Journal of Mathematical Research & Exposition
May, 2010, Vol. 30, No. 3, pp. 381–389
DOI:10.3770/j.issn:1000-341X.2010.03.001
Http://jmre.dlut.edu.cn

Galois Connections in A Topos

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Abstract In this paper, we investigate the Galois connections between two partially ordered objects in an arbitrary elementary topos. Some characterizations of Galois adjunctions which is similar to the classical case are obtained by means of the diagram proof. This shows that the diagram method can be used to reconstruct the classical order theory in an arbitrary elementary topos.

Keywords partial order object; Galois connection; topos.

Document code A
MR(2000) Subject Classification 18B25; 06A15
Chinese Library Classification O153.1; O154

1. Introduction and preliminaries

The development of topos theory resulted from the confluence of two streams of mathematical thought since the 20th sixties. The first is the development of an axiomatic treatment of sheaf theory by Grothendieck. This axiomatic development culminated in the discovery by Giraud that a category is equivalent to a category of sheaves for a Grothendieck topology if and only if it satisfies the conditions of being a Grothendieck topos. The main purpose of the axiomatic development is to be able to define sheaf cohomology. The second stream is Lawvere's continuing search for a natural way of founding mathematics (universal algebra, set theory, category theory, etc.) on the basic notion of morphism and composition of morphisms. All formal (and naive) presentations of set theory up to then had taken as primitives the notions of elements and sets with membership as the primitive relation. Now a topos can be considered both as a "generalized space" and as a "generalized universe of sets". Topos theory unifies this two seemingly wholly distinct mathematical aspects.

Recall that a topos $\mathcal{E}$ is a category which has finite limits and every object of $\mathcal{E}$ has a power object. For a fixed object A of $\mathcal{E}$, the power object of A is an object PA which represents $\mathrm{Sub}(-\times A)$, so that $\mathrm{Hom}_{\mathcal{E}}(-,PA)\simeq \mathrm{Sub}(-\times A)$ naturally. It says precisely that for any arrow

Received May 27, 2008; Accepted October 25, 2008
Supported by the National Natural Science Foundation of China (Grant No. 10731050).
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$B' \xrightarrow{f} B$, the following diagram commutes, where φ is the natural isomorphism.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{E}}(B, PA) \arrow[r, "\varphi(A,B)"] \arrow[d, "\mathrm{Hom}_{\mathcal{E}}(f,PA)"'] & \mathrm{Sub}(B \times A) \arrow[d, "\mathrm{Sub}(f\times A)"] \\
\mathrm{Hom}_{\mathcal{E}}(B', PA) \arrow[r, "\varphi(A,B')"] & \mathrm{Sub}(B' \times A)
\end{tikzcd} \tag{1}$$

Figure 1 The nature of φ

An impotant example of toposes is the category of sheaves on a topological space. In particular, the category of sets is a topos. For details of toposes and sheaves please see Johnstone [1], Mac and Moerdijk [2], Joyal and Tierney [3], Johnstone and Joyal [4]. For a general background on category theory please refer to [5], [6].

In [2], lattice and Heyting Algebra objects in a topos are well defined. In this paper we will investigate the more general concept of partially ordered objects and Galois connections between partially ordered objects in an arbitrary topos by means of diagram method. More details about lattices and locales please see [8–12].

2. Main results

Throughout this paper, we work with a fixed topos $\mathcal{E}$. All objects mentioned belong to the topos $\mathcal{E}$. We begin with some definitions.

Definition 1 ([2]) *A subobject $\leq_L \rightarrowtail L \times L$ is called an internal partial order on L, provided that the following conditions are satisfied*

1) Reflexivity: The diagonal $L \xrightarrow{\delta} L \times L$ *factors through* $\leq_L \overset{e_L}{\rightarrowtail} L \times L$, *as in*

$$\begin{tikzcd}
L \arrow[r, "\delta"] \arrow[dr, dotted] & L \times L \\
& \leq_L \arrow[u, tail, "e_L"']
\end{tikzcd} \tag{2}$$

Figure 2 Reflexivity

2) Antisymmetry: The intersection $\leq_L \cap \geq_L$ is contained in the diagonal, as in the following pullback

$$\begin{tikzcd}
\leq_L \cap \geq_L \arrow[rr, tail] \arrow[dd] \arrow[dr, dotted] & & \leq_L \arrow[dd, tail, "e_L"] \\
& L \arrow[dr, tail, "\delta"] & \\
\geq_L \arrow[rr, tail] & & L \times L
\end{tikzcd} \tag{3}$$

Figure 3 Antisymmetry

Where $\geq_L$ is defined as the composite $\leq_L \overset{e_L}{\rightarrowtail} L \times L \xrightarrow{\tau} L \times L$ *with τ as the twist map interchanging the factors of the product.*

3) Transitivity: The subobject $C \overset{\langle \pi_1 ev, \pi_2 eu \rangle}{\rightarrowtail} L \times L$ *factors through* $\leq_L \overset{e_L}{\rightarrowtail} L \times L$*, as in*

$$\begin{tikzcd} C \arrow[rr, tail, "\langle \pi_1 ev, \pi_2 eu \rangle"] \arrow[dr, dotted] & & L \times L \\ & \leq_L \arrow[ur, tail, "e_L"'] & \end{tikzcd} \tag{4}$$

Figure 4 Transitivity

where C is the following pullback

$$\begin{tikzcd} C \arrow[rr, "u"] \arrow[dd, "v"'] & & \leq_L \arrow[d, tail, "e_L"] \\ & & L \times L \arrow[d, "\pi_1"] \\ \leq_L \arrow[r, tail, "e_L"'] & L \times L \arrow[r, "\pi_2"'] & L \end{tikzcd} \tag{5}$$

Figure 5 The definition of C

An object L endowed with an internal partial order $\leq_L$ is called a partially ordered object.

Let L and M be two partially ordered objects. We can define the product of partially ordered object $L \times M$ of L and M as the product object $L \times M$ endowed with the "pointwise order" $\leq_L \times \leq_M \rightarrowtail L \times L \times M \times M \simeq L \times M \times L \times M$. Also, a subobject B of a partially ordered object $(L, \leq_L)$ is again a partial order object endowed with the induced partial order $\leq_B$, as in the pullback

$$\begin{tikzcd} \leq_B \arrow[r, tail] \arrow[d, tail] & \leq_L \arrow[d, tail] \\ B \times B \arrow[r, tail] & L \times L \end{tikzcd} \tag{6}$$

Figure 6 The induced partial order

We now turn to the discussion of morphisms between partial order objects.

In [2], for morphisms $L \underset{g}{\overset{f}{\rightrightarrows}} M$ between two objects in a topos, $f \leq g$ is defined to be $L \xrightarrow{\langle f, g \rangle} M \times M$ factors through $\leq_M \overset{e_M}{\rightarrowtail} M \times M$, as in

$$\begin{tikzcd} L \arrow[rr, "\langle f, g \rangle"] \arrow[dr, dotted] & & M \times M \\ & \leq_M \arrow[ur, tail, "e_M"'] & \end{tikzcd} \tag{7}$$

Figure 7 The first definition of $f \leq g$

In [7], the author define $f \le g$ if and only if for all generalized element x in L, $f(x) \le g(x)$ in M. We show these two definitions are equivalent.

Lemma 1 *Let L, M be two partially ordered objects with a pair of morphisms $L \overset{f}{\underset{g}{\rightrightarrows}} M$. Then $f \le g$ if and only if $fr \le gr$ for every morphism $A \xrightarrow{r} L$.*

Proof $\Rightarrow$. Suppose $f \le g$, then there exists a morphism $L \xrightarrow{k} \le_M$ such that $\langle f, g\rangle = e_M k$. So $\langle fr, gr\rangle = \langle f, g\rangle r = e_M kr$, which means the outer triangle of Figure 8 below is commutative, i.e., $\langle fr, gr\rangle$ factors through $\le_M \overset{e_M}{\rightarrowtail} M \times M$.

$$\begin{tikzcd}
A \arrow[rr, "<fr{,}gr>"] \arrow[dr, "r"] \arrow[ddr, bend right] & & M \times M \\
& L \arrow[ur, "<f{,}g>"] \arrow[d, dotted, "k"] & \\
& \le_M \arrow[uur, bend right, tail, "e_M"'] &
\end{tikzcd} \tag{8}$$

Figure 8 Equivelence of two definitions

$\Leftarrow$. Indeed, in order to verify this, we can take the fixed identity morphism $L \xrightarrow{1_L} L$, then $f \le g$ is obvious. $\Box$

Corollary 1 *Let L, M be two partially ordered objects and $L \xrightarrow{f} M$ be a morphism. Then $f \le f$.*

Proof Since $p_i\langle f, f\rangle = p_i \delta f$ with $p_i : M \times M \to M$ $(i = 1, 2)$ being projections, $\langle f, f\rangle = \delta f$. And by Definition 1, we know δ factors through $\le_M \overset{e_M}{\rightarrowtail} M \times M$. It follows that the outer square is commutative as in the following Figure 9.

$$\begin{tikzcd}
L \arrow[r, "\langle f{,}f\rangle"] \arrow[dr, shift left, "f"] \arrow[dr, shift right, "f"'] & M \times M & \\
& M \arrow[u, "\delta"] \arrow[r, dotted] & \le_M \arrow[ul, tail, "e_M"']
\end{tikzcd} \tag{9}$$

Figure 9 Reflexivity of f

So we have that $\langle f, f\rangle$ factors through $\le_M \overset{e_M}{\rightarrowtail} M \times M$, thus $f \le f$.

Corollary 2 *Let L, M be two partially ordered objects and f, g, h morphisms between L and M. Then $f \le g$ and $g \le h$ imply $f \le h$.*

Corollary 3 *Let L, M be two partially ordered objects and $f : L \to M, g : M \to L$ be morphisms. Then $f \le g$ and $g \le f$ imply $f = g$.*

Proof $g \le f$ implies that $\langle g, f\rangle : L \to M \times M$ can be factored through $\le_M \rightarrowtail M \times M$, equivalently, $\langle f, g\rangle$ can be factored through $\geq_M \rightarrowtail M \times M$. Thus $\langle f, g\rangle : L \to M \times M$ can be factored through $\delta_M = \leq_M \cap \geq_M \rightarrowtail M \times M$. This shows $f = g$. $\Box$

The above argument shows that for two partially ordered objects L and M, the relation $\leq$ defined on the morphism set $\mathrm{Mor}(L, M)$ is a partial order relation.

Definition 2 ([2]) *Let L, M be two partially ordered objects in $\mathcal{E}$. A morphism $L \xrightarrow{f} M$ is called order-preserving or monotone if the composite $\leq_L \xrightarrow{e_L} L \times L \xrightarrow{f\times f} M \times M$ factors through $\leq_M$, as in*

$$\begin{tikzcd} \leq_L \arrow[r, tail, "e_L"] \arrow[d, dotted] & L \times L \arrow[d, "f\times f"] \\ \leq_M \arrow[r, tail, "e_M"'] & M \times M \end{tikzcd} \tag{10}$$

Figure 10 The definition of a monotone morphism

In [7], the author defines a function $f : L \to M$ to be order-preserving whenever $x \leq y$ in L implies $f(x) \leq g(x)$ in M. We show that it is equivalent to the above definition.

Lemma 2 *A morphism $L \xrightarrow{f} M$ between two partial ordered objects is order-preserving if and only if $r \leq s$ implies $fr \leq fs$ for every pair of parallel morphisms $A \overset{r}{\underset{s}{\rightrightarrows}} L$.*

Proof $\Rightarrow$. We first show $\langle fr, fs\rangle = f \times f\langle r, s\rangle$. This may be pictured as in the following Figure 11, where p_1, p_2, π_1, π_2 are projections.

$$\begin{tikzcd} A \arrow[rr, "\langle fr,fs\rangle"] \arrow[dr, "\langle r,s\rangle"'] \arrow[ddd, shift right, bend right, "r"'] \arrow[ddd, shift left, bend right, "s"] & & M \times M \arrow[dd, shift right, "\pi_1"'] \arrow[dd, shift left, "\pi_2"] \\ & L \times L \arrow[ur, "f\times f"] \arrow[dd, shift right, "p_1"'] \arrow[dd, shift left, "p_2"] & \\ & & M \\ & L \arrow[ur, "f"'] & \end{tikzcd} \tag{11}$$

Figure 11 Universal property of product

By the universal property of $M \times M$, it follows that $fp_i = \pi_i f \times f$, $i = 1, 2$. Similarly, $r = p_1\langle r, s\rangle, s = p_2\langle r, s\rangle$. Then $fp_i\langle r, s\rangle = \pi_i f \times f\langle r, s\rangle$, so $fr = \pi_1 f \times f\langle r, s\rangle, fs = \pi_2 f \times f\langle r, s\rangle$. By the universal property of $M \times M$, we also have $fr = \pi_1\langle fr, fs\rangle, fs = \pi_2\langle fr, fs\rangle$. So, $\pi_1 < fr, fs >= \pi_1 f \times f\langle r, s\rangle, \pi_2\langle fr, fs\rangle = \pi_2 f \times f\langle r, s\rangle$, thus $\langle fr, fs\rangle = f \times f\langle r, s\rangle$.

Now suppose $r \leq s$, then there exists a morphism $A \xrightarrow{k} \leq_L$ with $\langle r, s\rangle = e_L k$. It follows that the left triangle of in Figure 12 is commutative. Since f is monotone, the right square of the Figure 12 is commutative, i.e., there exists $\leq_L \xrightarrow{m} \leq_M$ such that $f \times fe_L = e_M m$. So $\langle fr, fs\rangle = f \times f\langle r, s\rangle = f \times fe_L k = e_M mk$, which means the outer of the Figure 12 is commutative.

$$\begin{tikzcd}
A \arrow[rr, "\langle fr,fs\rangle"] \arrow[dr, "\langle r,s\rangle"'] \arrow[dddr, bend right, "k"'] & & M \times M \\
& L \times L \arrow[ur, "f\times f"] & \\
& & \leq_M \arrow[uu, tail, "e_M"'] \\
& \leq_L \arrow[uu, tail, "e_L"'] \arrow[ur, "m"'] &
\end{tikzcd} \tag{12}$$

Figure 12 The relation between $\langle fr, fs\rangle$ and e_M

Thus, $\langle fr, fs\rangle$ factors through $\leq_M \overset{e_M}{\rightarrowtail} M \times M$.

$\Leftarrow$. It suffices to show there exists $\leq_L \xrightarrow{m} \leq_M$ with $f \times f e_L = e_M m$, as in the Figure 13.

$$\begin{tikzcd}
A \arrow[d, dotted, "k"] \arrow[dr, bend left, "\langle r,s\rangle"] \arrow[dd, bend right] & \\
\leq_L \arrow[r, tail, "e_L"] \arrow[d, dotted, "m"] & L \times L \arrow[d, "f\times f"] \\
\leq_M \arrow[r, tail, "e_M"] & M \times M
\end{tikzcd} \tag{13}$$

Figure 13 The existence of m

By Lemma 1, it is obvious that m exists.

Definition 3 ([2]) *Let L, M be two partially ordered objects. We say a pair (g, d) of morphisms $L \xrightarrow{g} M$ and $M \xrightarrow{d} L$ is a Galois connection or an adjunction between L and M provided that*

1) both g and d are monotone, and

2) $dg \leq 1_L$ and $1_M \leq gd$, that is, $\langle dg, 1_L\rangle$ and $\langle 1_M, gd\rangle$ factor through $\leq_L$ and $\leq_M$ respectively, as in the following diagrams

$$\begin{tikzcd}
L \arrow[rr, "\langle dg,1_L\rangle"] \arrow[dr] & & L \times L \\
& \leq_L \arrow[ur, tail, "e_L"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[rr, "\langle 1_M,gd\rangle"] \arrow[dr] & & M \times M \\
& \leq_M \arrow[ur, tail, "e_M"'] &
\end{tikzcd} \tag{14}$$

Figure 14 The definitions of $dg \leq 1_L$ and $1_M \leq gd$

We are now in a position to state the main theorem of Galois theory in categorical sense.

Theorem 1 *For every pair of order-preserving morphisms $L \underset{d}{\overset{g}{\rightleftarrows}} M$ between partially ordered objects, the following conditions are equivalent:*

1) (g, d) is an adjunction;

2) $t \leq gs$ implies $dt \leq s$ for all morphisms $A \xrightarrow{s} L$ and $A \xrightarrow{t} M$, and $dt \leq s$ implies $t \leq gs$ for all morphisms $B \xrightarrow{t} M$ and $B \xrightarrow{s} L$. Moreover, these conditions imply

3) $d = dgd$, and $g = gdg$;

4) gd and dg are idempotent.

Proof 1) $\Rightarrow$ 2). Suppose $t \leq gs$, then $\langle t, gs\rangle$ factors through $\leq_M \overset{e_M}{\rightarrowtail} M \times M$. Since d is monotone, we have $d \times d\langle t, gs\rangle = \langle dt, dgs\rangle = e_L lk$. Thus $dt \leq dgs$, as shown in the Figure 15.

$$\begin{tikzcd}
A \arrow[r, "\langle t,gs\rangle"'] \arrow[rr, bend left, "\langle dt,dgs\rangle"] \arrow[dr, bend right] & M \times M \arrow[r, "d\times d"'] & L \times L \\
& \leq_M \arrow[u, tail, "e_M"'] \arrow[r, "l"] & \leq_L \arrow[u, tail, "e_L"']
\end{tikzcd} \tag{15}$$

Figure 15 The proof of $dt \leq dgs$

And by the definition of adjunction, $dg \leq 1_L$, so $dgs \leq s$ for every $s : A \to L$. Whence, $dt \leq s$. The rest is similar.

2) $\Rightarrow$ 1). For every morphism $A \xrightarrow{s} L$ one has $gs \leq gs$, then $dgs \leq s$, thus $dg \leq 1_L$. The rest is similar.

1) $\Rightarrow$ 3). $dg \leq 1_L$ implies $dgd \leq d$ and $1_T \leq gd$ implies $d \leq dgd$ since d is monotone. Then we have $d = dgd$. Similarly, the rest is obvious.

3) $\Rightarrow$ 4). Trivial.

□

Definition 4 *Let L be a partially ordered object.*

1) A projection is an idempotent, monotone morphism $L \xrightarrow{p} L$.

2) A closure operator is a projection c with $1_L \leq c$.

3) A kernel operator is a projection k with $k \leq 1_L$.

So, from Theorem 1, dg and gd are kernel operator and closure operator respectively.

It is well known that the image of an arrow f is the smallest subobject (of the codomain f) through which f can factor. And the factorization of f is unique "up to isomorphism" as the following two Lemmas show.

Lemma 3 ([2]) *In a topos, every morphism f has an image m and factors as $f = me$, with e epi.*

Lemma 4 ([2]) *If $f = me$ and $f' = m'e'$ with m, m' monic and e, e' epi, then each map of the arrow f to the arrow f' extends to a unique map of m, e to m', e'.*

Proposition 1 *If a monotone morphism $L \xrightarrow{f} M$ between two partially ordered objects factors as $f = me$ with image m. Then m and e are monotone morphisms.*

Proof Given $L \xrightarrow{f} M$, which factors as $L \xrightarrow{e} I \overset{m}{\rightarrowtail} M$. The proof is just a matter of observing the corresponding partial order on I. Construct the following commutative Figure 16

$$\begin{tikzcd}
\leq_L \arrow[rr, tail, "e_L"] \arrow[dd] \arrow[dr, dotted, "e'"] & & L\times L \arrow[rr, shift left, "p_1"] \arrow[rr, shift right, "p_2"'] \arrow[dd, "f\times f"] \arrow[dr, dotted, "e\times e"] & & L \arrow[dd, "f"] \arrow[dr, "e"] & \\
& \leq_I \arrow[rr, tail, dotted, "e_I"] \arrow[dl, dotted, "m'"] & & I\times I \arrow[rr, dotted, shift left, "t_1"] \arrow[rr, dotted, shift right, "t_2"'] \arrow[dl, dotted, "m\times m"] & & I \arrow[dl, "m"] \\
\leq_M \arrow[rr, tail, "e_M"] & & M\times M \arrow[rr, shift left, "\pi_1"] \arrow[rr, shift right, "\pi_2"'] & & M &
\end{tikzcd} \tag{16}$$

Figure 16 The partial order on I

By the definition of product $L \times L$, $M \times M$, $I \times I$ with projections p_i, π_i, t_i $(i = 1, 2)$ respectively, we have $fp_i = \pi_i f{\times}f$, $ep_i = t_i e{\times}e$, $mt_i = \pi_i m{\times}m$, i.e., the front, back, bottom faces of the right side of the diagram are all commutative. Then, $\pi_i f{\times}f = fp_i = mep_i = \pi_i m{\times}m{\cdot}e{\times}e$, so $f \times f = m \times m \cdot e \times e$, which means the middle triangle is commutative. Since the smallestness of $m \times m$ is obvious, $f \times f = m \times m \cdot e \times e$ is again an epi-momo factorization, i.e., $m \times m$ is the image of $f \times f$.

We take $\leq_I$ as the pullback of $I \times I \to M \times M$ along e_M, that is, $\leq_I = (I \times I) \cap \leq_M$. It is easy to prove that $\leq_I$ is just both the induced partial order on I and the image of $\leq_L$. This shows the back and the bottom faces of the left side of the diagram are commutative, in other words, $\leq_I \xrightarrow{e_L} I \times I \xrightarrow{m\times m} M \times M$ and $\leq_L \xrightarrow{e_L} L \times L \xrightarrow{e\times e} I \times I$ factor through $\leq_M \xrightarrow{e_M} M \times M$ and $\leq_I \xrightarrow{e_I} I \times I$ respectively. So m, e are all monotone morphisms.

We now make use of the Galois theorem to give the relations between the closure (kernel) operator and its image.

Proposition 2 *Let L be a partially ordered object and $L \xrightarrow{f} L$ a monotone morphism. Then the following statements are equivalent:*

1) f is a projection;

2) If $f = me$ with e epi and m monic, as in

$$\begin{tikzcd}
L \arrow[rr, "f"] \arrow[dr, "e"'] & & L \\
& M \arrow[ur, tail, "m"'] &
\end{tikzcd} \tag{17}$$

Figure 17 The epi-momo factorization of f

then $em = 1_M$;

3) There exist a partially ordered object T and a monotone epi morphism $L \xrightarrow{e} T$ and a monotone monic morphism $T \xrightarrow{m} L$ such that $f = me$ and $1_M = em$.

Proof 1) $\Rightarrow$ 2). If f is a projection, then $meme = me$, so $em = 1_M$ since m is a monomorphism.

2) $\Rightarrow$ 3). Subobject M of L is also a partial order object endowed with the induced order. The left is trivial by Proposition 1.

3) $\Rightarrow$ 1). Trivial.

Proposition 3 *Let L be a partially ordered object and $L \xrightarrow{f} L$ a monotone morphism. Then the following statements are equivalent:*

1) f is a closure operator;

2) (m, e) is an adjunction between M and L, where M is the image of f;

3) There is an adjunction (g, d) between some S and L with $f = gd$.

Proof 1) $\Rightarrow$ 2). As Figure 17 shows, if f is a closure operator, then $em = 1_M$, which implies $em \leq 1_M$; in addition, $1_L \leq me$. Thus (m, e) is an adjunction by Theorem 1.

2) $\Rightarrow$ 3). Trivial.

3) $\Rightarrow$ 1). By Theorem 1(4), the morphism $f = gd$ is a projection. By Definition 3, we have that $1_L \leq f = gd$.

$\square$

Proposition 4 *Let L be a partially ordered object and $L \xrightarrow{f} L$ a monotone morphism. Then the following statements are equivalent:*

1) f is a kernel operator;

2) (e, m) is an adjunction between L and M, where M is the image of f;

3) There is an adjunction (g, d) between L and some T with $f = dg$.

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