt{-\lambda}^3) - (\sqrt{\Delta})^3 + \\ & 24(\mu^2 - 2\mu - 7) - 120\sqrt{-\lambda} + 56 \\ =& 2(\sqrt{\Delta})^3 - 16(\sqrt{-\lambda})^3 + 24 - \lambda - 120\sqrt{-\lambda} + 56. \end{aligned} \tag{19}$$

Let $u = \sqrt{-\lambda}$. Then $D = 8M$, where

$$M = 2(\sqrt{u^2 - u + 7})^3 - (2u^3 - 3u^2 + 15u - 7). \tag{20}$$

To check the minimum of M, we take the derivative with respect to u:

$$\frac{\mathrm{d}M}{\mathrm{d}u} = 3((2u-1)\sqrt{u^2-u+7} - (2u^2-2u-5)). \tag{21}$$

$\frac{\mathrm{d}M}{\mathrm{d}u}$ has a unique real root $u=2$ where M takes its minimum 27. Then we have

$$g_2(-\frac{3\mu+1+\sqrt{-\lambda}+\sqrt{\Delta}}{6}) = \frac{D}{216} = \frac{8M}{216} \geq 1 > 0. \tag{22}$$

Therefore, $g_2(z)$ has three different roots located in three open intervals $(-\mu, -\frac{3\mu+1+\sqrt{-\lambda}+\sqrt{\Delta}}{6})$, $(-\frac{3\mu+1+\sqrt{-\lambda}+\sqrt{\Delta}}{6}, -1)$, and $(-1, 0)$.

It remains to show that $g_1(z)$ and $g_2(z)$ share no common root. In fact, assume $g_1(z)$ and $g_2(z)$ have a common root z_0. We have $g_2(z_0) - g_1(z_0) = 0$. Then $(z_0+\mu)(z_0+1) = 0$. Therefore $z_0 = -\mu$ or $z_0 = -1$. However we have shown that $g_1(-\mu) = g_1(-1) = -1$ and $g_2(-\mu) = g_2(-1) = -1$, leading to a contradiction.

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