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# Some Symmetry Identities for the Euler Polynomials

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**Abstract** Using the generating functions, we prove some symmetry identities for the Euler polynomials and higher order Euler polynomials, which generalize the multiplication theorem for the Euler polynomials. Also we obtain some relations between the Bernoulli polynomials, Euler polynomials, power sum, alternating sum and Genocchi numbers.

**Keywords** Euler polynomial; Bernoulli number; Bernoulli polynomial; Genocchi number; power sum; alternating sum.

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## 1. Introduction

The Bernoulli numbers $B_n$ and the Bernoulli polynomials $B_n(x)$ are defined by the exponential generating functions

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}, \tag{1}$$

and

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{te^{tx}}{e^t - 1}, \tag{2}$$

respectively. The Bernoulli numbers $B_n$ satisfy the recurrence relation $\sum_{i=0}^{n} \binom{n+1}{i} B_i = 0$ for all $n > 0$ with $B_0 = 1$, and the explicit formula for the Bernoulli polynomial is $B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i}$.

For each integer $k \geq 0$, $S_k(n) = 0^k + 1^k + 2^k + \cdots + (n-1)^k$ is called sum of integer powers, or simply power sum. It is well known that $S_k(n) = \sum_{i=0}^{n-1} i^k$ is a polynomial in $n$ of degree $k+1$ (see [1, 2]):

$$S_k(n) = \frac{1}{k+1} \sum_{i=0}^{k} B_i \binom{k+1}{i} n^{k+1-i}, \tag{3}$$

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which is called the power sum polynomial. The exponential generating function for $S_k(n)$ is [2]

$$\sum_{k=0}^{\infty} S_k(n)\frac{t^k}{k!} = 1 + e^t + e^{2t} + \cdots + e^{(n-1)t} = \frac{e^{nt}-1}{e^t-1}. \tag{4}$$

Deeba and Rodriguez [3] and Gessel [4] proved the following recurrence

$$B_n = \frac{1}{a(1-a^n)}\sum_{k=0}^{n-1} a^k \binom{n}{k} B_k S_{n-k}(a), \tag{5}$$

which is true for any positive integer $n$ and any positive integer $a > 1$.

Howard [5] showed that (5) is a consequence of the multiplication theorem for the Bernoulli polynomials. The multiplication theorem can be stated this way: If $n$ and $a$ are positive integers, then

$$a^{1-n}B_n(ax) = \sum_{i=0}^{a-1} B_n(x + \frac{i}{a}). \tag{6}$$

Tuenter [6] obtained a relation of symmetry between the power sum and the Bernoulli numbers, and also showed that the recurrence (5) is a special case of the relation. This relation can be stated as the following identity, which is symmetric in $a$ and $b$

$$\sum_{i=0}^{n}\binom{n}{i} a^{i-1}B_i b^{n-i} S_{n-i}(a) = \sum_{i=0}^{n}\binom{n}{i} b^{i-1}B_i a^{n-i} S_{n-i}(b),\ a,b>0, n\geq 0. \tag{7}$$

In [7], we generalize this relation of symmetry between the power sum polynomials and the Bernoulli numbers to the relations between the power sum polynomials and the Bernoulli polynomials in two ways. The aim of the present paper is to generalize the symmetric relation to the Euler polynomials, alternating sum, and Genocchi numbers.

For integers $k \geq 0$ and $n \geq 1$, the alternating sum $T_k(n)$ is defined by

$$T_k(n) = \sum_{r=0}^{n-1}(-1)^r r^k = 0^k - 1^k + 2^k - \cdots + (-1)^{n-1}(n-1)^k, \tag{8}$$

and the generating function is

$$\sum_{k=0}^{\infty} T_k(n)\frac{t^k}{k!} = \frac{1-(-1)^n e^{nt}}{1+e^t}. \tag{9}$$

The Euler polynomials $E_n(x)$ and Genocchi numbers $G_n$ are defined by means of the following generating functions:

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2e^{tx}}{e^t+1}, \tag{10}$$

and

$$\sum_{n=0}^{\infty} G_n\frac{t^n}{n!} = \frac{2t}{e^t+1}, \tag{11}$$

respectively. The following formulas (12)–(14) are well known [5]:

$$G_n = 2(1-2^n)B_n, \tag{12}$$

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{G_{k+1}}{k+1} x^{n-k}, \tag{13}$$

$$(n - n^m)G_m = \sum_{k=0}^{m-1} \binom{m}{k} n^k G_k T_{m-k}(n),\ m > 1,\ n > 1,\ \text{and } n \text{ odd}. \tag{14}$$

The multiplication theorem for $E_m(x)$, for $a$ odd, is [5]:

$$\sum_{i=0}^{a-1} (-1)^i E_m(x + \frac{i}{a}) = a^{-m} E_m(ax), \tag{15}$$

and the multiplication theorem for $E_m(x)$, for $a$ even, is [5]:

$$\sum_{i=0}^{a-1} (-1)^i B_{m+1}(x + \frac{i}{a}) = -\frac{1}{2}(m+1)a^{-m} E_m(ax). \tag{16}$$

In this paper, we prove some symmetry identities for the Euler polynomials and higher order Euler polynomials, which generalize the multiplication theorems for the Euler polynomials. We obtain some relations between the Bernoulli polynomials, Euler polynomials, power sum, alternating sum, and Genocchi numbers.

## 2. Some symmetry identities for the Euler polynomials

We shall prove the following theorem for the Euler polynomials, which are symmetric in $a$ and $b$.

**Theorem 2.1** *Let $a$ and $b$ be positive integers with the same parity. Then*

$$\sum_{i=0}^{a-1} (-1)^i a^m E_m(bx + \frac{b}{a} i) = \sum_{i=0}^{b-1} (-1)^i b^m E_m(ax + \frac{a}{b} i). \tag{17}$$

**Proof** Let $f(t) = \frac{2e^{abxt}}{e^{at}+1} \cdot \frac{1+(-1)^{a+1}e^{abt}}{e^{bt}+1}$. Then

$$f(t) = \frac{2e^{abxt}}{e^{at}+1} \cdot \frac{1-(-e^{bt})^a}{e^{bt}+1} = \frac{2e^{abxt}}{e^{at}+1} \cdot \sum_{i=0}^{a-1} (-e^{bt})^i = \sum_{i=0}^{a-1} (-1)^i \frac{2e^{(bx+\frac{b}{a}i)at}}{e^{at}+1}$$
$$= \sum_{i=0}^{a-1} (-1)^i \sum_{m=0}^{\infty} E_m(bx + \frac{b}{a} i) a^m \frac{t^m}{m!} = \sum_{m=0}^{\infty} \Big( \sum_{i=0}^{a-1} (-1)^i a^m E_m(bx + \frac{b}{a} i) \Big) \frac{t^m}{m!}.$$

Since $(-1)^{a+1} = (-1)^{b+1}$, the expression for $f(t) = \frac{2e^{abxt}}{e^{at}+1} \cdot \frac{1+(-1)^{a+1}e^{abt}}{e^{bt}+1}$ is symmetric in $a$ and $b$. Therefore, we obtain the following power series expansion for $f(t)$ by symmetry:

$$f(t) = \sum_{m=0}^{\infty} \Big( \sum_{i=0}^{b-1} (-1)^i b^m E_m(ax + \frac{a}{b} i) \Big) \frac{t^m}{m!}.$$

Equating the coefficients of $\frac{t^m}{m!}$ in the two expressions for $f(t)$ gives us the desired result. □

Replacing $b = 1$ in (17) gives us the multiplication theorem (15), for odd $a$.

We can also use (9) and (10) to expand $f(t)=\frac{2e^{abxt}}{e^{at}+1}\cdot\frac{1+(-1)^{a+1}e^{abt}}{e^{bt}+1}$ as

$$f(t)=\Big(\sum_{m=0}^{\infty}E_m(bx)a^m\frac{t^m}{m!}\Big)\Big(\sum_{m=0}^{\infty}T_m(a)b^m\frac{t^m}{m!}\Big)=\sum_{m=0}^{\infty}\Big(\sum_{k=0}^{m}\binom{m}{k}a^kb^{m-k}E_k(bx)T_{m-k}(a)\Big)\frac{t^m}{m!}.$$

By considering the symmetry of $f(t)$ in $a$ and $b$, we obtain the following (18) which is an analogue of (7) for the Euler polynomials.

**Theorem 2.2** *Let $a$ and $b$ be positive integers with the same parity. Then*

$$\sum_{k=0}^{m}\binom{m}{k}a^kb^{m-k}E_k(bx)T_{m-k}(a)=\sum_{k=0}^{m}\binom{m}{k}b^ka^{m-k}E_k(ax)T_{m-k}(b). \tag{18}$$

**Theorem 2.3** *Let $a$ and $b$ be positive integers, and $a$ be even. Then*

$$\sum_{i=0}^{a-1}(-1)^{i+1}a^m\frac{2}{m+1}B_{m+1}(bx+\frac{b}{a}i)=\sum_{i=0}^{b-1}b^mE_m(ax+\frac{a}{b}i). \tag{19}$$

**Proof** Let $g(t)=\frac{2e^{abxt}}{e^{bt}+1}\cdot\frac{1-e^{abt}}{1-e^{at}}$. Then

$$\begin{aligned}g(t)&=\frac{2e^{abxt}}{e^{bt}+1}\cdot\frac{1-e^{abt}}{1-e^{at}}=\frac{2e^{abxt}}{e^{bt}+1}\cdot\sum_{i=0}^{b-1}e^{ait}\\&=\sum_{i=0}^{b-1}\frac{2e^{abxt+ait}}{e^{bt}+1}=\sum_{i=0}^{b-1}\sum_{m=0}^{\infty}E_m(ax+\frac{a}{b}i)b^m\frac{t^m}{m!}\\&=\sum_{m=0}^{\infty}\Big(\sum_{i=0}^{b-1}E_m(ax+\frac{a}{b}i)b^m\Big)\frac{t^m}{m!}.\end{aligned}$$

On the other hand, considering $a$ is even, we have

$$\begin{aligned}g(t)&=\frac{2e^{abxt}}{e^{bt}+1}\cdot\frac{1-e^{abt}}{1-e^{at}}=-\frac{2e^{abxt}}{e^{at}-1}\cdot\frac{1-(-e^{bt})^a}{1-(-e^{bt})}\\&=\frac{-2}{at}\frac{ate^{abxt}}{e^{at}-1}\cdot\sum_{i=0}^{a-1}(-e^{bt})^i=\frac{-2}{at}\sum_{i=0}^{a-1}(-1)^i\frac{ate^{abxt+bit}}{e^{at}-1}\\&=\frac{-2}{at}\sum_{i=0}^{a-1}(-1)^i\sum_{m=0}^{\infty}B_m(bx+\frac{b}{a}i)a^m\frac{t^m}{m!}\\&=\frac{-2}{at}\sum_{m=0}^{\infty}\sum_{i=0}^{a-1}(-1)^iB_m(bx+\frac{b}{a}i)a^m\frac{t^m}{m!}\\&=-2\sum_{m=0}^{\infty}\sum_{i=0}^{a-1}(-1)^iB_m(bx+\frac{b}{a}i)a^{m-1}\frac{t^{m-1}}{m!}\\&=-2\sum_{m=0}^{\infty}\sum_{i=0}^{a-1}(-1)^i\frac{1}{m+1}B_{m+1}(bx+\frac{b}{a}i)a^m\frac{t^m}{m!}.\end{aligned}$$

Equating the coefficients of $\frac{t^m}{m!}$ in the two expressions for $g(t)$ gives us the desired result. □

Putting $b=1$ in (20) gives us the multiplication theorem (14), for even $a$.

For a real or complex parameter $r$, the generalized Euler polynomials $E_n^{(r)}(x)$, with each being of degree $n$ ($n \geq 0$) in $x$ as well as in $r$, are defined by means of the following generating function [8]:

$$\sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!} = (\frac{2}{e^t+1})^r e^{xt}. \tag{20}$$

**Theorem 2.4** *Let $a$ and $b$ be positive integers with the same parity. Then*

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}a^k b^{m-k}E_{m-k}^{(r-1)}(ay)\sum_{i=0}^{a-1}(-1)^i E_k^{(r)}(bx+\frac{b}{a}i)\\ &=\sum_{k=0}^{m}\binom{m}{k}b^k a^{m-k}E_{m-k}^{(r-1)}(by)\sum_{i=0}^{b-1}(-1)^i E_k^{(r)}(ax+\frac{a}{b}i).\end{aligned} \tag{21}$$

**Proof** Let $h(t) = \frac{2^{2r-1}e^{abxt}e^{abyt}(1+(-1)^{a+1}e^{abt})}{(e^{at}+1)^r(e^{bt}+1)^r}$. Then we can expand $h(t)$ as

$$\begin{aligned}h(t) &= (\frac{2}{e^{at}+1})^r e^{abxt}\cdot\frac{1-(-e^{bt})^a}{e^{bt}+1}\cdot(\frac{2}{e^{bt}+1})^{r-1}e^{abyt}\\ &=\sum_{i=0}^{a-1}(-1)^i(\frac{2}{e^{at}+1})^r e^{abxt+bit}\cdot(\frac{2}{e^{bt}+1})^{r-1}e^{abyt}\\ &=\sum_{i=0}^{a-1}(-1)^i\sum_{m=0}^{\infty}E_m^{(r)}(bx+\frac{b}{a}i)a^m\frac{t^m}{m!}\sum_{m=0}^{\infty}E_m^{(r-1)}(ay)b^m\frac{t^m}{m!}\\ &=\Big(\sum_{m=0}^{\infty}\sum_{i=0}^{a-1}(-1)^i E_m^{(r)}(bx+\frac{b}{a}i)a^m\frac{t^m}{m!}\Big)\Big(\sum_{m=0}^{\infty}E_m^{(r-1)}(ay)b^m\frac{t^m}{m!}\Big)\\ &=\sum_{m=0}^{\infty}c_m\frac{t^m}{m!},\end{aligned}$$

where by multiplying rule of formal power series

$$c_m=\sum_{k=0}^{m}\binom{m}{k}a^k b^{m-k}E_{m-k}^{(r-1)}(ay)\sum_{i=0}^{a-1}(-1)^i E_k^{(r)}(bx+\frac{b}{a}i).$$

We may also expand $h(t)$ as

$$\begin{aligned}h(t) &= (\frac{2}{e^{bt}+1})^r e^{baxt}\cdot\frac{1-(-e^{at})^b}{e^{at}+1}\cdot(\frac{2}{e^{at}+1})^{r-1}e^{bayt}\\ &=\sum_{i=0}^{b-1}(-1)^i(\frac{2}{e^{bt}+1})^r e^{baxt+ait}\cdot(\frac{2}{e^{at}+1})^{r-1}e^{bayt}\\ &=\sum_{i=0}^{b-1}(-1)^i\sum_{m=0}^{\infty}E_m^{(r)}(ax+\frac{a}{b}i)b^m\frac{t^m}{m!}\sum_{m=0}^{\infty}E_m^{(r-1)}(by)a^m\frac{t^m}{m!}\\ &=\Big(\sum_{m=0}^{\infty}\sum_{i=0}^{b-1}(-1)^i E_m^{(r)}(ax+\frac{a}{b}i)b^m\frac{t^m}{m!}\Big)\Big(\sum_{m=0}^{\infty}E_m^{(r-1)}(by)a^m\frac{t^m}{m!}\Big)\\ &=\sum_{m=0}^{\infty}c_m\frac{t^m}{m!},\end{aligned}$$

where $c_m = \sum_{k=0}^{m}\binom{m}{k}b^k a^{m-k}E_{m-k}^{(r-1)}$ (by) $\sum_{i=0}^{b-1}(-1)^i E_k^{(r)}(ax+\frac{a}{b}i)$. That is, $h(t)$ is symmetric in $a$ and $b$, so is $c_m$. Equating the two expressions for $c_m$ gives the identity of the theorem. $\Box$

## 3. Some identities for the power sum and Genocchi numbers

**Theorem 3.1** *Let $a$ and $b$ be positive integers with the same parity. Then*

$$\sum_{k=0}^{m}\binom{m}{k}G_k a^k b^{m-k+1}T_{m-k}(a)=\sum_{k=0}^{m}\binom{m}{k}G_k b^k a^{m-k+1}T_{m-k}(b). \tag{22}$$

**Proof** Let $f(t)=\frac{2abt}{e^{at}+1}\cdot\frac{1+(-1)^{a+1}e^{abt}}{e^{bt}+1}$. Then

$$\begin{aligned}
f(t)&=\frac{2abt}{e^{at}+1}\cdot\frac{1-(-e^{bt})^a}{e^{bt}+1}=\frac{2abt}{e^{at}+1}\cdot\sum_{i=0}^{a-1}(-e^{bt})^i\\
&=\sum_{i=0}^{a-1}(-1)^i b\frac{2at}{e^{at}+1}e^{bit}=\sum_{i=0}^{a-1}(-1)^i b\Big(\sum_{m=0}^{\infty}G_m a^m\frac{t^m}{m!}\Big)\Big(\sum_{m=0}^{\infty}(bi)^m\frac{t^m}{m!}\Big)\\
&=\sum_{i=0}^{a-1}(-1)^i b\sum_{m=0}^{\infty}\Big(\sum_{k=0}^{m}\binom{m}{k}G_k a^k(bi)^{m-k}\Big)\frac{t^m}{m!}\\
&=\sum_{m=0}^{\infty}\Big(\sum_{k=0}^{m}\binom{m}{k}G_k a^k b^{m-k+1}\sum_{i=0}^{a-1}(-1)^i i^{m-k}\Big)\frac{t^m}{m!}\\
&=\sum_{m=0}^{\infty}\Big(\sum_{k=0}^{m}\binom{m}{k}G_k a^k b^{m-k+1}T_{m-k}(a)\Big)\frac{t^m}{m!}.
\end{aligned}$$

Since $(-1)^{a+1}=(-1)^{b+1}$, the expression for $f(t)=\frac{2abt}{e^{at}+1}\cdot\frac{1+(-1)^{a+1}e^{abt}}{e^{bt}+1}$ is symmetric in $a$ and $b$. Therefore, we obtain the following power series expansion for $f(t)$ by symmetry: $f(t)=\sum_{m=0}^{\infty}(\sum_{k=0}^{m}\binom{m}{k}G_k b^k a^{m-k+1}T_{m-k}(b))\frac{t^m}{m!}$. Compairing the coefficients of $\frac{t^m}{m!}$ in the two expressions for $f(t)$, we get the desired result (22). $\Box$

Since $T_0(1)=1$, and $T_k(1)=0$ for all $k\geq 1$, setting $b=1$ in (22) will yield (14).

**Theorem 3.2** *Let $m$ and $n$ be positive integers. Then*

$$\sum_{k=0}^{m}\binom{m}{k}T_k(n)S_{m-k}(n)=2^m S_m(n),\text{ where } n \text{ is odd}, \tag{23}$$

$$\sum_{k=0}^{m}\binom{m}{k}T_k(n)S_{m-k}(n)=2^{m+1}S_m(\frac{n}{2})-2^m S_m(n),\text{ where } n \text{ is even}. \tag{24}$$

**Proof** If $n$ is odd, then

$$\begin{aligned}
\Big(\sum_{m=0}^{\infty}S_m(n)\frac{t^m}{m!}\Big)\Big(\sum_{m=0}^{\infty}T_m(n)\frac{t^m}{m!}\Big)&=\frac{e^{nt}-1}{e^t-1}\cdot\frac{(-1)^{n+1}e^{nt}+1}{e^t+1}\\
&=\frac{e^{2nt}-1}{e^{2t}-1}=\sum_{m=0}^{\infty}2^m S_m(n)\frac{t^m}{m!}.
\end{aligned}$$

By the multiplying rule of formal power series,

$$\Big(\sum_{m=0}^{\infty} S_m(n)\frac{t^m}{m!}\Big)\Big(\sum_{m=0}^{\infty} T_m(n)\frac{t^m}{m!}\Big) = \sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}T_k(n)S_{m-k}(n)\frac{t^m}{m!}.$$

Compairing the coefficients of $\frac{t^m}{m!}$ in the two expressions, we obtain the desired result (23).

Similarly, considering

$$\begin{aligned}\Big(\sum_{m=0}^{\infty} S_m(n)\frac{t^m}{m!}\Big)\Big(\sum_{m=0}^{\infty} T_m(n)\frac{t^m}{m!}\Big) &= \frac{e^{nt}-1}{e^t-1}\cdot\frac{1-e^{nt}}{e^t+1} = 2\frac{e^{nt}-1}{e^{2t}-1} - \frac{e^{2nt}-1}{e^{2t}-1} \\ &= 2\sum_{m=0}^{\infty} 2^m S_m(\frac{n}{2})\frac{t^m}{m!} - \sum_{m=0}^{\infty} 2^m S_m(n)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}(2^{m+1}S_m(\frac{n}{2}) - 2^m S_m(n))\frac{t^m}{m!}\end{aligned}$$

for $n$ even, we obtain the result (24). □

**Theorem 3.3** *Let $m$ and $n$ be positive integers. Then*

$$\sum_{k=0}^{m}\binom{m}{k}B_k(n)T_{m-k}(n) = 2^{m-1}(B_m(\frac{n}{2}) + (-1)^{n+1}B_m(n)), \tag{25}$$

$$\sum_{k=0}^{m}\binom{m}{k}E_k(n)S_{m-k}(n) = 2^m(E_m(n) - E_m(\frac{n}{2})). \tag{26}$$

**Proof** Let $f(t) = \frac{te^{nt}}{e^t-1}\cdot\frac{1+(-1)^{n+1}e^{nt}}{e^t+1}$. Then

$$\begin{aligned}f(t) &= \frac{te^{nt}}{e^t-1}\cdot\frac{1+(-1)^{n+1}e^{nt}}{e^t+1} = \Big(\sum_{m=0}^{\infty} B_m(n)\frac{t^m}{m!}\Big)\Big(\sum_{m=0}^{\infty} T_m(n)\frac{t^m}{m!}\Big) \\ &= \sum_{m=0}^{\infty}\Big(\sum_{k=0}^{m}\binom{m}{k}B_k(n)T_{m-k}(n)\Big)\frac{t^m}{m!}.\end{aligned}$$

On the other hand,

$$\begin{aligned}f(t) &= \frac{te^{nt}+(-1)^{n+1}te^{2nt}}{e^{2t}-1} = \frac{1}{2}\sum_{m=0}^{\infty} B_m(\frac{n}{2})\frac{(2t)^m}{m!} + \frac{(-1)^{n+1}}{2}\sum_{m=0}^{\infty} B_m(n)\frac{(2t)^m}{m!} \\ &= \sum_{m=0}^{\infty}(2^{m-1}(B_m(\frac{n}{2}) + (-1)^{n+1}B_m(n)))\frac{t^m}{m!}.\end{aligned}$$

Equating coefficients of $\frac{t^m}{m!}$ in the two expressions of $f(t)$ yields the identity (25).

Similarly, by considering

$$\begin{aligned}g(t) &= \frac{2e^{nt}}{e^t+1}\cdot\frac{e^{nt}-1}{e^t-1} = \Big(\sum_{m=0}^{\infty} E_m(n)\frac{t^m}{m!}\Big)\Big(\sum_{m=0}^{\infty} S_m(n)\frac{t^m}{m!}\Big) \\ &= \sum_{m=0}^{\infty}\Big(\sum_{k=0}^{m}\binom{m}{k}E_k(n)S_{m-k}(n)\Big)\frac{t^m}{m!}\end{aligned}$$

and

$$g(t) = \frac{2e^{2nt} - 2e^{2nt}}{e^{2t} - 1} = \sum_{m=0}^{\infty} E_m(n)\frac{(2t)^m}{m!} - \sum_{m=0}^{\infty} E_m(\frac{n}{2})\frac{(2t)^m}{m!}$$
$$= \sum_{m=0}^{\infty}(2^m(E_m(n) - E_m(\frac{n}{2})))\frac{t^m}{m!},$$

we obtain (26). $\square$

**Theorem 3.4** *Let $m$ and $n$ be positive integers. Then*

$$\sum_{k=0}^{m}\binom{m}{k}G_k S_{m-k}(n) = 2^m(B_m(\frac{n}{2}) - B_m). \tag{27}$$

**Proof** Let $f(t) = \frac{2t}{e^t+1}\cdot\frac{e^{nt}-1}{e^t-1}$. Then

$$f(t) = \frac{2t}{e^t+1}\cdot\frac{e^{nt}-1}{e^t-1} = \Big(\sum_{m=0}^{\infty} G_m(n)\frac{t^m}{m!}\Big)\Big(\sum_{m=0}^{\infty} S_m(n)\frac{t^m}{m!}\Big)$$
$$= \sum_{m=0}^{\infty}\Big(\sum_{k=0}^{m}\binom{m}{k}G_k(n)S_{m-k}(n)\Big)\frac{t^m}{m!},$$

and

$$f(t) = \frac{2te^{nt} - 2t}{e^{2t} - 1} = \sum_{m=0}^{\infty} B_m(\frac{n}{2})\frac{(2t)^m}{m!} - \sum_{m=0}^{\infty} B_m\frac{(2t)^m}{m!}$$
$$= \sum_{m=0}^{\infty}(2^m(B_m(\frac{n}{2}) - B_m(n)))\frac{t^m}{m!}.$$

Equating coefficients of $\frac{t^m}{m!}$ in the two expressions of $f(t)$ yields the identity (27). $\square$

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