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The Frattini Subsystem of a Lie Supertriple System

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Abstract In the present paper, we develop initially the Frattini theory for Lie supertriple systems, obtain some properties of the Frattini subsystem and show that the intersection of all maximal subsystems of a solvable Lie supertriple system is its ideal. Moreover, we give the relationship between ϕ-free and complemented for Lie supertriple system.

Keywords Lie supertriple systems; solvable Lie supertriple systems; the Frattini subsystem; ϕ-free; complemented.

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1. Introduction

As a natural generalization of Lie triple systems, Lie supertriple systems is becoming an efficient tool for analyzing the properties of physical systems. The concept of Lie supertriple systems was introduced gradually in the course of study of the Yang-Baxter equations [1]. In last ten years, many important results of Lie supertriple systems have been obtained [1–10]. In particular, Lie supertriple systems was applied in [9] to obtain some new solutions of Yang-Baxter equation and a simple solution of the Yang-Baxter equation, that is, the Yang-Baxter equation can be reduced to a triple product relation. Moreover, Lie supertriple systems have close connection with Lie superalgebras as the relationship between Lie triple systems and Lie algebras [7].

Frattini theory was initiated in the study of finite groups by Frattini in the paper "Intorno alla generazione dei gruppi di operazioni, Atti della Accademic dei Lincei, Rendicondi Ser.4, Vol. 1, 281-285 (1885)". Frattini described the elements of a finite group into two classes, generators and non-generators, and noticed that the non-generators form a normal subgroup, called the Frattini subgroup, which equals the intersection of all subgroups of the given group. The theory of the Frattini subgroup of a group has been well developed since that time and has proved useful in the study of various problems in the group theory [11–14]. It therefore seems natural to study this concept in some other algebraic structures. Because of the connection between finite groups and

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Lie algebras of finite dimension, Marshall in 1967 made the first attempt which was followed by Barnes, Chao, Gastineau-Hill, Stitzinger, and such investigations have been successfully carried out by Barnes, Bechtell, Schwarck and Towers, et al. [15–24]. For associative algebras the Frattini subalgebra has been mentioned initially by Stitzinger and used by Knopfmacher in 1970 to solve some number-theoretic problem. The first extensive work about Frattini subalgebra of a Malcev algebra has been published by Malek [25, 26]. The Frattini theory has been developped initially for restricted Lie superalgebras and n-Lie algebras [27–29].

In this paper, we shall define T as a finite dimensional Lie supertriple system over a field $\mathbf{F}$ of arbitrary characteristic, $\mathrm{Aut}(T)$ as the automorphism group of T, $J(T)$ as Jacobson radical which is the intersection of all maximal ideals of T, $F(T)$ as the Frattini subsystem of T which is the intersection of all maximal subsystems of T and $\phi(T)$ as Frattini ideal of T is the largest ideal of T that is contained in $F(T)$. We let $C(T) = \{x \in T | L(x,T) = R(T,x) = 0\}$ be the center of T and $\mathbf{N}$ be the set of all positive integers. Our notation and terminology are standard as may be found in [4, 9, 16–29]. For background material on Lie supertriple systems we refer to [1–10].

The purpose of the present paper is to mention initially and develop Frattini theory for Lie supertriple systems. Definitions are given in Section 1. In Section 2 we obtain some properties of the Frattini subsystem and show that the intersection of all maximal subsystems of a solvable Lie supertriple system is its ideal. In Section 3, we give the relationship between ϕ-free and complemented for Lie supertriple system.

Definition 1.1 ([9]) *A Lie supertriple system $T = T_{\bar{0}} \oplus T_{\bar{1}}$ is a $\mathbf{Z}_2$-graded vector space with a ternary product $[\,,\,,\,]$ satisfying the following identities:*

(1) $\sigma([x,y,z]) \equiv (\sigma(x) + \sigma(y) + \sigma(z)) \pmod 2$;

(2) $[x,y,z] = -(-1)^{xy}[y,x,z]$;

(3) $(-1)^{xz}[x,y,z] + (-1)^{yx}[y,z,x] + (-1)^{zy}[z,x,y] = 0$;

(4) $[u,v,[x,y,z]] = [[u,v,x],y,z] + (-1)^{(u+v)x}[x,[u,v,y],z] + (-1)^{(u+v)(x+y)}[x,y,[u,v,z]]$

for any x, y, z, u, $v \in T$, where we denote $\sigma(x)$ as the graded degree of x, for simplicity of the degree, and denote $(-1)^{\sigma(x)\sigma(y)}$ as $(-1)^{xy}$.

Throughout this paper, if $(-1)^x$ occurs in an expression, then it is assumed that x is homogeneous.

Definition 1.2 ([30]) *Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be a superalgebra whose multiplication is denoted by $[\,,\,]$. This implies in particular that $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbf{Z}_2$. We call L a Lie superalgebra if the multiplication satisfies the following identities:*

(1) $[a,b] = -(-1)^{\alpha\beta}[b,a]$,

(2) $[a,[b,c]] = [[a,b],c] + (-1)^{\alpha\beta}[b,[a,c]]$,

for any $a \in L_\alpha$, $b \in L_\beta$, $c \in L$; α, $\beta \in \mathbf{Z}_2$.

Remark 1.3 (1) $T_{\bar{0}}$ is an ordinary Lie triple system and $L_{\bar{0}}$ is an ordinary Lie algebra.

(2) Let T be a Lie superalgebra (Lie algebra). It is clear that if we introduce a triple product

$[x,y,z]$ by $[x,y,z]=[[x,y],z]$ for any $x,\ y,\ z\in T$, then T becomes a Lie supertriple system (Lie triple system). So in this sense every Lie superalgebra (Lie algebra) is a Lie supertriple system (Lie triple system). Many important features of Lie algebras are not necessarily true for Lie superalgebras, for instance, Lie's theorem and Levi's theorem of Lie algebras are not true, in general, for Lie superalgebras. Thus, many important features of Lie triple systems are not necessarily true for Lie supertriple systems.

Definition 1.4 ([9]) *A superderivation of a Lie supertriple system T is a homogeneous linear mapping D of T into T such that*

$$D([x,y,z])=[D(x),y,z]+(-1)^{Dx}[x,D(y),z]+(-1)^{(x+y)D}[x,y,D(z)],\quad \forall x,y,z\in T.$$

$\mathrm{Der}(T)$ *denotes the set of all superderivations of T.*

It can be shown that $\mathrm{Der}(T)$ is a Lie superalgebra under the bracket operation $[D_1,D_2]=D_1D_2-(-1)^{D_1D_2}D_2D_1$ taken in $\mathrm{End}_{\mathbf{F}}(T)$.

Definition 1.5 ([9]) *Let $L(x,y)z=[x,y,z]$ and $R(x,y)z=(-1)^{z(x+y)}[z,x,y]$. Then $L(x,y)$ and $R(x,y)$ are called the left and right multiplication operator of T, respectively.*

Clearly, $L(x,y)=(-1)^{xy}R(y,x)-R(x,y)$. Define $\mathrm{InnDer}(T)=\{D|D=\Sigma L(x,y)\}$ for any $x,y\in T$, then $\mathrm{InnDer}(T)$ is a subset of $\mathrm{Der}(T)$ and $\mathrm{InnDer}(T)$ is closed with respect to $[,]$. $\mathrm{InnDer}(T)$ is called the inner derivation algebra of T. It is easy to see that $\mathrm{InnDer}(T)$ is an ideal of $\mathrm{Der}(T)$. So $\mathrm{InnDer}(T)$ is a Lie superalgebra.

Definition 1.6 ([9]) *Let B be an ideal of T, $T^2=[T,T,T]=T^{(2)}$, $B^{(1)}=B$ and $B^{(k+1)}=[T,B^{(k)},B^{(k)}]$. Clearly, $B^{(k)}$ is an ideal of T for any $k\in\mathbf{N}$. An ideal B of T is called T-solvable in T if there is some positive integer k such that $B^{(k)}=\{0\}$. T is called solvable if $T^{(k)}=\{0\}$. In particular, an ideal B of T is called abelian if $B^{(2)}=\{0\}$.*

2. Some properties of the Frattini subsystem

Lemma 2.1 *Let T be a Lie supertriple system. Then the following statements hold:*

(1) If B is a subsystem of T such that $B+F(T)=T$, then $B=T$.

(2) If B is a subsystem of T such that $B+\phi(T)=T$, then $B=T$.

Proof (1) Suppose that $B\neq T$. Then there is a maximal subsystem M of T such that $B\subseteq M$ since B is a subsystem of T. Now $F(T)\subseteq M$, and so $T=M$ since $B+F(T)=T$, contradicting the maximality of M. Thus $B=T$.

(2) It is similar to Lemma 2.1 (1). □

Lemma 2.2 *Let T be a Lie supertriple system. If B is an ideal of T, then there is a proper subsystem C of T such that $T=B+C$ if and only if $B\not\subseteq F(T)$.*

Proof Let C be a proper subsystem of T such that $T=B+C$. If $B\subseteq F(T)$, then $T=B+C\subseteq F(T)+C\subseteq T$. So $T=F(T)+C$. In the light of Lemma 2.1, we have $C=T$ and a contradiction. Hence $B\not\subseteq F(T)$. Conversely, if there is no proper subsystems C of T such that $T=B+C$, then every maximal subsystem of T contains B. So $B\subseteq F(T)$ and we have a contradiction. The result holds. □

Lemma 2.3 *Let T be a Lie supertriple system. Let C be a subsystem of T and B be an ideal of T. Then the following statements hold:*

(1) If $B\subseteq F(C)$, then $B\subseteq F(T)$;

(2) If $B\subseteq \phi(C)$, then $B\subseteq \phi(T)$.

Proof (1) If $C=T$, then it is clear. If $C\neq T$ and $B\nsubseteq F(T)$, then there is a proper subsystem M of T such that $T=B+M=C+M$ by Lemma 2.2. We can obtain $\dim C=\dim(B+C\cap M)$ by virtue of the dimensional formula. Then $C=B+C\cap M$ since $C\supseteq B+C\cap M$, i.e., $C=B+C\cap M\subseteq F(C)+C\cap M\subseteq C$. So $F(C)+C\cap M=C$ and $C=M\cap C$ by means of Lemma 2.1, i.e., $C\subseteq M$. Thus $T=B+M\subseteq C+M\subseteq M$ and we have a contradiction. Hence $B\subseteq F(T)$.

(2) It is similar to Lemma 2.3 (1). □

Lemma 2.4 *Let T be a Lie supertriple system. Then $F(T)\subseteq T^2$ and $J(T)\subseteq T^2$.*

Proof If $T=T^2=[T,T,T]$, then it is clear that $F(T)\subseteq T^2$. If $T\neq[T,T,T]$, then suppose that $x\in F(T)$ and $x\notin T^2$, and that $\dim T=n$. We can construct an $(n-1)$-dimensional subspace of T containing T^2 but not x, and this subspace is clearly a maximal subsystem of T. But x belongs to every such that subsystem since $x\in F(T)$. Then we have a contradiction, so $F(T)\subseteq T^2$. Similarly, it is easy to show that $J(T)\subseteq T^2$. □

Lemma 2.5 *Let T be a Lie supertriple system. If $T=A_1\oplus A_2\oplus\cdots\oplus A_n$, where each A_i $(1\le i\le n)$ is an ideal of T, then $\phi(T)=\phi(A_1)\oplus\phi(A_2)\oplus\cdots\oplus\phi(A_n)$.*

Proof We show first that $F(T)\subseteq F(A_1)+F(A_2)+\cdots+F(A_n)$. If M_i is a maximal subsystem of A_i $(1\le i\le n)$, then $M_i+(A_1\oplus A_2\oplus\cdots\oplus \hat{A}_i\oplus\cdots\oplus A_n)$ is a maximal subsystem of T, where $\hat{A}_i$ indicates that A_i is omitted from the direct sum. The result follows by taking intersections.

We will show that $\phi(A_i)=\phi(T)\cap A_i$ $(1\le i\le n)$. Since $F(T)\subseteq F(A_1)+F(A_2)+\cdots+F(A_n)$, we have $\phi(T)\cap A_i\subseteq F(T)\cap A_i\subseteq F(A_i)$. Then $\phi(T)\cap A_i\subseteq\phi(A_i)$. It is easy to show that $\phi(A_i)$ is an ideal of T. So $\phi(A_i)\subseteq\phi(T)$ by Lemma 2.3. Consequently, $\phi(A_i)\subseteq\phi(T)\cap A_i$, which means that $\phi(A_i)=\phi(T)\cap A_i$ as claimed. So clearly $\phi(T)\supseteq\phi(A_1)+\phi(A_2)+\cdots+\phi(A_n)$.

Now suppose that $x\in\phi(T)$. Then $x=x_1+x_2+\cdots+x_n$ and $[x,A_i,A_i]=[x_i,A_i,A_i]\subseteq A_i\cap\phi(T)=\phi(A_i)$, where $x_i\in F(A_i), 1\le i\le n$. If $x_i\notin\phi(A_i)$, then $[\phi(A_i)\dot{+}\, \mathrm{F}x_i,A_i,A_i]\subseteq\phi(A_i)\subset\phi(A_i)\dot{+}\,\mathrm{F}x_i$. So $\phi(A_i)\dot{+}\,\mathbf{F}x_i$ is an ideal of A_i. Since $x_i\in F(A_i)$ and $\phi(A_i)\subseteq F(A_i)$, it is clear that $\phi(A_i)\dot{+}\,\mathbf{F}x_i$ is an ideal of A_i contained in $F(A_i)$ and strictly bigger than $\phi(A_i)$. Thus we have a contradiction since $\phi(A_i)$ is the largest ideal of A_i which is contained in $F(A_i)$. Hence $x_i\in\phi(A_i)$ and $x\in\phi(A_1)+\phi(A_2)+\cdots+\phi(A_n)$ and $\phi(T)=\phi(A_1)\dot{+}\,\phi(A_2)\dot{+}\cdots\dot{+}\,\phi(A_n)$. The proof is completed since $\phi(A_i)$ is an ideal of $\phi(T)$. □

Lemma 2.6 *Let T be solvable. Then the following statements hold:*

(1) If A is a minimal ideal of T, then A is abelian;

(2) $J(T) = T^2$. In particular, if T is abelian, then $F(T) = \phi(T) = J(T) = \{0\}$.

Proof (1) Let A be an ideal of T. It is clear that $[T, A, A]$ is also an ideal of T. Since A is a minimal ideal of T, $[T, A, A] = A$ or $[T, A, A] = \{0\}$. If $[T, A, A] = A^{(2)} = A$, then $A^{(k+1)} = [T, A^{(k)}, A^{(k)}] = [T, A, A]$. Since T is solvable, $[T, A, A] = A^{(k+1)} = \{0\}$ for some $k \in \mathbf{N}$. Hence A is abelian.

(2) Let I be a maximal ideal of T. Since the quotient algebra T/I is solvable, T/I contains no proper ideals. So T/I is one-dimensional and abelian. Hence we have $I \supseteq T^2$. This applies for all maximal ideals I of T, so that $J(T) \supseteq T^2$. By Lemma 2.4, the result follows. □

Theorem 2.7 *Let T be solvable. Then $F(T)$ is an ideal of T.*

Proof We use induction over $\dim T$. The result is trivial for $\dim T = 1$.

Let A be a minimal ideal of T. Put $F(T : A) = \bigcap\{M : A \subseteq M,\ M$ is a maximal subsystem of $T\}$. Then $F(T : A)/A = F(T/A)$ and $F(T : A)$ is an ideal of T by induction hypothesis.

If $A \subseteq F(T)$, then $F(T) = F(T : A)$ is an ideal of T. Suppose $A \nsubseteq F(T)$. So there is a maximal subsystem M of T such that $T = M + A$ by Lemma 2.2. Since A is a minimal ideal of T, $[T, A, A] = \{0\}$ by Lemma 2.6 (1). Then $A \subseteq C_T(A)$ and $[A \cap M, T, T] = [A \cap M, M + A, M + A] \subseteq A \cap M$, i.e., $A \cap M$ is an ideal of T contained in A, hence $T = A \dot{+} M$. If $A \subset C_T(A)$, then $\{0\} \subset M \cap C_T(A) \lhd T$ and every maximal subsystem M of T contains some minimal ideal B of T. In this case, $F(T) = \bigcap\{F(T : B)$: B is a minimal ideal of $T\}$, which is an ideal of T.

Suppose $C_T(A) = A$. We will show that $F(T) = \{0\}$. Let M be a maximal subsystem not containing A and suppose $m \in M, m \notin A$. We prove $m \notin F(T)$. Since $m \notin C_T(A) = A$, there exists $a \in A$ such that $[m, a, T] \neq 0$. Define $\theta : T \to T$ by putting $\theta = \mathrm{id}_T + L(m, a)$. Since $L(m, a)^2 = 0$, $(\mathrm{id}_T + L(m, a))(\mathrm{id}_T - L(m, a)) = \mathrm{id}_T$. Then θ is an automorphism of T. Put $M_1 = \theta(M)$. Clearly, M_1 is a maximal subsystem of T. If $m \in M_1$, then there is $m' \in M$ such that $m = m' + [a, m, m']$, so $[a, m, m'] \in M$.

Since $[a, m, [a, m, m']] \in [A, T, A] = \{0\}$, $[a, m, m] = [a, m, m' + [a, m, m']] = [a, m, m'] + [a, m, [a, m, m']] = [a, m, m'] \in M$. Then $[a, m, m] = [a, m, m'] \in A \cap M = \{0\}$, i.e., there is an element $m \in T$ such that $[m, a, m] = 0$. But $[m, a, T] \neq 0$, a contradiction. So $m \notin M_1 = \theta(M)$, namely, $m \notin F(T)$. It follows that $F(T) = \{0\}$. Thus $F(T)$ is an ideal of T. □

3. On ϕ-free Lie supertriple systems

Definition 3.1 *T is called ϕ-free if $\phi(T) = \{0\}$.*

Lemma 3.2 *Let A be an ideal of T and let B be a subsystem of T which is minimal with respect to $T = A + B$. Then $A \cap B \subseteq \phi(B)$.*

Proof Suppose that $A \cap B \nsubseteq \phi(B)$. Then $A \cap B \nsubseteq F(B)$ since $A \cap B$ is an ideal of B and $\phi(B)$ is the largest ideal of B which is contained in $F(B)$. It follows that there is a maximal subalgebra M of B such that $A \cap B \nsubseteq M$. Clearly, $B = A \cap B + M$, and so $T = A + (A \cap B + M) = A + M$, which contradicts the minimality of B. Hence $A \cap B \subseteq \phi(B)$. $\square$

Lemma 3.3 *Let I be an abelian ideal of T. If $I \cap \phi(T) = \{0\}$, then there is a subsystem B of T such that $T = I \dot{+} B$.*

Proof Choose a subsystem B of T to be minimal with respect to $T = I + B$. Then $I \cap B \subseteq \phi(B)$ by virtue of Lemma 3.2. It is clear that $I \cap B$ is an ideal of T since $[T, I, I] = \{0\}$. Hence $I \cap B \subseteq \phi(T)$ by means of Lemma 2.3. It follows that $I \cap B \subseteq I \cap \phi(T) = \{0\}$ and the result follows. $\square$

Definition 3.4 *T is called complemented if for each subsystem A of T, there exists a subsystem B of T such that $T = \langle A, B\rangle$ and $A \cap B = \{0\}$, where $\langle A, B\rangle$ denotes T generated by A and B.*

Lemma 3.5 *If T is complemented, then T is ϕ-free.*

Proof If $F(T) \neq \{0\}$, then there is a subsystem A of T such that $T = \langle F(T), A\rangle$. Let B be a maximal subsystem of T containing A. So $T = \langle F(T), A\rangle \subseteq \langle F(T), B\rangle \subseteq B$, a contradiction. Hence $F(T) = \phi(T) = \{0\}$ and T is ϕ-free. $\square$

Lemma 3.6 *If T is complemented and B is an ideal of T, then T/B is complemented.*

Proof Consider T/B. Let $H/B \subseteq T/B$. Then there exists a proper subsystem K of T such that $T = \langle H, K\rangle$ and $H \cap K = \{0\}$. Therefore, $T/B = \langle H/B, (K + B)/B\rangle$. It is easy to show that $H \cap (K + B) = B$, so $(H/B) \cap ((K + B)/B) = \{0\}$. Hence T/B is complemented. $\square$

Theorem 3.7 *If A is a minimal abelian ideal of T, then T is complemented if and only if there exists a subsystem B of T such that $T = A \dot{+} B$ and B is complemented.*

Proof ($\Rightarrow$). If T is complemented, then T is ϕ-free in the light of Lemma 3.5. So there exists a subsystem B of T such that $T = A \dot{+} B$ by means of Lemma 3.3. It is clear that B is complemented by virtue of Lemma 3.6.

($\Leftarrow$). Suppose that $T = A \dot{+} B$ and B is complemented, where B is a subsystem of T. Let U be a subsystem of T. We need to find a subsystem W of T such that $T = \langle U, W\rangle$ and $U \cap W = \{0\}$. Since B is complemented, there is a subsystem V of T with respect to $A \subseteq V \subseteq T$ such that $B \cong T/A = \langle (U + A)/A, V/A\rangle$ and $((U + A)/A) \cap (V/A) = \{0\}$. So $\langle U + A, V\rangle = T$ and $(U + A) \cap V = A$. We consider three cases.

(1) If $A \cap U = \{0\}$, then $T \supseteq \langle U, V\rangle = \langle\langle U, A\rangle, V\rangle \supseteq \langle U + A, V\rangle = T$, that is, $T = \langle U, V\rangle$. Since $(U + A) \cap V = A$, we have $\dim((U + A) \cap V) = \dim A$, that is, $\dim U + \dim A - \dim(U \cap A) + \dim V - \dim(U + V) = \dim A$ and $\dim U + \dim A - \dim(U \cap A) + \dim V - \dim U - \dim V + \dim(U \cap V) = \dim A$. So $\dim(U \cap A) = \dim(U \cap V)$. Since $U \cap A \subseteq U \cap V$, we have $U \cap A = U \cap V = \{0\}$. So we can put $W = V$. Hence $\langle U, W\rangle = T$ and $U \cap W = \{0\}$.

(2) If $A \cap U \neq \{0\}$ and $A \subseteq U$, then we put $W = V \cap B$. Since $\langle U, W\rangle = \langle U, V \cap B\rangle = \langle U, A, V \cap B\rangle = \langle U, \langle A, V \cap B\rangle\rangle \supseteq \langle U, A + V \cap B\rangle = \langle U, V\rangle = T$, we have $\langle U, W\rangle = T$. So $U \cap W = U \cap (V \cap B) = (U \cap V) \cap B \subseteq A \cap B = \{0\}$ since $(U + A) \cap V = A$. Hence $\langle U, W\rangle = T$

and $U \cap W = \{0\}$.

(3) If $A \cap U \neq \{0\}$ and $A \not\subseteq U$, then we put $W = V \cap B$. So $A \subseteq \langle U, W\rangle$. For, if not, let M be a maximal subsystem of T with respect to $\langle U, W\rangle \subseteq M$. Since $\dim(V \cap B + A) = \dim(V \cap B) + \dim A - \dim(V \cap B \cap A) = \dim(V \cap B) + \dim A = \dim V + \dim A + \dim B - \dim(V + B) = \dim V + \dim T - \dim T = \dim V$ by means of $T = A \dot{+} B$ and $A \subseteq V$, we have $V \cap B + A = V$ by virtue of $A + V \cap B \subseteq V$. So $\langle U, W, A\rangle = \langle U, V \cap B, A\rangle = \langle U + A, V \cap B, A\rangle = \langle U + A, \langle V \cap B, A\rangle\rangle = \langle U + A, V\rangle = T$. If $A \subseteq M$, then $\langle U, W, A\rangle \subseteq M$, a contradiction. Hence $A \not\subseteq M$ and this implies $T = A + M$.

If $A \cap M \neq \{0\}$, then $[A \cap M, T, T] = [A \cap M, A + M, A + M] \subseteq A \cap M$ by $[T, A, A] = \{0\}$, so $A \cap M$ is an abelian ideal of T. Since A is a minimal abelian ideal of T, we have a contradiction. Then $A \cap M = \{0\}$ and hence $U \cap A \subseteq M \cap A = \{0\}$, this contradicts $A \cap U \neq \{0\}$. Thus $\langle U, W\rangle = \langle U + A, W + A\rangle = \langle U + A, V \cap B + A\rangle = \langle U + A, V\rangle = T$ and $U \cap W = U \cap (V \cap B) = U \cap V \cap B = U \cap A \cap B = \{0\}$ since $U \cap A \subseteq U \cap V$. The result follows. $\square$

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