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A Property of Continuous Mapping from a Sphere to the Euclidean Space and Its Applications

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Abstract In this paper, we study the property of continuous mappings from a sphere to the Euclidean space. By using the theory of the periodic transformation in algebraic topology, we obtain a generalized Borsuk-Ulam theorem and then give some applications of the theorem.

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1. Introduction

The classical Borsuk-Ulam theorem says that [1]: for every continuous map $f : S^m \to R^k$ (from the m-sphere to the Euclidean k-space), there exists a point $x \in S^m$ such that $f(x) = f(-x)$, if $k \le m$. It is a well-known theorem and has been generalized in many ways. Conner and Floyd [2] generalized the Borsuk-Ulam theorem to one in which the Euclidean k-space was replaced by a differentiable k-manifold M^k. Pedro et al. [3] generalized the theorem to a situation where the Euclidean k-space was replaced by a wide class of topological spaces. Munkholm showed in [4] that all differentiability hypotheses in the theorem can be removed by changing the assumption that M^k is compact, differentiable manifold to that M^k is a closed topological manifold, and Carlos Biasi et al. [5] further extended the result of Munkholm to a more generalized form via replacing the compact k-manifold by a generalized manifold.

In the present paper, we focus on the study of the property of continuous mapping from a sphere to the Euclidean space, and obtain a generalized Borsuk-Ulam theorem in a different form by using the theory of the periodic transformation. We then give some applications of the theorem.

2. Main results

Lemma 2.1 ([6]) *If t is a fixed point free periodic transformation on S^m whose period is n, p is a prime number and n can be divided by p, then $I(S^m, Z_p) = m + 1$.*

Lemma 2.2 ([6]) *If X and Y are arcwise connected Hausdorff space, T is a fixed point free*

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periodic transformation group on X and Y, and $f : X \to Y$ is an equivariant mapping about T, then $I(X,T) \le I(Y,T)$.

Theorem 2.1 *If $f : S^m \to R^k$ is a continuous mapping from an m-sphere to the Euclidean k-space and t is a fixed point free periodic transformation on S^m whose period is p^i, p is a prime number, $(i,p^i) = 1$ and $m \ge kp^i - 1$, then there exists a point $x \in S^m$ such that $f(x) = f(tx) = \cdots = f(t^{p^i-1}x)$.*

Proof Suppose the conclusion of theorem is not true. Let

$$\varphi : S^m \to R^k \times R^k \times \cdots \times R^k = R^{p^i k},$$

$$\varphi(x) = (f(x), f(tx), \ldots, f(t^{p^i-1}x)).$$

Then φ is a continuous mapping and $\varphi(S^m) \subset R^{p^i k} - \triangle^k_{p^i}$, where

$$\triangle^k_{p^i} = \{(a_1, a_2, \ldots, a_{p^i}) | a_j \in R^k, a_1 = a_2 = \cdots = a_{p^i},\ 1 \le j \le p^i\}.$$

For any $y \in R^{p^i k} - \triangle^k_{p^i}$, $y = (y_1, y_2, \ldots, y_{p^i})$, let

$$\begin{aligned}
&\psi : R^{p^i k} - \triangle^k_{p^i} \to \widetilde{S}^{kp^i-1},\\
&\psi(y) = \psi(y_1, y_2, \ldots, y_{p^i})\\
&\quad = (y_1 - \frac{y_1 + y_2 + \cdots + y_{p^i}}{p^i}, y_2 - \frac{y_1 + y_2 + \cdots + y_{p^i}}{p^i}, \ldots, y_{p^i} - \frac{y_1 + y_2 + \cdots + y_{p^i}}{p^i}).\\
&\qquad (y_1^2 + y_2^2 + \cdots + y_{p^i}^2 - \frac{(y_1 + y_2 + \cdots + y_{p^i})^2}{p^i})^{-\frac{1}{2}}\\
&\quad = (z_1, z_2, \ldots, z_{p^i}).
\end{aligned}$$

Then ψ is a continuous mapping and

$$z_1 + z_2 + \cdots + z_{p^i} = 0,\ |z_1|^2 + |z_2|^2 + \cdots + |z_{p^i}|^2 = 1.$$

Hence $\widetilde{S}^{kp^i-1}$ is a $(kp^i - 1)$-sphere. Let

$$g = \psi \circ \varphi : S^m \to \widetilde{S}^{kp^i-1}.$$

Then g is a continuous mapping. Let

$$t : \widetilde{S}^{kp^i-1} \to \widetilde{S}^{kp^i-1},$$

$$t(z_1, z_2, \ldots, z_{p^i}) = (z_2, z_3, \ldots, z_{p^i}, z_1).$$

Then t is a periodic transformation on $\widetilde{S}^{kp^i-1}$ whose period is p^i. Suppose for $1 < i < p^i$, t^i has a fixed point, i.e.,

$$t^i(z_1, z_2, \ldots, z_{p^i}) = (z_{i+1}, z_{i+2}, \ldots, z_{p^i}, z_1, z_2, \ldots, z_i) = (z_1, z_2, \ldots, z_{p^i}).$$

Then

$$z_1 = z_{i+1} = z_{2i+1} = \cdots = z_{p^i-i+1},$$

$$z_2 = z_{i+2} = z_{2i+2} = \cdots = z_{p^i-i+2},$$

$$\vdots$$

$$z_i = z_{2i} = z_{3i} = \cdots = z_{p^i}.$$

Since $(i, p^i) = 1$, there exists integral number a and b such that $ai + bp^i = 1$. For any $1 \leq j \leq p^i$,

$$z_j = z_{j+|ai|} = z_{j+|1-bp^i|} = \begin{cases} z_{j-1}, & b > 0 \\ z_{j+1}, & b < 0. \end{cases}$$

Then $z_1 = z_2 = \cdots = z_{p^i}$. This contradicts the assumption. Then t is a fixed point free periodic transformation on $\widetilde{S}^{kp^i-1}$ and $gt(x) = tg(x)$. By Lemmas 2.1 and 2.2

$$m + 1 = I(S^m, Z_p) \leq I(\widetilde{S}^{kp^i-1}, Z_p) = kp^i$$

i.e., $m < kp^i - 1$. This contradicts to the condition of Theorem 2.1 and the proof of Theorem 2.1 is completed. □

Remark While $i = 1$, $p = 2$, $t(x) = -x$ and $m = k$, Theorem 2.1 is the Borsuk-Ulam Theorem. Hence Theorem 2.1 generalizes the Borsuk-Ulam Theorem.

In the following we give two applications of the theorem.

Corollary 2.1 *Let $f : S^m \to M^k$ be a continuous mapping from the m-sphere S^m to a smooth k-manifold M^k and t is a fixed point free periodic transformation on S^m whose period is p^i. Suppose p is a prime number, $(i, p^i) = 1$ and $m \geq (2k + 1)p^i - 1$. Then there exists a point $x \in S^m$ such that $f(x) = f(tx) = f(t^2x) = \cdots = f(t^{p^{i-1}}x)$.*

Proof Whitney has proved [7] that any smooth k-manifold can be embedded in R^{2k+1} as a submanifold. Hence there exists a single smooth mapping $h : M^k \to R^{2k+1}$ such that (h, M^k) is an embedding submanifold of R^{2k+1}. Let

$$\widetilde{f} = h \circ f : S^m \to R^{2k+1}.$$

Then, by Theorem 2.1, there exists a point $x \in S^m$ such that $\widetilde{f}(x) = \widetilde{f}(tx) = \widetilde{f}(t^2x) = \cdots = \widetilde{f}(t^{p^i-1}x)$. Since h is single, there exists a point $x \in S^m$ such that $f(x) = f(tx) = f(t^2x) = \cdots = f(t^{p^i-1}x)$. Hence the conclusion of the corollary follows.

Corollary 2.2 *If S^m is covered by $k + 1$ closed sets and t is a fixed point free periodic transformation on S^m whose period is p^i, p is a prime number, $(i, p^i) = 1$, $m \geq kp^i - 1$, then there exists a point $x \in S^m$ such that one of the sets contains points x, tx, t^2x, ... , $t^{p^i-1}x$.*

Proof Suppose $A_1, A_2, \ldots, A_{k+1}$ are closed subsets of S^m whose union is all of S^m. Define $f : S^m \to R^k$ by

$$f(x) = (d(x, A_1), d(x, A_2), \ldots, d(x, A_k)), x \in S^m,$$

where $d(x, A_i)$ is the distance of the point x from A_i. Then f is continuous. By Theorem 2.1, there exists a point $x \in S^m$ such that $f(x) = f(tx) = f(t^2x) = \cdots = f(t^{p^i-1}x)$. In the other words, we can find a point x in S^m with the property $d(x, A_i) = d(tx, A_i) = \cdots = d(t^{p^i-1}x, A_i)$ for $1 \leq i \leq k$. If $d(x, A_i) > 0$ for $1 \leq i \leq k$, then x, tx, t^2x, ... , $t^{p^i-1}x$ lie in A_{k+1}, since A_1,

$A_2, \ldots, A_{k+1}$ cover S^m. If $d(x, A_i) = 0$ for some i, since each A_i is closed, we know that all x, tx, $t^2x, \ldots, t^{p^i-1}x$ are in A_i.

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