

# The Uniqueness of Decomposition of Symplectic Ternary Algebras with Trivial Center

Xi Mei BAI\*, Wen Li LIU

*College of Mathematics and Computer, Hebei University, Hebei 071002, P. R. China*

**Abstract** In this paper, the authors define the center of a Symplectic ternary algebra, and investigate the relationship between the center of a Symplectic ternary algebra and that of the Lie triple system associated with it. As an application of the relationship, the unique decomposition theorem for Symplectic ternary algebras with trivial center is obtained.

**Keywords** Symplectic ternary algebras; Lie triple system; center;  $U$ -endomorphism.

**Document code** A

**MR(2000) Subject Classification** 17A40; 17B05

**Chinese Library Classification** O152.5; O153.3

## 1. Introduction

The algebras studied here are a generalization of the class of ternary algebras given in [1], which are, in turn, a variation of Freudenthal triple systems [2]. The advantage of the latest algebras, which we call Symplectic ternary algebras, is that they are defined by identities and hence admit direct sums. Faulkner and Ferrar have investigated the structure of this algebra. They have proved that semisimple Symplectic ternary algebras can be decomposed into simple ideals and given a classification of simple algebras over algebraically closed fields of characteristic zero. They gave a construction of a Lie triple system from a Symplectic ternary algebra, and related notions of ideal and form of a symplectic ternary algebra to the parallel notions in a Lie triple system constructed from the symplectic ternary algebra.

It is well known that a Lie triple system  $T$  with trivial center can be decomposed into the direct sum of simple ideals, and the decomposition is unique. The purpose of the present paper is to investigate this problem for Symplectic ternary algebras. Firstly, we recall some fundamental notions and facts which can be found in [2] or [3].

If  $U$  is a ternary algebra with trilinear product  $\langle x, y, z \rangle$  satisfying

$$S(x, y) = L(x, y) - L(y, x) = R(x, y) - R(y, x), \quad (1)$$

$$S(x, y)R(z, w) = R(z, w)S(x, y) = R(S(x, y)z, w) = R(z, S(x, y)w), \quad (2)$$

$$[R(x, y), R(z, w)] = R(R(z, w)x, y) = R(x, R(w, z)y), \quad (3)$$

---

Received April 30, 2008; Accepted May 16, 2009

Supported by the National Natural Science Foundation of China (Grant No. 10871192).

\* Corresponding author

E-mail address: baixm\_101@163.com (X. M. BAI)

for all  $x, y, z, u, v \in U$ , where  $\langle x, y, z \rangle = L(x, y)z = R(y, z)x = U(x, z)y$ , we say that  $U$  is a Symplectic ternary algebra.

**Example 1.1** ([2]) If  $U$  is a vector space with a nondegenerate skew form  $\langle \cdot, \cdot \rangle$ , then  $\langle \cdot, \cdot, \cdot \rangle$  defined by

$$\langle x, y, z \rangle = 1/2(\langle x, y \rangle z + \langle y, z \rangle x + \langle x, z \rangle y), \quad x, y, z \in U,$$

satisfies (1)–(3).

The verification of Example 1.1 is straightforward, which means that  $U$  is a Symplectic ternary algebra.

A subspace  $I$  of a Symplectic ternary algebra  $U$  is called a subalgebra if  $\langle I, I, I \rangle \subseteq I$ . An ideal of a Symplectic ternary algebra  $U$  is a subspace  $I$  such that the product of an element of  $I$  with any two elements of  $U$  in any order lies in  $I$ . Indeed, if  $\langle U, I, U \rangle \subseteq I$  and either  $\langle I, U, U \rangle \subseteq I$  or  $\langle U, U, I \rangle \subseteq I$ , then  $I$  is an ideal of  $U$ .

An endomorphism of a Symplectic ternary algebra  $U$  is a linear transformation  $\varphi$  of  $U$  satisfying  $\varphi\langle x, y, z \rangle = \langle \varphi(x), \varphi(y), \varphi(z) \rangle$ . Clearly,  $\text{Ker}\varphi$  is an ideal of  $U$ .

A Lie triple system over a field  $F$  is a vector space  $T$  with a trilinear product  $[\cdot, \cdot, \cdot]$  satisfying

$$[x, y, y] = 0, \quad (4)$$

$$[x, y, z] + [y, z, x] + [z, x, y] = 0, \quad (5)$$

$$[[x, y, z], u, v] = [[x, u, v], y, z] + [x, [y, u, v], z] + [x, y, [z, u, v]], \quad (6)$$

for all  $x, y, z, u, v \in T$ , where  $[x, y, z] = L(x, y)z = R(y, z)x$ .

An ideal of a Lie triple system  $T$  is a subspace  $I$  for which  $[T, T, I] \subseteq I$ .

Finally, we recall the following fact [2] which will be needed in the sequel.

**Lemma 1.1** ([2]) If  $U$  is a Symplectic ternary algebra, then  $T(U) = U \oplus U$  will be a Lie triple system with the following product

$$\begin{aligned} \left[ \begin{pmatrix} a \\ b \end{pmatrix} \quad \begin{pmatrix} c \\ d \end{pmatrix} \quad \begin{pmatrix} e \\ f \end{pmatrix} \right] &= \begin{pmatrix} R(c, f) - R(e, d) & S(e, c) \\ S(d, f) & R(f, c) - R(d, e) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \begin{pmatrix} \langle a, c, f \rangle - \langle a, e, d \rangle + \langle e, c, b \rangle - \langle c, e, b \rangle \\ \langle d, f, a \rangle - \langle f, d, a \rangle + \langle b, f, c \rangle - \langle b, d, e \rangle \end{pmatrix}. \end{aligned} \quad (7)$$

We call  $T(U)$  the Lie triple system associated with  $U$ .

## 2. The center of Symplectic ternary algebras

In this section, we define the center,  $C(U)$ , of a Symplectic ternary algebra  $U$ , and enumerate several elementary results concerning  $C(U)$ .

**Definition 2.1** For a subspace  $S$  of a Symplectic ternary algebra  $U$ , we define the centralizer of  $S$  in  $U$ ,  $C_U(S) := \{x \in U \mid \langle x, S, U \rangle = \langle S, x, U \rangle = \langle x, U, S \rangle = \langle U, x, S \rangle = 0\}$ . In fact, by identity (1), we have  $\langle U, S, x \rangle = \langle S, U, x \rangle = 0, \forall x \in C_U(S)$ . In particular,  $C(U) := C_U(U) = \{x \in T \mid$

$\langle x, U, U \rangle = \langle U, x, U \rangle = 0\}$  is the center of  $U$ .

**Definition 2.2** The centralizer of a subspace  $S$  in a Lie triple system  $T$  is the set of

$$C_T(S) := \{x \in T \mid L(x, a) = R(a, x) = 0, \forall a \in S\}.$$

In particular,  $C(T) := C_T(T) = \{x \in T \mid [T, T, x] = 0\}$  is the center of  $T$ .

**Lemma 2.1** If  $U$  is a Symplectic ternary algebra, then the following statements hold:

- (i) If  $S$  is a subset of  $U$  and  $S \neq \emptyset$ , then  $C_U(S)$  is a subalgebra of  $U$ ;
- (ii) If  $S$  is an ideal of  $U$ , then  $C_U(S)$  is an ideal of  $U$ .

**Proof** 1) It is easily seen that  $C_U(S)$  is a subspace. Now we prove  $\langle x, y, z \rangle \in C_U(S)$ , for any  $x, y, z \in C_U(S)$ . For any  $s \in S$ , and  $u \in U$ , by identity (3) we have

$$\langle \langle x, y, z \rangle, s, u \rangle = \langle \langle x, s, u \rangle, y, z \rangle + \langle x, \langle y, s, u \rangle, z \rangle + \langle x, y, \langle z, u, s \rangle \rangle = 0, \quad (8)$$

and

$$\langle s, \langle x, y, z \rangle, u \rangle = \langle \langle s, x, u \rangle, y, z \rangle - \langle \langle x, y, z \rangle x, u \rangle - \langle s, x, \langle u, z, y \rangle \rangle = 0. \quad (9)$$

The similar argument shows that  $\langle \langle x, y, z \rangle, u, s \rangle = \langle u, \langle x, y, z \rangle, s \rangle = 0$ . The proof of (i) is completed.

2) By Definition 2.1, it suffices to prove that a permutation of  $\langle C_U(S), U, U \rangle$  is contained in  $C_U(S)$ . In order to show that  $C_U(S)$  contains  $\langle C_U(S), U, U \rangle$ , let us consider  $x \in C_U(S)$ ,  $u, u_1, u_2 \in U$ , and  $s \in S$ . By virtue of (3), we have

$$\langle \langle x, u, u_1 \rangle, s, u_2 \rangle = \langle \langle x, s, u_2 \rangle, u, u_1 \rangle + \langle x, \langle u, s, u_2 \rangle, u_1 \rangle + \langle x, u, \langle u_1, u_2, s \rangle \rangle = 0 \quad (10)$$

and

$$\langle s, \langle x, u, u_1 \rangle, u_2 \rangle = \langle \langle x, u, u_1 \rangle, u_2, s \rangle = \langle u_2, \langle x, u, u_1 \rangle, s \rangle = 0.$$

This means  $\langle C_U(S), U, U \rangle \subseteq C_U(S)$ . The proof for  $\langle U, C_U(S), U \rangle \subseteq C_U(S)$  is analogous, which finishes the proof.  $\square$

**Lemma 2.2** If  $T(U)$  is the Lie triple system associated with a symplectic ternary algebra  $U$ , then  $T(C(U)) = C(T(U))$ .

**Proof** Since the identity (7) is zero for  $e, f \in C(U)$ , we have  $T(C(U)) \subseteq C(T(U))$ . On the other hand, by applying (7) to  $\begin{pmatrix} e \\ f \end{pmatrix} \in C(T(U))$  and letting  $b = d = 0$ , we get  $\langle a, c, f \rangle = 0$ . Let  $a = d = 0$ . We have  $\langle b, f, c \rangle = 0$ . Hence  $f \in C(U)$ . A similar calculation shows that  $e \in C(U)$ . Therefore  $C(T(U)) \subseteq T(C(U))$ , which completes the proof.  $\square$

**Theorem 2.1** If a Symplectic ternary algebra  $U$  has the decomposition  $U = U_1 \oplus U_2$ , where  $U_1, U_2$  are ideals of  $U$ , then  $T(U) = T(U_1) \oplus T(U_2)$ .

**Proof** For  $\begin{pmatrix} e \\ f \end{pmatrix} \in T(U_1)$ , and  $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \in T(U)$ . Since  $U_1$  is an ideal of  $U$ , by (7), it is easy to prove that  $T(U_1)$  is an ideal of  $T(U)$ . A similar argument works for  $T(U_2)$ . Since  $U_1 \cap U_2 = \{0\}$ , we have  $T(U_1) \oplus T(U_2) \subseteq T(U)$ . Now we verify that  $T(U) \subseteq T(U_1) \oplus T(U_2)$ . Assume that

$a = a_1 + a_2, b = b_1 + b_2, a_i, b_i \in U_i$ . Since

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \in T(U_1) + T(U_2),$$

we have  $T(U) \subseteq T(U_1) \oplus T(U_2)$ . The proof is completed.  $\square$

The following two Lemmas (Lemma 2.3 and Lemma 2.4) were given in [3].

**Lemma 2.3** ([3]) *If a Lie triple system  $T$  has the decomposition  $T = T_1 \oplus T_2$ , where  $T_1, T_2$  are ideals of  $T$ , then  $C(T) = C(T_1) \oplus C(T_2)$ .*

**Theorem 2.2** *If a Symplectic ternary algebra  $U$  has the decomposition:  $U = U_1 \oplus U_2$ , where  $U_1, U_2$  are ideals of  $U$ , then  $C(U) = C(U_1) \oplus C(U_2)$ .*

**Proof** This is an immediate consequence of Theorem 2.1, Lemmas 2.2 and 2.3.  $\square$

**Lemma 2.4** ([3]) *If a Lie triple system  $T$  can be decomposed into the direct sum of two ideals:  $T = T_1 \oplus T_2$ , and  $A$  is a subsystem of  $T$  such that  $T_1 \subseteq A$ , then  $A = T_1 \oplus (T_2 \cap A)$ .*

Combining Theorem 2.1 with Lemma 2.4, we have the following lemma:

**Lemma 2.5** *If a Symplectic ternary algebra  $U$  can be decomposed into the direct sum of two ideals:  $U = U_1 \oplus U_2$ , and  $A$  is a subalgebra of  $U$  such that  $U_1 \subseteq A$ , then  $A = U_1 \oplus (U_2 \cap A)$ .*

### 3. The unique decomposition theorem

In this section, we will discuss the unique decomposition theorem for Symplectic ternary algebras.

**Definition 3.1** *If  $\varphi$  is an endomorphism of a Symplectic ternary algebra  $U$ , satisfying*

$$\varphi L(a, b) = L(a, b)\varphi, \varphi U(a, b) = U(a, b)\varphi,$$

*then we say that  $\varphi$  is a  $U$ -endomorphism of  $U$ .*

**Example 3.1** *If  $U$  is the direct sum of two ideals  $U = U_1 \oplus U_2$ , we define a linear map  $\pi: U \rightarrow U_1$  by  $\pi(u_1 + u_2) = u_1$ . By direct calculation, we have that  $\pi$  is a  $U$ -endomorphism of  $U$ .*

**Lemma 3.1** *Let  $\varphi$  be a  $U$ -endomorphism of  $U$ . Then the following statements hold:*

- 1)  $U = \text{Ker}\varphi^k \oplus \text{Im}\varphi^k$ , for some integer  $k$ ;
- 2) If  $U$  is indecomposable, then  $\varphi^k = 0$ , or  $\varphi \in \text{Aut } U$ .

**Proof** 1) Let  $f(\lambda)$  be the minimum polynomial of  $\varphi$  and  $f(\lambda) = \lambda^k g(\lambda)$ . As  $\lambda$  and  $g(\lambda)$  are coprime polynomials, there are two polynomials  $u(\lambda), v(\lambda)$  such that  $u(\lambda)g(\lambda) + v(\lambda)\lambda^k = 1$ . Hence we have

$$u(\varphi)g(\varphi)y + v(\varphi)\varphi^k y = y, \quad \forall y \in U$$

and

$$u(\varphi)g(\varphi)y \in \text{Ker}\varphi^k, v(\varphi)\varphi^k y \in \text{Im}\varphi^k.$$

It then follows that  $U = \text{Ker}\varphi^k + \text{Im}\varphi^k$ .

For any  $y \in \text{Ker}\varphi^k \cap \text{Im}\varphi^k$ , clearly,  $\varphi^k y = 0$ , and there is  $y_0 \in U$  satisfying  $\varphi^k y_0 = y$ . Thus

$$y = u(\varphi)g(\varphi)y + v(\varphi)\varphi^k y = \varphi^k u(\varphi)g(\varphi)y_0 = 0.$$

Hence the Symplectic ternary algebra  $U$  can be decomposed into the direct sum of subspaces:

$$U = \text{Ker}\varphi^k \dot{+} \text{Im}\varphi^k.$$

It remains to show that both  $\text{Ker}\varphi^k$  and  $\text{Im}\varphi^k$  are ideals. As  $\varphi$  is a  $U$ -endomorphism, so is  $\varphi^k$ . It is easy to prove that  $\text{Ker}\varphi^k$  is an ideal of  $U$ . Now if we take a vector  $y$  in  $\text{Im}\varphi^k$ , since  $\varphi^k$  is a  $U$ -endomorphism, for  $x_1, x_2 \in U$ , we have

$$\begin{aligned} \langle x_1, x_2, y \rangle &= \langle x_1, x_2, \varphi^k y_0 \rangle = L(x_1, x_2)\varphi^k(y_0) = \varphi^k \langle x_1, x_2, y_0 \rangle \in \text{Im}\varphi^k, \\ \langle x_1, y, x_2 \rangle &= \langle x_1, \varphi^k y_0, x_2 \rangle = U(x_1, x_2)\varphi^k(y_0) = \varphi^k \langle x_1, y_0, x_2 \rangle \in \text{Im}\varphi^k. \end{aligned}$$

These two equations yield that  $\text{Im}\varphi^k$  is an ideal of  $U$ . Then 1) holds.

2) If  $U$  is indecomposable, then  $\text{Ker}\varphi^k = U$  or  $\text{Im}\varphi^k = U$ . By the result 1), we get that  $\varphi^k = 0$  or  $\varphi \in \text{Aut } U$ .

**Lemma 3.2** *If  $U$  is indecomposable and  $\varphi_1, \varphi_2, \dots, \varphi_n, \sum_{i=1}^j \varphi_i$  ( $j = 1, 2, \dots, n$ ) are all  $U$ -endomorphisms and if  $\sum_{i=1}^n \varphi_i = \text{id}$ , then  $\varphi_i$  is an automorphism for some integer  $i$ .*

**Proof** We use induction on  $n$ . The result is trivial for the case  $n = 1$ . Suppose the result holds for the case  $n - 1$ . Now we prove the case  $n$ .

Set  $\sum_{i=1}^{n-1} \varphi_i = \psi$ . Since  $\psi + \varphi_n = \text{id}$ , we have  $\psi\varphi_n = \varphi_n\psi$ . If  $\varphi_n, \psi$  are not automorphisms, by Lemma 3.1, then there are two positive integers  $k, h$  such that  $\psi^k = \varphi_n^h = 0$ .

Set  $m = k + h + 1$ . We have

$$(\psi + \varphi_n)^m = \sum_{j=0}^m C_m^j \psi^j \varphi_n^{m-j} = 0.$$

But this leads to a contradiction for  $\psi + \varphi_n = \text{id}$ . Therefore,  $\varphi_n \in \text{Aut } U$  or  $\psi \in \text{Aut } U$ . If  $\varphi_n \in \text{Aut } U$ , then the result is proved. If  $\psi \in \text{Aut } U$ , since

$$\langle x, y, z \rangle = \psi\psi^{-1}\langle x, y, z \rangle = \psi\langle x, y, \psi^{-1}z \rangle = \psi\langle x, \psi^{-1}y, z \rangle,$$

we have  $\psi^{-1}$  is a  $U$ -endomorphism of  $U$ , and so is  $\varphi_i\psi^{-1}$  for  $i < n$ . Since

$$\varphi_1\psi^{-1} + \varphi_2\psi^{-1} + \dots + \varphi_{n-1}\psi^{-1} = \text{id},$$

by the induction, we have  $\psi^{-1}\varphi_i \in \text{Aut } U$ , which means that  $\varphi_i \in \text{Aut } U$ . The proof is completed.

□

**Theorem 3.1** *If  $U$  is a symplectic ternary algebra with trivial center, and can be decomposed into the direct sum of ideals, i.e.,*

$$U = M_1 \oplus M_2 \oplus \dots \oplus M_s = N_1 \oplus N_2 \oplus \dots \oplus N_t, \quad (11)$$

where  $M_i, N_i$  are indecomposable, then  $s = t$  and  $M_i = N_i$ ,  $i = 1, 2, \dots, s$  by adjusting their orders.

**Proof** We prove the result by induction on  $s$ . If  $s = 1$ , we are done. Suppose the result holds for  $s - 1$ . Now we prove the case  $s$ . Set  $M = M_2 \oplus \cdots \oplus M_s, N = N_2 \oplus \cdots \oplus N_t$ , then  $U = M_1 \oplus M = N_1 \oplus N$ . By Theorem 2.2, it is clear that  $C(M) = C(N) = 0$ , and  $M = C_U(M_1), M_1 = C_U(M), N = C_U(N_1), N_1 = C_U(N)$ . To prove the result, we need to prove that  $M_1 = N_1$  and  $M = N$ .

We denote by  $\pi, \sigma, \rho_i$  and  $\tau_i$  a projection from  $U$  into  $M_1$ , an imbedding from  $M_1$  into  $U$ , the projection from  $U$  into  $N_i$  and an imbedding from  $N_i$  to  $U$ , respectively. It is easy to see that  $\pi, \rho_1, \rho_2, \dots, \rho_n$  and  $\sum_{i=1}^k \rho_i$  ( $k = 1, 2, \dots, n$ ) are all  $U$ -endomorphisms of  $U$ , and  $\sum_{i=1}^n \rho_i = \text{id}_U$ .

Set  $\pi_i^* = \pi\tau_i = \pi|_{N_i}, \rho_i^* = \rho_i\sigma = \rho_i|_{M_1}$ . Since

$$\begin{aligned}\pi_i^* \rho_i^* L(x, y)z &= \pi_i^* L(x, y) \rho_i z = \pi L(x, y) \rho_i z \\ &= L(x, y) \pi \rho_i z = L(x, y) \pi_i^* \rho_i^* z, \quad \forall x, y, z \in M_1,\end{aligned}\tag{12}$$

we have  $\pi_i^* \rho_i^* L(x, y) = L(x, y) \pi_i^* \rho_i^*$ . Similarly, we can verify that  $\pi_i^* \rho_i^* U(x, y) = U(x, y) \pi_i^* \rho_i^*$ , for  $x, y \in M_1$ . Hence,  $\pi_i^* \rho_i^*$  is an  $M_1$ -endomorphism of  $M_1$ .

We may define a projection  $\sum_{i=1}^j \tau_i \rho_i$  by  $(\sum_{i=1}^j \tau_i \rho_i)(x) = \sum_{i=1}^j \tau_i \rho_i(x), \forall x \in U$ . It is easy to see that this projection is a  $U$ -endomorphism of  $U$ , and then  $\pi(\sum_{i=1}^j \tau_i \rho_i)\sigma = \sum_{i=1}^j \pi_i^* \rho_i^*$  is an  $M_1$ -endomorphism of  $M_1$ . For  $m \in M_1$ , since

$$m = \pi(m) = \pi \sum_{i=1}^n \rho_i(m) = \sum_{i=1}^n \pi_i^* \rho_i^*(m),$$

we have  $\sum_{i=1}^n \pi_i^* \rho_i^* = \text{id}_{M_1}$ . By Lemma 3.2, there is an integer  $i$  such that  $\pi_i^* \rho_i^* \in \text{Aut } M_1$ . If we adjust the order of the sequence  $N_1, N_2, \dots, N_t$ , suppose  $i = 1$ , then we have that  $\rho_1^*$  is a  $1 - 1$  projection. Thus  $\text{Ker } \rho_1^* = M_1 \cap \text{Ker } \rho_1 = M_1 \cap N = \{0\}$ . Since  $\langle M_1, N, U \rangle, \langle N, M_1, U \rangle, \langle M_1, U, N \rangle, \langle U, M_1, N \rangle$  are all contained in  $M_1 \cap N = \{0\}$ , we have  $M_1 \subseteq C_U(N) = N_1$ . By Lemma 2.5, we have  $N_1 = M_1 \oplus (N_1 \cap M)$ . But  $N_1$  is indecomposable, hence  $N_1 = M_1$  and  $M = N$ .  $\square$

## References

- [1] FAULKNER J R. A construction of Lie algebras from a class of ternary algebras [J]. Trans. Amer. Math. Soc., 1971, **155**: 397–408.
- [2] FAULKNER J R, FERRAR J C. On the structure of symplectic ternary algebras [J]. Indag. Math., 1972, **34**: 247–256.
- [3] LISTER W G. A structure theory of Lie triple systems [J]. Trans. Amer. Math. Soc., 1952, **72**: 217–242.
- [4] MENG Daoji, SHI Yiqian. The uniqueness of decomposition of Lie triple systems [J]. Adv. Math. (China), 2004, **33**(1): 52–56.
- [5] BAI Ruipu, MENG Daoji. Decomposition of  $n$ -Lie algebras and its uniqueness [J]. Chinese Ann. Math. Ser. A, 2004, **25**(2): 147–152. (in Chinese)
- [6] BAI Ximei, LIU Wenli. Decomposition of unital symplectic ternary algebras [J]. J. Hebei Univ. Nat. Sci., 2002, **22**(3): 291–293.
- [7] BAI Ximei, ZHANG Zhixue. On symplectic ternary algebras [J]. J. Hebei Univ. Nat. Sci., 2001, **21**(4): 360–362.
- [8] LIU Wenli, ZHANG Zhixue. On the Peirce decomposition for symplectic algebras [J]. J. Hebei Univ. Nat. Sci., 2001, **21**(4): 363–365.