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Vector-Valued Multilinear Commutators of Singular Integrals with Mixed Homogeneity

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Abstract The boundedness of vector-valued multilinear commutators of singular integral operators whose kernels are variable with mixed homogeneity on Lebesgue spaces is obtained.

Keywords singular integral; BMO; maximal function; commutator; Lebesgue space.

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1. Introduction

Recently, Softova [1] considered the boundedness on generalized Morrey spaces of the following singular integral operator

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} k(x; x-y) f(y) \mathrm{d}y \tag{1.1}$$

and its commutator with a function $b \in \mathrm{BMO}(\mathbb{R}^n)$ defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x) = \text{p.v.} \int_{\mathbb{R}^n} (b(x) - b(y)) k(x; x-y) f(y) \mathrm{d}y. \tag{1.2}$$

In (1.1) and (1.2), the kernel $k(x;\xi) : \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is a variable kernel with mixed homogeneity; for its definition see the next section. This class of kernels was firstly studied by Fabes and Rivière in [2]. They generalized the classical kernels of Calderón-Zygmund $k(\xi) = \frac{\Omega(\xi)}{|\xi|^n}$ having homogeneity of degree $-n$ and those studied by Jones in [3] satisfying homogeneity property of the form $k(\lambda\xi, \lambda^m\tau) = \lambda^{-n-m} k(\xi,\tau)$, $\xi \in \mathbb{R}^n$, $\tau \in (0,\infty)$, $m \geq 1$. Introducing a new metric ρ and using the Fourier transform in $L^2(\mathbb{R}^n)$ and the Marcinkiewicz interpolation theorem, Fabes and Rivière obtained the boundedness of (1.1) in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, where $\mathbb{R}^n$ is endowed with the topology induced by ρ and defined by ellipsoids.

In this paper, we consider the vector-valued singular operator of (1.1). Let $1 < q < \infty$. We

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define the vector-valued singular operator T_q associated with the operator T by

$$T_qF(x) = |TF(x)|_q = \Big(\sum_{k=1}^{\infty} |Tf_k(x)|^q\Big)^{1/q},$$

where for simplicity, we denote a sequence functions $\{f_k\}_{k=1}^{\infty} \equiv F$.

Now we can state our first result.

Theorem 1.1 *For $1 < q < \infty$, T_q is of type-(δ, δ), $1 < \delta < \infty$.*

Since the commutators of singular integral operators with $\mathrm{BMO}(\mathbb{R}^n)$ functions play a key role in the study of the regularity of solutions to nondivergence elliptic equations with $\mathrm{VMO}(\mathbb{R}^n)$ coefficients [4–6]. The commutators attract more attention recently [7–11] and references therein. In fact, as a generalization of m-th commutator of singular integral and BMO function, Pérez and Trujillo-Gozalez introduced multilinear commutators in [9], they obtained sharp weighted and vector-valued estimates of multilinear commutators respectively in [9] and [10].

In [12], one of the authors of the paper discussed the generalization of (1.2), namely, multilinear commutator,

$$[\vec{b}, T]f(x) = \text{p.v.} \int_{\mathbb{R}^n} \prod_{i=1}^{m} (b_i(x) - b_i(y)) k(x; x-y) f(y) \mathrm{d}y, \tag{1.3}$$

where $\vec{b} = \{b_1, \ldots, b_m\}$, and $\{b_i\}_{i=1}^{m}$ are BMO functions.

Motivated by [9] and [10], as a continuation of [12] we consider in this paper vector-valued extensions of the multilinear commutators of (1.1) defined by the formula

$$[\vec{b}, T]_q F(x) = |[\vec{b}, T]F(x)|_q = \Big(\sum_{k=1}^{\infty} |[\vec{b}, T]f_k(x)|^q\Big)^{1/q},$$

where $\vec{b} = \{b_1, \ldots, b_m\}$, $\{b_i\}_{i=1}^{m}$ are $\mathrm{BMO}(\mathbb{R}^n)$ functions, and $F \in L^{\delta}(\ell^q)(\mathbb{R}^n)$. Here is another main result of this paper.

Theorem 1.2 *Let $1 < \delta < \infty$. Then there exists a positive constant C such that*

$$\|[\vec{b}, T]_q F\|_{L^{\delta}(\mathbb{R}^n)} \le C \prod_{i=1}^{m} \|b_i\|_{\mathrm{BMO}} \|F\|_{L^{\delta}(\ell^q)(\mathbb{R}^n)}$$

holds for all smooth vector functions $F = \{f_k\}_{k=1}^{\infty}$ with f_k having compact support and $F \in L^{\delta}(\ell^q)(\mathbb{R}^n)$.

The paper is organized as follows. In Section 2, we recall some definitions and preliminary results. In Section 3, we give the proofs of Theorems 1.1 and 1.2.

Throughout this paper, we use C to denote a positive constant that may vary from lines to lines.

2. Preliminary results

Let $\alpha_1, \ldots, \alpha_n$ be real numbers, where $\alpha_i \ge 1$ and set $\alpha = \sum_{i=1}^{n} \alpha_i$. Following Fabes and Rivière [2], the function $F(x, \rho) = \sum_{i=1}^{n} x_i^2 \rho^{-2\alpha_i}$, considered for any fixed $x \in \mathbb{R}^n$, is a decreasing one with respect to $\rho > 0$ and the equation $F(x,\rho) = 1$ has a unique solution $\rho(x)$. It is easy to check that $\rho(x-y)$ defines a distance between any two points $x, y \in \mathbb{R}^n$. Thus $\mathbb{R}^n$, endowed with the metric ρ, results in a homogeneous metric space, see [2, 7] for details. The balls with respect to ρ, centered at the origin and of radius r, are simply the ellipsoids

$$\mathcal{E}_r(0) = \Big\{x \in \mathbb{R}^n : \quad \frac{x_1^2}{r^{2\alpha_1}} + \cdots + \frac{x_n^2}{r^{2\alpha_n}} < 1\Big\}$$

with Lebesgue measure $|\mathcal{E}_r(0)| = C(n)r^{\alpha}$.

Definition 2.1 *The function $k(x;\xi) : \ \mathbb{R}^n \times \{\mathbb{R}^n \setminus \{0\}\} \to \mathbb{R}$ is called a variable kernel with mixed homogeneity if: (i) For every fixed x, $k(x;\cdot)$ is a constant kernel satisfying*

$i_a)$ $k(x;\cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\})$;

$i_b)$ $k(x; t^{\alpha_1}\xi_1, \ldots, t^{\alpha_n}\xi_n) = t^{-\alpha}k(x;\xi)$, $\forall t > 0$, $\alpha_i \geq 1$, $\alpha = \sum_{i=1}^n \alpha_i$;

$i_c)$ $\int_{\Sigma_n} k(x;\xi)\mathrm{d}\sigma_\xi = 0$ and $\int_{\Sigma_n} |k(x;\xi)|\mathrm{d}\sigma_\xi < \infty$, where Σ_n is the unit sphere with the Euclidean norm in $\mathbb{R}^n$ and $\mathrm{d}\sigma$ is the Lebesgue measure induced in Σ_n;

(ii) For every multiindex $\beta : \sup_{\xi\in\mathbb{R}^n} |D_\xi^\beta k(x;\xi)| \leq C(\beta)$ independently of x.

For a given function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ define the Hardy-Littlewood maximal operator Mf and the sharp maximal operator $f^\#$ by setting for all $x \in \mathbb{R}^n$,

$$Mf(x) = \sup_{x\in\mathcal{E}} \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} |f(y)|\mathrm{d}y, \quad f^\#(x) = \sup_{x\in\mathcal{E}} \frac{1}{|\mathcal{E}|}\int |f(y) - f_{\mathcal{E}}|\mathrm{d}y,$$

where the supremum is taken over all ellipsoids $\mathcal{E}$ centered at x, and $f_{\mathcal{E}} = \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} f(y)\mathrm{d}y$. Define also the vector-valued maximal operator $M_qF(x) = (\sum_{k=1}^\infty (Mf_k(x))^q)^{1/q}$ for $q \geq 1$.

Lemma 2.1 *Let f be a measurable function. Then Mf is a type-(p,p) operator, that is, there exists a constant C such that for all $f \in L^p(\mathbb{R}^n)$, $Mf \in L^p(\mathbb{R}^n)$ and $\|Mf\|_p \leq C\|f\|_p$.*

Lemma 2.2 *Let $1 < p < \infty$. Then there exists a constant C such that*

$$\|f^\#\|_p \leq C\|f\|_p$$

for all functions $f \in L^p(\mathbb{R}^n)$.

Definition 2.2 *For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, if $f^\# \in L^\infty(\mathbb{R}^n)$, we call $f \in \mathrm{BMO}(\mathbb{R}^n)$. $\|f^\#\|_{L^\infty}$ is defined to be the norm of f in $\mathrm{BMO}(\mathbb{R}^n)$ and denoted by $\|f\|_{\mathrm{BMO}}$.*

Lemma 2.3 *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $1 < p < \infty$. Then there exists a constant C such that for any ellipsoid $\mathcal{E}$*

$$\left(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} |b(y) - b_{\mathcal{E}}|^p \mathrm{d}y\right)^{1/p} \leq C\|b\|_{\mathrm{BMO}}.$$

Lemma 2.3 is a John-Nirenberg type inequality. Lemmas 2.1–2.3 are the well known maximal and sharp inequalities obtained in Lebesgue spaces [13, 14].

Definition 2.3 *Let $1 < p < \infty$. A function $F = \{f_k\}_{k=1}^\infty \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to $L^p(\ell^q)(\mathbb{R}^n)$*

space if the following norm is finite

$$\|F\|_{L^p(\ell^q)(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} \Big(\sum_{k=1}^{\infty} |f_k(x)|^q \Big)^{p/q} \mathrm{d}x \Big)^{1/p}.$$

Moreover, we need spherical harmonics and their properties. Denote by $\mathcal{Y}_j$ the space of all n-dimensional spherical harmonics of degree j. It is a finite dimensional linear space with $g_j = \dim \mathcal{Y}_j$ such that $g_0 = 1$, $g_1 = n$ and

$$g_j = \binom{j+n-1}{n-1} - \binom{j+n-3}{n-1} \leq C(n) j^{n-2}, \quad j \geq 2. \tag{2.3}$$

Further, let $\{\mathcal{Y}_{sj}\}_{s=1}^{g_j}$ be an orthonormal base of $\mathcal{Y}_j$. Then $\{\mathcal{Y}_{sj} : j = 0, 1, \ldots; s = 1, \ldots, g_j\}$ is a complete orthonormal system in $L^2(\Sigma_n)$ and

$$\sup_{x \in \Sigma_n} |D_x^{\beta} \mathcal{Y}_{sj}(x)| \leq C(n) j^{|\beta| + (n-2)/2}, \quad j \geq 1. \tag{2.4}$$

If $\phi \in C^{\infty}(\Sigma_n)$, then $\sum_{j=0}^{\infty} \sum_{s=1}^{g_j} \psi_{sj} \mathcal{Y}_{sj}$ is the Fourier series expansion of ϕ with respect to $\{\mathcal{Y}_{sj} : j = 0, 1, \ldots; s = 1, \ldots, g_j\}$ and

$$\psi_{sj} = \int_{\Sigma_n} \phi(y) \mathcal{Y}_{sj}(y) \mathrm{d}\sigma, \quad |\psi_{sj}| \leq C(n,l) j^{-2l} \sup_{|\beta| = 2l, y \in \Sigma_n} |D_y^{\beta} \phi(y)| \tag{2.5}$$

for any integer l. In particular, the expansion of ϕ into spherical harmonics converges uniformly to ϕ. We refer to [15] for the proof of the above results.

We write

$$[\vec{b}, T] F(x) = \text{p.v.} \int_{\mathbb{R}^n} \prod_{i=1}^{m} (b_i(x) - b_i(y)) k(x, x-y) F(y) \mathrm{d}y = \Big\{ \lim_{\epsilon \to 0} [\vec{b}, T]_{\epsilon} f_d(x) \Big\}_{d=1}^{\infty},$$

where

$$\lim_{\epsilon \to 0} [\vec{b}, T]_{\epsilon} f_d(x) = \lim_{\epsilon \to 0} \int_{\rho(x-y) > \epsilon} \prod_{i=1}^{m} (b_i(x) - b_i(y)) k(x, x-y) f_d(y) \mathrm{d}y.$$

Let $x,\ y \in \mathbb{R}^n$ and $\overline{y} = y/\rho(y) \in \Sigma_n$. From the properties of the kernel with respect to the second variable and completeness of $\{\mathcal{Y}_{sj} : j = 0, 1, \ldots; s = 1, \ldots, g_j\}$ in $L^2(\Sigma_n)$, it follows that

$$k(x; x-y) = \rho(x-y)^{-\alpha} k(x; \overline{x-y}) = \rho(x-y)^{-\alpha} \sum_{s,j} \psi_{sj}(x) \mathcal{Y}_{sj}(\overline{x-y}).$$

By Definition 2.1 (ii) and (2.5), we then have

$$\|\psi_{sj}\|_{\infty} \leq C(n,l,k) j^{-2l} \tag{2.6}$$

for any integer $l > 1$. Substituting the kernel with its expansion, we obtain

$$\lim_{\epsilon \to 0} T_{\epsilon} f(x) = \lim_{\epsilon \to 0} \int_{\rho(x-y) > \epsilon} \sum_{s,j} \psi_{sj}(x) \mathcal{H}_{sj}(x-y) f(y) \mathrm{d}y,$$

$$\lim_{\epsilon \to 0} [\vec{b}, T]_{\epsilon} f(x) = \lim_{\epsilon \to 0} \int_{\rho(x-y) > \epsilon} \prod_{i=1}^{m} (b_i(x) - b_i(y)) \sum_{s,j} \psi_{sj}(x) \mathcal{H}_{sj}(x-y) f(y) \mathrm{d}y,$$

where $\mathcal{H}_{sj}(x-y)$ stands for $\mathcal{Y}_{sj}(\overline{x-y}) \rho(x-y)^{-\alpha}$. It is known that $\mathcal{H}_{sj}(\cdot)$ are constant kernels in the sense of Definition 2.1 (i), see [1]. By the same argument stated in the proof of [1, Theorem

3.1], we deduce

$$\lim_{\epsilon\to 0}[\vec{b},T]_\epsilon f(x)=\lim_{\epsilon\to 0}\sum_{s,j}\psi_{sj}(x)[\vec{b},K_{sj,\epsilon}]f(x)$$

exists, where

$$K_{sj,\epsilon}f(x)=\int_{\rho(x-y)>\epsilon}\mathcal{H}_{sj}(x-y)f(y)\mathrm{d}y,$$

and

$$[\vec{b},K_{sj,\epsilon}]f(x)=\int_{\rho(x-y)>\epsilon}\prod_{i=1}^{m}(b_i(x)-b_i(y))\mathcal{H}_{sj}(x-y)f(y)\mathrm{d}y.$$

Thus we can write

$$[\vec{b},T]F(x)=\sum_{s,j}\psi_{sj}(x)[\vec{b},K_{sj}]F(x)$$

with

$$K_{sj}F(x)=\lim_{\epsilon\to 0}\int_{\rho(x-y)>\epsilon}\mathcal{H}_{sj}(x-y)F(y)\mathrm{d}y,$$

and

$$[\vec{b},K_{sj}]F(x)=\lim_{\epsilon\to 0}\int_{\rho(x-y)>\epsilon}\prod_{i=1}^{m}(b_i(x)-b_i(y))\mathcal{H}_{sj}(x-y)F(y)\mathrm{d}y.$$

For $\mathcal{H}_{sj}(x-y)$, we have the following lemma which is Lemma 3.2 in [1].

Lemma 2.4 *Let $\mathcal{E}$ and $2\mathcal{E}$ be ellipsoids centered at x_0 and of radius r and $2r$, respectively. Then*

$$|\mathcal{H}_{sj}(x-y)-\mathcal{H}_{sj}(x_0-y)|\le C(n,\alpha)j^{n/2}\frac{\rho(x_0-x)}{\rho(x_0-y)^{\alpha+1}}$$

for each $x\in\mathcal{E}$ and $y\notin 2\mathcal{E}$.

For convenience, we introduce some notations. Given any positive integer m, for all $1\le i\le m$, we denote by C_i^m the family of all finite subsets $\sigma=\{\sigma(1),\dots,\sigma(i)\}$ of i different elements of $\{1,2,\dots,m\}$. For any $\sigma\in C_i^m$, we associate the complementary sequence σ' given by $\sigma'=\{1,2,\dots,m\}\backslash\sigma$. For any $\sigma\in C_i^m$, we set $\vec{b}_\sigma=\prod_{t\in\sigma}b_t$, $\|\vec{b}_\sigma\|=\prod_{t\in\sigma}\|b_t\|_{\mathrm{BMO}}$ and

$$[\vec{b}_\sigma,K_{sj}]F(x)=\int_{R^n}(\vec{b}(x)-\vec{b}(y))_\sigma\mathcal{H}_{sj}(x-y)F(y)\mathrm{d}y.$$

In the case of $\sigma=\{1,2,\dots,m\}$, we denote $[\vec{b}_\sigma,T]$ by $[\vec{b},T]$.

3. Proofs of main results

To give the proofs, we adopt the method used in [1] and [10]. Firstly, we have the following lemma, which is derived from Corollary 4.6.3 on page 328 in [13], since $\{K_{sj}\}_{sj}$ satisfy the conditions therein according to [2] .

Lemma 3.1 *Let $1<\delta,\ q<\infty$. There exists constant C such that for all $F\in L^\delta(\ell^q)(\mathbb{R}^n)$,*

$$\|K_{sj}F\|_{L^\delta(\ell^q)(\mathbb{R}^n)}\le C\|F\|_{L^\delta(\ell^q)(\mathbb{R}^n)}.$$

By Lemma 3.1, we can easily obtain Theorem 1.1. In fact,

$$\begin{aligned}\|T_qF\|_\delta &\le \sum_{j=1}^{\infty}\sum_{s=1}^{g_j}\|\psi_{sj}\|_\infty\|K_{sj}F\|_{L^\delta(\ell^q)(\mathbb{R}^n)}\\ &\le C\sum_{j=1}^{\infty}\sum_{s=1}^{g_j} j^{-2l}\|F\|_{L^\delta(\ell^q)(\mathbb{R}^n)}\\ &\le C\sum_{j=1}^{\infty} j^{-2l+n-2}\|F\|_{L^\delta(\ell^q)(\mathbb{R}^n)}\\ &\le C\|F\|_{L^\delta(\ell^q)(\mathbb{R}^n)},\end{aligned}$$

provided that we take $l > (n-2)/2$.

Lemma 3.2 *Let $[\vec{b}, K_{sj}]$ be as above and $1 < p < \infty$. Then there exists a constant C such that*

$$([\vec{b}, K_{sj}]_qF)^{\#}(x) \le C\Big[j^{n/2}\|\vec{b}\|\left(M(|F|_q^p)(x)\right)^{1/p} + \sum_{i=1}^{m}\sum_{\sigma\in C_i^m}\|\vec{b}_\sigma\|\left(M(|[\vec{b}_{\sigma'}, K_{sj}]_qF|^p)(x)\right)^{1/p}\Big]$$

for all smooth vector functions $F = \{f_k\}_{k=1}^{\infty}$, $f_k \in L_c^\infty$ for $k \ge 1$, and for all $x \in \mathbb{R}^n$.

Proof To prove the lemma, we make use of induction on m. First let us consider $m = 1$. In this case,

$$[b, K_{sj}]F = (b-\lambda)K_{sj}F - K_{sj}((b-\lambda)F).$$

For fixed $x_0 \in \mathbb{R}^n$, $\mathcal{E}$ denotes an ellipsoid at x_0 with radius r, $2\mathcal{E}$ denotes the ellipsoid concentric with $\mathcal{E}$ and radius two times the radius of $\mathcal{E}$. Decompose $F = F^1 + F^2$, where $F^1 = F\chi_{2\mathcal{E}} = \{f_k\chi_{2\mathcal{E}}\}_{k=1}^{\infty}$, with χ being the characteristic function of the respective set. We write $[b, K_{sj}]F$ as follows: $[b, K_{sj}]F = (b-\lambda)K_{sj}(F) - [K_{sj}((b-\lambda)F^1) + K_{sj}((b-\lambda)F^2)]$. Setting $\lambda = b_{\mathcal{E}} = \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} b(y)\mathrm{d}y$, and $A = |K_{sj}((b-\lambda)F^2)(x_0)|_q$, we have

$$\begin{aligned}([b, K_{sj}]_qF)^{\#}(x_0) \le&\frac{2}{|\mathcal{E}|}\int_{\mathcal{E}}|[b, K_{sj}]_qF(x) - A|\mathrm{d}x\\ \le&\frac{C}{|\mathcal{E}|}\int_{\mathcal{E}}\left|[b, K_{sj}]F(x) + K_{sj}\left((b-\lambda)\,F^2\right)(x_0)\right|_q\mathrm{d}x\\ \le&\frac{C}{|\mathcal{E}|}\int_{\mathcal{E}}|(b(x)-\lambda)K_{sj}(F)(x)|_q\,\mathrm{d}x + \frac{C}{|\mathcal{E}|}\int_{\mathcal{E}}\left|K_{sj}\left((b-\lambda)F^1\right)(x)\right|_q\mathrm{d}x +\\ &\frac{C}{|\mathcal{E}|}\int_{\mathcal{E}}\left|K_{sj}\left((b-\lambda)F^2\right)(x) - K_{sj}\left((b-\lambda)F^2\right)(x_0)\right|_q\mathrm{d}x\\ =&C(\mathrm{I}+\mathrm{II}+\mathrm{III}).\end{aligned}$$

We first estimate I. Here and in what follows, p' is the conjugate exponent to p. Using Hölder's inequality, we have

$$\begin{aligned}\mathrm{I} &= \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}\Big(\sum_{k=1}^{\infty}|(b(x)-\lambda)K_{sj}(f_k)(x)|^q\Big)^{1/q}\mathrm{d}x\\ &= \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|b(x)-\lambda||K_{sj}(F)(x)|_q\mathrm{d}x\end{aligned}$$

$$\begin{aligned}&\le C\Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|b(x)-\lambda|^{p'}\mathrm{d}x\Big)^{1/p'}\Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|K_{sj}(F)(x)|_q^p\mathrm{d}x\Big)^{1/p}\\&\le C\|b\|_{\mathrm{BMO}}\left(M(|K_{sj}F|_q^p)(x_0)\right)^{1/p},\end{aligned}$$

where in the last inequality, we used Lemma 2.3.

To estimate II, let $1<\tau<p$ and τ' be the conjugate exponent to τ. We have

$$\begin{aligned}\mathrm{II}&=\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}\left|K_{sj}\left((b-\lambda)F^1\right)(x)\right|_q\mathrm{d}x\\&\le\frac{1}{|\mathcal{E}|}\Big(\int_{\mathcal{E}}\left|K_{sj}\left((b-\lambda)F^1\right)(x)\right|_q^\tau\mathrm{d}x\Big)^{1/\tau}\Big(\int_{\mathcal{E}}1\mathrm{d}x\Big)^{1/\tau'}\\&\le\frac{C}{|\mathcal{E}|^{1/\tau}}\Big(\int_{\mathcal{E}}\left|(b-\lambda)F^1(x)\right|_q^\tau\mathrm{d}x\Big)^{1/\tau}\\&\le\frac{C}{|\mathcal{E}|^{1/\tau}}\Big(\int_{2\mathcal{E}}|F(x)|_q^p\,\mathrm{d}x\Big)^{1/p}\Big(\int_{2\mathcal{E}}|b(x)-\lambda|^{p\tau/(p-\tau)}\mathrm{d}x\Big)^{(p-\tau)/p\tau}\\&\le C\|b\|_{\mathrm{BMO}}\left(M(|F|_q^p)(x_0)\right)^{1/p},\end{aligned}$$

where we used that $(K_{sj})_q$ is of type (p,p) for $1<q,p<\infty$ (see Lemma 3.1) and Lemma 2.3.

Now we turn to estimate III. Since x, $x_0\in\mathcal{E}$, $y\notin 2\mathcal{E}$, by Lemma 2.4, we have

$$\begin{aligned}&\left|K_{sj}\left((b-\lambda)F^2\right)(x)-K_{sj}\left((b-\lambda)F^2\right)(x_0)\right|_q\\&\le\Big(\sum_{k=1}^{\infty}\Big|\int_{(2\mathcal{E})^c}|\mathcal{H}_{sj}(x-y)-\mathcal{H}_{sj}(x_0-y)||b(y)-\lambda|f_k^2(y)|\mathrm{d}y\Big|^q\Big)^{1/q}\\&\le Cj^{n/2}\rho(x_0-x)\int_{(2\mathcal{E})^c}\frac{|b(y)-\lambda||F(y)|_q}{\rho(x_0-y)^{\alpha+1}}\mathrm{d}y\\&\le Cj^{n/2}r\Big(\int_{(2\mathcal{E})^c}\frac{|F(y)|_q^p}{\rho(x_0-y)^{\alpha+1}}\mathrm{d}y\Big)^{1/p}\Big(\int_{(2\mathcal{E})^c}\frac{|b(y)-\lambda|^{p'}}{\rho(x_0-y)^{\alpha+1}}\mathrm{d}y\Big)^{1/p'},\end{aligned}\tag{3.2}$$

where in the second inequality, we used Minkowski's inequality. Here we have

$$\int_{(2\mathcal{E})^c}\frac{|F(y)|_q^p}{\rho(x_0-y)^{\alpha+1}}\mathrm{d}y=\sum_{k=1}^{\infty}\int_{2^{k+1}\mathcal{E}\setminus 2^k\mathcal{E}}\frac{|F(y)|_q^p}{\rho(x_0-y)^{\alpha+1}}\mathrm{d}y\le\frac{2^{\alpha+1}}{r}M(|F|_q^p)(x_0).\tag{3.3}$$

Since $|b_{2^k\mathcal{E}}-b_{\mathcal{E}}|\le C(n,\alpha)k\|b\|_{\mathrm{BMO}}$ for $k\in\mathbb{N}$, by Lemma 2.3, we have

$$\begin{aligned}\int_{(2\mathcal{E})^c}\frac{|b(y)-\lambda|^{p'}}{\rho(x_0-y)^{\alpha+1}}\mathrm{d}y&=\sum_{k=1}^{\infty}\int_{2^{k+1}\mathcal{E}\setminus 2^k\mathcal{E}}\frac{|b(y)-\lambda|^{q_ip'}}{\rho(x_0-y)^{\alpha+1}}\mathrm{d}y\\&\le\sum_{k=1}^{\infty}\frac{1}{(2^kr)^{\alpha+1}}\int_{2^{k+1}\mathcal{E}\setminus 2^k\mathcal{E}}|b(y)-\lambda|^{p'}\mathrm{d}y\\&\le\sum_{k=1}^{\infty}\frac{2^{p'-1}}{(2^kr)^{\alpha+1}}\int_{2^{k+1}\mathcal{E}\setminus 2^k\mathcal{E}}(|b(y)-b_{2^{k+1}\mathcal{E}}|^{p'}+|b_{2^{k+1}\mathcal{E}}-b_{\mathcal{E}}|^{p'})\mathrm{d}y\\&\le\sum_{k=1}^{\infty}\frac{|2^{k+1}\mathcal{E}|}{(2^kr)^{\alpha+1}}(1+k^{p'})\|b\|_{\mathrm{BMO}}^{p'}\\&\le C\frac{\|b\|_{\mathrm{BMO}}^{p'}}{r}.\end{aligned}\tag{3.4}$$

Combining (3.2), (3.3), and (3.4), we obtain

$$\mathrm{III} \le C j^{n/2} \|b\| (M(|F|_q^p)(x_0))^{1/p}.$$

Summing up I to III, and taking the supremum over all ellipsoids $\mathcal{E}$, we obtain the lemma for the case of $m = 1$.

Now suppose $m > 1$. For any $\vec{\lambda} = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$, we have

$$\begin{aligned}
[\vec{b}, K_{sj}]F(x) =& \int_{\mathbb{R}^n} (b_1(x) - b_1(y)) \cdots (b_m(x) - b_m(y)) \mathcal{H}_{sj}(x-y) F(y) \mathrm{d}y \\
=& \int_{\mathbb{R}^n} \prod_{i=1}^m ((b_i(x) - \lambda_i) - (b_i(y) - \lambda_i)) \mathcal{H}_{sj}(x-y) F(y) \mathrm{d}y \\
=& \sum_{i=0}^m \sum_{\sigma \in C_i^m} (-1)^{m-i} (\vec{b}(x) - \vec{\lambda})_\sigma \int_{\mathbb{R}^n} (\vec{b}(y) - \vec{\lambda})_{\sigma'} \mathcal{H}_{sj}(x-y) F(y) \mathrm{d}y \\
=& (\prod_{i=1}^m (b_i(x) - \lambda_i) K_{sj} F(x) + (-1)^m K_{sj} (\prod_{i=1}^m (b_i - \lambda_i) F)(x) + \\
& \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} (-1)^{m-i} (\vec{b}(x) - \vec{\lambda})_\sigma \int_{\mathbb{R}^n} (\vec{b}(y) - \vec{\lambda})_{\sigma'} \mathcal{H}_{sj}(x-y) F(y) \mathrm{d}y. \qquad (3.5)
\end{aligned}$$

Now for $(\vec{b}(y) - \vec{\lambda})_{\sigma'}$, we write $(\vec{b}(y) - \vec{\lambda})_{\sigma'} = (\vec{b}(y) - \vec{b}(x) + \vec{b}(x) - \vec{\lambda})_{\sigma'}$. Thus,

$$\begin{aligned}
& \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} (-1)^{m-i} (\vec{b}(x) - \vec{\lambda})_\sigma \int_{\mathbb{R}^n} (\vec{b}(y) - \vec{\lambda})_{\sigma'} \mathcal{H}_{sj}(x-y) F(y) \mathrm{d}y \\
&= \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} (-1)^{m-i} (\vec{b}(x) - \vec{\lambda})_\sigma \sum_{j=0}^{m-i} \sum_{\xi \in C_j^{m-i}} (\vec{b}(x) - \vec{\lambda})_\xi \int_{\mathbb{R}^n} (\vec{b}(y) - \vec{b}(x))_{\xi'} \mathcal{H}_{sj}(x-y) F(y) \mathrm{d}y \\
&= \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} (-1)^{m-i} (\vec{b}(x) - \vec{\lambda})_\sigma \times \\
&\quad \sum_{j=0}^{m-i} \sum_{\substack{\xi \in C_j^{m-i} \\ \xi \cup \xi' = \sigma'}} (-1)^{m-i-j} (\vec{b}(x) - \vec{\lambda})_\xi \int_{\mathbb{R}^n} (\vec{b}(x) - \vec{b}(y))_{\xi'} \mathcal{H}_{sj}(x-y) F(y) \mathrm{d}y \\
&= \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} (-1)^{m-i} \prod_{t=1}^m (b_t(x) - \lambda_t) \int_{\mathbb{R}^n} \mathcal{H}_{sj}(x-y) F(y) \mathrm{d}y + \\
&\quad \sum_{i=1}^{m-1} \sum_{\sigma \in C_i^m} (-1)^{m-i} (\vec{b}(x) - \vec{\lambda})_\sigma \times \\
&\quad \sum_{j=0}^{m-i-1} \sum_{\substack{\xi \in C_j^{m-i} \\ \xi \cup \xi' = \sigma'}} (-1)^{m-i-j} (\vec{b}(x) - \vec{\lambda})_\xi \int_{\mathbb{R}^n} (\vec{b}(x) - \vec{b}(y))_{\xi'} \mathcal{H}_{sj}(x-y) F(y) \mathrm{d}y \\
&= \sum_{i=1}^{m-1} \binom{m}{i} (-1)^{m-i} \prod_{t=1}^m (b_t(x) - \lambda_t) \int_{\mathbb{R}^n} \mathcal{H}_{sj}(x-y) F(y) \mathrm{d}y +
\end{aligned}$$

$$\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m} C_{m,i}(\vec{b}(x)-\vec{\lambda})_\sigma \int_{\mathbb{R}^n}(\vec{b}(x)-\vec{b}(y))_{\sigma'}\mathcal{H}_{sj}(x-y)F(y)\mathrm{d}y$$
$$= (-1+(-1)^{m+1})\prod_{t=1}^{m}(b_t(x)-\lambda_t)\int_{\mathbb{R}^n}\mathcal{H}_{sj}(x-y)F(y)\mathrm{d}y+$$
$$\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m} C_{m,i}(\vec{b}(x)-\vec{\lambda})_\sigma \int_{\mathbb{R}^n}(\vec{b}(x)-\vec{b}(y))_{\sigma'}\mathcal{H}_{sj}(x-y)F(y)\mathrm{d}y, \tag{3.6}$$

where $C_{m,i}$ are 1 or -1. Substituting (3.6) into (3.5), we have

$$\begin{aligned}[\vec{b},K_{sj}]F(x) =&(-1)^{m+1}\prod_{i=1}^{m}(b_i(x)-\lambda_i)K_{sj}F(x)+(-1)^m K_{sj}(\prod_{i=1}^{m}(b_i-\lambda_i)F)(x)+\\ &\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m} C_{m,i}(\vec{b}(x)-\vec{\lambda})_\sigma \int_{\mathbb{R}^n}(\vec{b}(x)-\vec{b}(y))_{\sigma'}\mathcal{H}_{sj}(x-y)F(y)\mathrm{d}y.\end{aligned}$$

For fixed $x_0\in\mathbb{R}^n$, let $\mathcal{E}$, $2\mathcal{E}$, F^1 and F^2 be the same as above. We write $[\vec{b},K_{sj}]F$ as follows

$$\begin{aligned}[\vec{b},K_{sj}]F(x) =&(-1)^{m+1}\prod_{i=1}^{m}(b_i(x)-\lambda_i)K_{sj}(F)(x)+(-1)^m K_{sj}(\prod_{i=1}^{m}(b_i-\lambda_i)F^1)(x)+\\ &(-1)^m K_{sj}(\prod_{i=1}^{m}(b_i-\lambda_i)F^2)(x)+\\ &\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m} C_{m,i}(\vec{b}(x)-\vec{\lambda})_\sigma \int_{\mathbb{R}^n}(\vec{b}(x)-\vec{b}(y))_{\sigma'}\mathcal{H}_{sj}(x-y)F(y)\mathrm{d}y.\end{aligned}$$

Setting $\lambda_i=b_{i\mathcal{E}}=\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}b_i(y)\mathrm{d}y$, and $A=|(-1)^mK_{sj}(\prod_{i=1}^m(b_i-\lambda_i)F^2)(x_0)|_q$, we then have

$$\begin{aligned}&([\vec{b},K_{sj}]_qF)^{\#}(x_0)\\ &\le \frac{2}{|\mathcal{E}|}\int_{\mathcal{E}}|[\vec{b},K_{sj}]_qF(x)-A|\mathrm{d}x\\ &\le \frac{C}{|\mathcal{E}|}\int_{\mathcal{E}}\Big|[\vec{b},K_{sj}]F(x)-(-1)^mK_{sj}\Big(\prod_{i=1}^{m}(b_i-\lambda_i)F^2\Big)(x_0)\Big|_q\mathrm{d}x\\ &\le \frac{C}{|\mathcal{E}|}\int_{\mathcal{E}}\Big|\prod_{i=1}^{m}(b_i(x)-\lambda_i)K_{sj}(F)(x)\Big|_q\mathrm{d}x+\frac{C}{|\mathcal{E}|}\int_{\mathcal{E}}\Big|K_{sj}\Big(\prod_{i=1}^{m}(b_i-\lambda_i)F^1\Big)(x)\Big|_q\mathrm{d}x+\\ &\quad\frac{C}{|\mathcal{E}|}\int_{\mathcal{E}}\Big|K_{sj}\Big(\prod_{i=1}^{m}(b_i-\lambda_i)F^2\Big)(x)-K_{sj}\Big(\prod_{i=1}^{m}(b_i-\lambda_i)F^2\Big)(x_0)\Big|_q\mathrm{d}x+\\ &\quad\frac{C}{|\mathcal{E}|}\int_{\mathcal{E}}\Big|\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}(-1)^{m-i+1}(\vec{b}(x)-\vec{\lambda})_\sigma\int_{\mathbb{R}^n}(\vec{b}(x)-\vec{b}(y))_{\sigma'}\mathcal{H}_{sj}(x-y)F(y)\mathrm{d}y\Big|_q\mathrm{d}x\\ &= C(\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}).\end{aligned}$$

To estimate I, similar to the case of $m=1$, using Hölder's inequality for finitely many functions, we have

$$\begin{aligned}
I &= \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}\Big(\sum_{k=1}^{\infty}|\prod_{i=1}^{m}(b_i(x)-\lambda_i)K_{sj}(f_k)(x)|^q\Big)^{1/q}\mathrm{d}x\\
&= \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}\prod_{i=1}^{m}|b_i(x)-\lambda_i||K_{sj}(F)(x)|_q\mathrm{d}x\\
&\le C\Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}\prod_{i=1}^{m}|b_i(x)-\lambda_i|^{p'}\mathrm{d}x\Big)^{1/p'}\Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|K_{sj}(F)(x)|_q^p\mathrm{d}x\Big)^{1/p}\\
&\le C\prod_{i=1}^{m}\|b_i\|_{\mathrm{BMO}}\left(M(|K_{sj}F|_q^p)(x_0)\right)^{1/p},
\end{aligned}$$

where in the last inequality, we used Lemma 2.3.

To estimate II, let $1<\tau<p$ and τ' be the conjugate exponent to τ again. We have

$$\begin{aligned}
\mathrm{II} =&\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}\Big|K_{sj}\Big(\prod_{i=1}^{m}(b_i-\lambda_i)F^1\Big)(x)\Big|_q\mathrm{d}x\\
\le&\frac{1}{|\mathcal{E}|}\Big(\int_{\mathcal{E}}\Big|K_{sj}(\prod_{i=1}^{m}(b_i-\lambda_i)F^1(x)\Big|_q^{\tau}\mathrm{d}x\Big)^{1/\tau}\Big(\int_{\mathcal{E}}1\mathrm{d}x\Big)^{1/\tau'}\\
\le&\frac{C}{|\mathcal{E}|^{1/\tau}}\Big(\int_{\mathcal{E}}\Big|\prod_{i=1}^{m}(b_i-\lambda_i)F^1(x)\Big|_q^{\tau}\mathrm{d}x\Big)^{1/\tau}\\
\le&\frac{C}{|\mathcal{E}|^{1/\tau}}\Big(\int_{2\mathcal{E}}|F(x)|_q^p\mathrm{d}x\Big)^{1/p}\Big(\int_{2\mathcal{E}}\prod_{i=1}^{m}|b_i(x)-\lambda_i|^{p\tau/(p-\tau)}\mathrm{d}x\Big)^{(p-\tau)/p\tau}\\
\le&C\prod_{i=1}^{m}\|b_i\|_{\mathrm{BMO}}\left(M(|F|_q^p)(x_0)\right)^{1/p},
\end{aligned}$$

where we used that $(K_{sj})_q$ is of type (p,p) for $1<q$, $p<\infty$ (see Lemmas 2.3 and 3.1).

Now we turn to estimate III. Since $x,x_0\in\mathcal{E},\ y\notin 2\mathcal{E}$, by Lemma 2.4, we have

$$\begin{aligned}
&\Big|K_{sj}\Big(\prod_{i=1}^{m}(b_i-\lambda_i)F^2\Big)(x)-K_{sj}\Big(\prod_{i=1}^{m}(b_i-\lambda_i)F^2\Big)(x_0)\Big|_q\\
&\le\Big(\sum_{k=1}^{\infty}\Big|\int_{(2\mathcal{E})^c}|\mathcal{H}_{sj}(x-y)-\mathcal{H}_{sj}(x_0-y)|\prod_{i=1}^{m}|b_i(y)-\lambda_i||f_k^2(y)|\mathrm{d}y\Big|^q\Big)^{1/q}\\
&\le Cj^{n/2}\rho(x_0-x)\int_{(2\mathcal{E})^c}\frac{\prod_{i=1}^m|b_i(y)-\lambda_i||F(y)|_q}{\rho(x_0-y)^{\alpha+1}}\mathrm{d}y\\
&\le Cj^{n/2}r\Big(\int_{(2\mathcal{E})^c}\frac{|F(y)|_q^p}{\rho(x_0-y)^{\alpha+1}}\mathrm{d}y\Big)^{1/p}\Big(\int_{(2\mathcal{E})^c}\frac{\prod_{i=1}^m|b_i(y)-\lambda_i|^{p'}}{\rho(x_0-y)^{\alpha+1}}\mathrm{d}y\Big)^{1/p'},
\end{aligned}\tag{3.7}$$

where in the second inequality, we used Minkowski's inequality.

Let $1<q_1,\dots,q_m<\infty$, and $1/q_1+\cdots+1/q_m=1$. By Hölder's inequality for finitely many functions, we obtain

$$\Big(\int_{(2\mathcal{E})^c}\frac{\prod_{i=1}^m|b_i(y)-\lambda_i|^{p'}}{\rho(x_0-y)^{\alpha+1}}\mathrm{d}y\Big)^{1/p'}\le\prod_{i=1}^{m}\Big(\int_{(2\mathcal{E})^c}\frac{|b_i(y)-\lambda_i|^{q_ip'}}{\rho(x_0-y)^{\alpha+1}}\mathrm{d}y\Big)^{1/q_ip'}.\tag{3.8}$$

Similarly to (3.4), for $1 \le i \le m$, we have

$$\int_{(2\mathcal{E})^c} \frac{|b_i(y)-\lambda_i|^{q_i p'}}{\rho(x_0-y)^{\alpha+1}}\mathrm{d}y \le C\frac{\|b_i\|_{\mathrm{BMO}}^{q_i p'}}{r}. \tag{3.9}$$

Combining (3.7) to (3.9), we obtain

$$\mathrm{III} \le Cj^{n/2}\|\vec{b}\|(M(|F|_q^p)(x_0))^{1/p}. \tag{3.10}$$

Finally, we estimate IV. Let $1 < \tau < p$ and τ' be the conjugate exponent to τ again. We have

$$\begin{aligned}\mathrm{IV} &\le \frac{1}{|\mathcal{E}|}\sum_{i=1}^{m-1}\sum_{\sigma\in\sigma_i}\Big(\int_{\mathcal{E}}\Big|\prod_{t\in\sigma}|b_t(x)-\lambda_t|[b_{\sigma'},K_{sj}]_qF(x)\Big|^{\tau}\mathrm{d}x\Big)^{1/\tau}\Big(\int_{\mathcal{E}}1\mathrm{d}x\Big)^{1/\tau'}\\ &\le \sum_{i=1}^{m-1}\sum_{\sigma\in\sigma_i}\Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}|[b_{\sigma'},K_{sj}]_qF(x)|^p\mathrm{d}x\Big)^{\frac{1}{p}}\Big(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}}\prod_{t\in\sigma}|b_t(x)-\lambda_t|^{p\tau/(p-\tau)}\mathrm{d}x\Big)^{\frac{(p-\tau)}{p\tau}}\\ &\le C\sum_{i=1}^{m-1}\sum_{\sigma\in\sigma_i}\|b_\sigma\|\left(M|[b_{\sigma'},K_{sj}]_q|^pF(x_0)\right)^{1/p}.\end{aligned}$$

Summing up I to IV, and taking the supremum over all ellipsoids $\mathcal{E}$, we obtain Lemma 3.2 for $m > 1$. This finishes the proof of Lemma 3.2. □

Lemma 3.3 *Let $1 < \delta,\ q < \infty$. Then*

$$\|[\vec{b},K_{sj}]F\|_{L^\delta(\ell^q)(\mathbb{R}^n)} \le Cj^{n/2}\|\vec{b}\|_{\mathrm{BMO}}\|F\|_{L^\delta(\ell^q)(\mathbb{R}^n)} \tag{3.11}$$

holds for all smooth vector functions $F=\{f_k\}_{k=1}^\infty$ and $F\in L_c^\infty$ for $1\le k<\infty$ and for all $x\in\mathbb{R}^n$ with constants C depending only on n, p and α.

Proof We use induction on m. First we suppose $([\vec{b},K_{sj}]_qF)^\#$ in $L^\delta(\mathbb{R}^n)$. Let $m=1$. In this case $\vec{b}=b_1$. Choose p such that $1<p<\delta$. By Lemmas 3.1 and 3.2, we have

$$\begin{aligned}\int_{\mathbb{R}^n}|([b_1,K_{sj}]_qF)^\#(x)|^\delta\mathrm{d}x \le& C\int_{\mathbb{R}^n}\Big[j^{n/2}\|b_1\|_{\mathrm{BMO}}\left(M(|F|_q^p)(x)\right)^{1/p}+\\ &\|b_1\|_{\mathrm{BMO}}\left(M(|K_{sj}F|_q^p)(x)\right)^{1/p}\Big]^\delta\mathrm{d}x\\ \le& C[j^{n/2}\|b_1\|_{\mathrm{BMO}}\|F\|_{L^\delta(\ell^q)(\mathbb{R}^n)}]^\delta.\end{aligned}$$

Power $1/\delta$ to the last inequality, we obtain (3.11). Assume that (3.11) holds for $1,2,\ldots,m-1$. Then by Lemmas 3.1 and 3.2 again, we have

$$\begin{aligned}\int_{\mathbb{R}^n}|([\vec{b},K_{sj}]_qF)^\#(x)|^\delta\mathrm{d}x \le& C\int_{\mathbb{R}^n}\Big[(j^{n/2}\|\vec{b}\|)^\delta\left(M(|F|_q^p)(x)\right)^{\delta/p}+\\ &\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}\|b_\sigma\|^\delta\Big(M(|[\vec{b}_{\sigma'},K_{sj}]F|_q^p)(x)\Big)^{\delta/p}\Big]\mathrm{d}x\\ \le& C[(j^{n/2}\|\vec{b}\|)^\delta\|(M(|F|_q^p)\|_{\delta/p})^{\delta/p}+\\ &\sum_{i=1}^{m-1}\sum_{\sigma\in C_i^m}(j^{n/2}\|\vec{b}_\sigma\|\|\vec{b}_{\sigma'}\|)^\delta(\||F|_q^p\|_{\delta/p})^{\delta/p}]\end{aligned}$$

$$\le C(j^{n/2}\|\vec{b}\|\|F\|_{L^\delta(\ell^q)(\mathbb{R}^n)})^\delta,$$

where $1 < p < \delta$. Power $1/\delta$ to the inequality again, we obtain (3.11). To end the proof of the lemma, we need to check that $([\vec{b}, K_{sj}]_q F)^\#$ in $L^\delta(\mathbb{R}^n)$, for $F \in L^\infty_C$ and F in $L^\delta(\ell^q)(\mathbb{R}^n)$.

First, suppose that the symbols $\{b_i\}_{i=1}^m$ are all bounded functions. Since F has compact support, we can assume that the support of F is contained in the $\mathcal{E} = \mathcal{E}(0,\ R)$. Then we can write

$$\int_{\mathbb{R}^n} |[\vec{b}, K_{sj}]_q F(x)|^\delta \mathrm{d}x = \int_{2\mathcal{E}} |[\vec{b}, K_{sj}]_q F(x)|^\delta dx + \int_{(2\mathcal{E})^c} |[\vec{b}, K_{sj}]_q F(x)|^\delta \mathrm{d}x.$$

The first integral can easily be estimated by making use of the L^∞-boundedness of the b_i and the L^q-boundedness for $q > 1$ of the operator K_{sj}, since K_{sj} is a Calderón-Zygmund operator. For the second term, by the properties of $\mathcal{H}_{sj}$, since $\rho(x) > 2R$, we have the following pointwise estimate:

$$\begin{aligned} |[\vec{b}, K_{sj}]_q F(x)| &\le C\Big(\sum_{k=1}^\infty \Big|\int_{\mathcal{E}} \frac{\prod_{i=1}^m |b_i(x) - b_i(y)||f_k(y)|}{\rho(x-y)^\alpha}\mathrm{d}y\Big|^q\Big)^{1/q} \\ &\le C\Big(\sum_{k=1}^\infty \Big|\frac{1}{\rho(x-0)^\alpha}\int_{\mathcal{E}(0,\rho(x-0))} |f_k(y)|_q \mathrm{d}y\Big|^q\Big)^{1/q} \le CM_q F(x). \end{aligned}$$

According to our supposition $F \in L^\infty_C$ and F in $L^\delta(\ell^q)(\mathbb{R}^n)$, then $F \in L^\delta$ for $1 < \delta < \infty$. By the well known fact that M_q is bounded in L^δ for $1 < \delta < \infty$, we obtain the second integral is finite. This means $[\vec{b}, K_{sj}]_q F \in L^\delta$ for $1 < \delta < \infty$.

For the general case, letting $N > 0$, we truncate the symbols b_i as $b_i^N = \max(\min(b_i,\ N),\ -N)$ and denote $\vec{b}^N = (b_1^N, \dots, b_m^N)$. It is easy to see that $\|b_i^N\|_{\mathrm{BMO}} \le C\|b_i\|_{\mathrm{BMO}}$. According to the above, we have

$$\int_{\mathbb{R}^n} |[\vec{b}^N, K_{sj}]_q|^\delta \mathrm{d}x \le C\|\vec{b}\| \int_{\mathbb{R}^n} |F(x)|_q^\delta \mathrm{d}x.$$

Taking into account the fact that F has compact support, we obtain that any product $b_{i_1}^N \dots b_{i_k}^N F$ converges in any L^q for $q > 1$ to $b_{i_1} \dots b_{i_k} F$ as $N \to \infty$. Hence, at least a subsequence, $|[\vec{b}^N, K_{sj}]_q F|^\delta$ converges pointwise almost everywhere to $|[\vec{b}, K_{sj}]_q F|^\delta$. By Fatou's lemma we conclude that the lemma holds for this general case. Thus, by Lemma 2.2, we obtain Lemma 3.3 for $F \in L^\infty_C$. Then, by the standard limit process, we obtain Lemma 3.3. This completes the proof of Lemma 3.3. □

Now we can complete the proof of Theorem 1.2. By (3.1) and Lemma 3.3, we have

$$\|[\vec{b}, T]F\|_{L^\delta(\ell^q)(\mathbb{R}^n)} \le \sum_{j=1}^\infty \sum_{s=1}^{g_j} \|\psi_{sj}(x)\|_\infty \|[\vec{b}, K_{sj}]_q F\|_\delta \le C\|\vec{b}\|\|F\|_{L^\delta(\ell^q)(\mathbb{R}^n)} \sum_{j=1}^\infty j^{-2l+n-2+n/2}.$$

So choosing $l > (3n-2)/4$, we obtain that

$$\|[\vec{b}, T]_q F\|_\delta \le C\|\vec{b}\|\|F\|_{L^\delta(\ell^q)(\mathbb{R}^n)}.$$

This completes the proof of Theorem 1.2. □

Remark The main results of the paper can also be obtained by the similar method in [16].

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