

Local Jordan Derivations and Local Jordan Automorphisms of Upper Triangular Matrix Algebras

Yan Xia ZHAO^{1,*}, Rui Ping YAO², Deng Yin WANG²

1. School of Mathematics and Information Science, Henan Polytechnic University,
Henan 454000, P. R. China;

2. Department of Mathematics, China University of Mining and Technology,
Jiangsu 221008, P. R. China

Abstract Let R be a commutative ring with identity, $T_n(R)$ the R -algebra of all upper triangular n by n matrices over R . In this paper, it is proved that every local Jordan derivation of $T_n(R)$ is an inner derivation and that every local Jordan automorphism of $T_n(R)$ is a Jordan automorphism. As applications, we show that local derivations and local automorphisms of $T_n(R)$ are inner.

Keywords local Jordan derivations; local Jordan automorphisms; local derivations; local automorphisms; upper triangular matrix algebras.

Document code A

MR(2000) Subject Classification 17C30; 13C10; 15A30

Chinese Library Classification O153

1. Introduction

A problem which several authors have considered recently is that of finding sufficient conditions on a linear map to ensure it preserves a particular algebraic or function property. One aspect of this is the notion of a local mapping—a function that agrees at each point with some map (the map possibly changing from point to point) that has the desired property. Many authors have examined those functions which agree with derivations or automorphisms at each point. Formally, a linear mapping δ from an algebra \mathcal{A} into itself is called a *local derivation* (resp., *local automorphism*) if for every $a \in \mathcal{A}$, there exists a derivation (resp., an automorphism) δ_a of \mathcal{A} , depending on a , such that $\delta(a) = \delta_a(a)$. A remarkable fact concerning the algebra is the question whether it then follows that δ is necessarily a derivation (resp., an automorphism). If every linear local derivation or local automorphism of an algebra is a derivation or an automorphism, then we can say that the derivations (resp., automorphisms) of those structures are, in a certain sense, completely determined by their local actions.

Local derivations, local automorphisms and other local maps have been studied in a variety of contexts. Larson [1] initially considered local maps in his examination of reflexivity and interpolation for subspace of $\mathcal{B}(\mathbb{H})$, where \mathbb{H} is a Hilbert space. The notion of local derivations (resp.,

Received March 12, 2008; Accepted January 5, 2009

Supported by the Doctor Foundation of Henan Polytechnic University (Grant No. B2010-93).

* Corresponding author

E-mail address: zhaoyanxia001@126.com (Y. X. ZHAO)

local automorphisms) was introduced independently by Larson and Sourour [2] and Kadison [3] (resp., Larson and Sourour [2]). Larson and Sourour [2] showed every local derivation on $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators on a Banach space \mathcal{H} , is a derivation. Kadison [3] showed that every norm continuous linear local derivation from a von Neumann algebra into its dual normal bimodule is a derivation. Johnson [4] proved that this result holds for all C^* -algebra \mathcal{U} and all Banach \mathcal{U} bimodules \mathcal{H} . Kadison also presented an example, constructed by C. Jensen, of an algebra, but not an operator algebra, which has nontrivial local derivations. A nontrivial local derivation on an operator algebra was found in [5]. Other recent work that showed all local derivations or local automorphisms are actually global derivations or automorphisms can be found in [5-7]. Crist [5] proved that for an algebra which is the direct limit of finite CSL algebra via $*$ -extendable embedding, any norm continuous linear local derivation is a derivation. Zhang, Ji and Cao [6] showed that every norm continuous linear local derivation of a nest subalgebra of a factor von Neumann algebra is a derivation. In [7], Zhang, Yang and Pan showed that every surjective weakly continuous linear local automorphism of nest subalgebras with non-trivial nests of factor Von Neumann algebras is an automorphism. Larson and Sourour [2] also showed that every surjective linear local automorphism of $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators on an infinite-dimensional Banach space \mathcal{H} , is an automorphism. For a separable infinite dimensional Hilbert space \mathbb{H} , Brešar and Šemrl [8] proved that the above conclusion holds true without the assumption on surjection. But in the finite-dimensional case, the situation is somewhat different. Namely, anti-automorphisms of $M_n(\mathbb{C})$, the algebra of all $n \times n$ complex matrices, are also local automorphisms [2, Theorem 2.2]. Crist [9] showed that any linear local automorphism of a finite dimensional CSL algebra \mathcal{A} is either an automorphism or can be factored as an automorphism and the transpose of a self-adjoint summand of \mathcal{A} . In this article, another nontrivial local automorphism on a subalgebra of $M_3(\mathbb{C})$ was constructed.

Example 1.1 ([9, Example 3.1]) Let $\mathcal{A} \subset M_3(\mathbb{C})$ be the algebra of finite dimensional upper triangular matrices constant on each diagonal, i.e., $\mathcal{A} = \{a(\sum_{i=1}^3 E_{ii}) + b(E_{12} + E_{23}) + cE_{13} \mid a, b, c \in \mathbb{C}\}$, where $\{E_{ij}\}_{ij}$ are the standard matrix units. Let $I = \sum_{i=1}^3 E_{ii}$ and $T = E_{12} + E_{23}$. Then $\{I, T, T^2\}$ is a basis for \mathcal{A} . Define the function $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ on the basis and extend linearly: $\alpha(I) = I$, $\alpha(T) = e^{\frac{1}{2}}T$, $\alpha(T^2) = T^2$. Then the map is proved to be a nontrivial local automorphism of \mathcal{A} .

Although, as Crist said in [9], it is somewhat difficult to construct nontrivial local automorphisms for certain algebra system, yet we constructed an example of nontrivial local automorphism for an F -algebra, where F is an arbitrary field.

Example 1.2 Let F be a field, and \mathcal{A} be the F -algebra consisting of all 3×3 strictly upper triangular matrices. For fixed $k \neq 0, 1$, we define $\phi : \mathcal{A} \rightarrow \mathcal{A}$ by

$$a_{12}E_{12} + a_{13}E_{13} + a_{23}E_{23} \mapsto a_{12}E_{12} + ka_{13}E_{13} + a_{23}E_{23}.$$

Then we see that ϕ is a bijective linear map of \mathcal{A} . Since E_{12} , E_{23} are both fixed by ϕ , we see that ϕ cannot be an automorphism of \mathcal{A} . Otherwise, $E_{13} = E_{12}E_{23}$ is also fixed by ϕ , in

contradiction with $\phi(E_{13}) = kE_{13}$. However, one may verify that ϕ is a local automorphism of \mathcal{A} . For any given $x = a_{12}E_{12} + a_{13}E_{13} + a_{23}E_{23} \in \mathcal{A}$, if $a_{23} = 0$, then the action of ϕ on x agrees with that of the inner automorphism induced by $y = \sum_{i=1}^3 E_{ii} + a_{23}^{-1}(k-1)a_{13}E_{12}$, that is, $\phi(x) = \text{Int } y(x) = yxy^{-1}$. If $a_{23} \neq 0$, then the action of ϕ on x identifies that of the diagonal automorphism induced by $z = \text{diag}(1, 1, k^{-1})$, that is, $\phi(x) = zxz^{-1}$.

Motivated by those and the concept of local maps introduced by Kadison [3] and Larson and Sourour [2], we define the notions of local Jordan derivations and local Jordan automorphisms. A linear mapping φ of \mathcal{A} into itself is called a *local Jordan derivation* (resp., *local Jordan automorphism*) if for every $a \in \mathcal{A}$, there exists a Jordan derivation (resp., Jordan automorphism) φ_a of \mathcal{A} , depending on a , such that $\varphi(a) = \varphi_a(a)$. Recall that a linear mapping $\delta : \mathcal{A} \rightarrow \mathcal{A}$ is a Jordan derivation if $\delta(a^2) = \delta(a)a + a\delta(a)$ for all $a \in \mathcal{A}$, and a bijective linear mapping $\psi : \mathcal{A} \rightarrow \mathcal{A}$ is called a Jordan automorphism if $\psi(ab + ba) = \psi(a)\psi(b) + \psi(b)\psi(a)$ for all $a, b \in \mathcal{A}$.

It is natural that every Jordan automorphism (resp., Jordan derivation) is a local Jordan automorphism (resp., local Jordan derivation). But the converse is not true. The map defined in Example 1.2 is also a local Jordan automorphism, but not a Jordan automorphism. Since each automorphism (resp., derivation) is a Jordan automorphism (resp., Jordan derivation), we have that automorphisms and derivations are local Jordan automorphisms and local Jordan derivations, respectively. The converse is not true either. For this we can also see the Example 1.2.

The algebra $T_n(R)$ of all upper triangular matrices over R is an interesting topic for many researchers. Significant research has been done in studying various linear mapping of $T_n(R)$ (see [10–15]). Benkovič [10] described Jordan derivations on triangular matrices over commutative rings and showed that every Jordan derivation from the algebra of all upper triangular matrices into its arbitrary bimodule is the sum of a derivation and an anti-derivation. In 1997, Cao [12] gave a description of the Lie automorphisms of upper triangular matrices over commutative rings. In 1990, Kezlan [13] proved that every R -algebra automorphism of the upper triangular matrices over commutative rings is inner. Tang, Cao and Zhang [14] determined all Jordan isomorphisms of triangular matrices over commutative rings. Zhang and Yu [15] showed that every Jordan derivation of triangular algebra is a derivation. Inspired by these, in this paper, we will study the local Jordan derivations and local Jordan automorphisms of upper triangular matrix algebras over commutative rings. As applications, we will also show that local derivations and local automorphisms of $T_n(R)$ are inner.

Throughout this paper, let R be a commutative ring with identity, R^* the group of invertible elements of R . Let $T_n(R)$ denote the R -algebra of all upper triangular n by n matrices over R , $T_n^*(R)$ the set of all invertible elements in $T_n(R)$. We denote by \mathbf{n} the subalgebra of $T_n(R)$ consisting of all strictly upper triangular matrices. Let $I_{k \times k}$ denote the $k \times k$ identity matrix ($I_{n \times n}$ is abbreviated to I), $E_{i,j}$ the matrix with 1 at the position (i, j) and zeros elsewhere for $1 \leq i, j \leq n$. Let

$$\mathfrak{S}_1 = \left\{ \begin{pmatrix} 0 & \\ & X \end{pmatrix} \in T_n(R) \mid X \in T_{n-1}(R) \right\}, \text{ and}$$

$$\mathfrak{S}_2 = \left\{ \begin{pmatrix} 0 & & \\ & Y & \\ & & 0 \end{pmatrix} \in T_n(R) \mid Y \in T_{n-2}(R) \right\}.$$

Obviously, \mathfrak{S}_1 and \mathfrak{S}_2 are subalgebras of $T_n(R)$.

2. Local Jordan derivations

We first develop some lemmas that will allow us to calculate with local Jordan derivations.

Lemma 2.1. *If δ is a local Jordan derivation on $T_n(R)$, then $\delta(E) = \delta(E)E + E\delta(E)$ for every idempotent $E \in T_n(R)$.*

Proof Let E be an idempotent in $T_n(R)$. Then $E^2 = E$. Since δ is a local Jordan derivation, there exists a Jordan derivation δ_E , depending on E , such that $\delta(E) = \delta_E(E)$. Thus we have

$$\delta(E) = \delta_E(E) = \delta_E(E^2) = \delta_E(E)E + E\delta_E(E) = \delta(E)E + E\delta(E). \quad \square$$

Lemma 2.2 *Let δ be a local Jordan derivation of $T_n(R)$. Then there exists an inner derivation $\text{ad } M$ for some $M \in T_n(R)$ such that $(\delta + \text{ad } M)(E_{ii}) = 0, i = 1, 2, \dots, n$.*

Proof Let δ be a local Jordan derivation of $T_n(R)$. Suppose that $\delta(E_{ii}) = (a_{st}^{(i)})_{n \times n}, i = 1, 2, \dots, n$, where $a_{st}^{(i)} \in R$. Since $E_{ii}^2 = E_{ii}$, by Lemma 2.1, we have $\delta(E_{ii}) = \delta(E_{ii})E_{ii} + E_{ii}\delta(E_{ii})$, it follows that

$$\delta(E_{ii}) = \sum_{k>i}^n a_{ik}^{(i)} E_{ik} + \sum_{l \geq 1}^{i-1} a_{li}^{(i)} E_{li}.$$

For $1 \leq i \neq j \leq n$, from $(E_{ii} + E_{jj})^2 = E_{ii} + E_{jj}$, and by Lemma 2.1, we have

$$\delta(E_{ii} + E_{jj}) = \delta(E_{ii} + E_{jj})(E_{ii} + E_{jj}) + (E_{ii} + E_{jj})\delta(E_{ii} + E_{jj}).$$

From this equality, we get $a_{ij}^{(i)} + a_{ij}^{(j)} = 0$. Set $M = \sum_{1 \leq i < j \leq n} a_{ij}^{(i)} E_{ij}$. Then by calculation, we have $(\delta + \text{ad } M)(E_{ii}) = 0, i = 1, 2, \dots, n$. \square

Lemma 2.3 *Let δ be a local Jordan derivation of $T_n(R)$ satisfying $\delta(E_{ii}) = 0$ for $i = 1, 2, \dots, n$. Then $\delta(E_{ij}) \in RE_{ij}$ for all $1 \leq i < j \leq n$.*

Proof Given $1 \leq i < j \leq n$. Consider the action of δ on the idempotent elements $E_{ii} + E_{ij}$ and $E_{jj} + E_{ij}$. By Lemma 2.1, we have

$$\begin{aligned} \delta(E_{ij}) &= \delta(E_{ij})(E_{ii} + E_{ij}) + (E_{ii} + E_{ij})\delta(E_{ij}) \\ &= \delta(E_{ij})(E_{jj} + E_{ij}) + (E_{jj} + E_{ij})\delta(E_{ij}), \end{aligned}$$

which implies that $\delta(E_{ij}) = b_{ij}E_{ij}$ for some $b_{ij} \in R$. That is to say $\delta(E_{ij}) \in RE_{ij}$ for all $1 \leq i < j \leq n$. \square

Lemma 2.4 *Let δ be as in Lemma 2.3. Then there exists some $D \in T_n(R)$ such that $(\delta + \text{ad } D)(E_{i,i+1}) = 0$ for $i = 1, 2, \dots, n-1$ and $(\delta + \text{ad } D)(E_{ii}) = 0$ for $i = 1, 2, \dots, n$.*

Proof By Lemma 2.3, we have $\delta(E_{i,i+1}) = b_{i,i+1}E_{i,i+1}$ with some $b_{i,i+1} \in R$. Let

$$D = \text{diag}(0, b_{12}, b_{12} + b_{23}, \dots, b_{12} + b_{23} + \dots + b_{n-1,n}).$$

Then $(\delta + \text{ad } D)(E_{i,i+1}) = 0$ for $i = 1, 2, \dots, n-1$, simultaneously, $(\delta + \text{ad } D)(E_{ii}) = 0$ for $i = 1, 2, \dots, n$. \square

Lemma 2.5 *Let δ be a local Jordan derivation of $T_n(R)$. If $\delta(E_{ii}) = 0$, $i = 1, 2, \dots, n$, and $\delta(E_{i,i+1}) = 0$, $i = 1, 2, \dots, n-1$, then $\delta(E_{i,i+k}) = 0$ for any $E_{i,i+k} \in T_n(R)$.*

Proof We will prove this lemma by induction on k ($k \geq 2$). By Lemma 2.3, we know $\delta(E_{i,i+k}) = b_{i,i+k}E_{i,i+k}$ for certain $b_{i,i+k} \in R$.

When $k = 2$, from the idempotence of $E_{i,i+1} + E_{i+1,i+2} + E_{i,i+2} + E_{i+1,i+1}$ and Lemma 2.1, we have

$$\begin{aligned} b_{i,i+2}E_{i,i+2} &= b_{i,i+2}E_{i,i+2}(E_{i,i+1} + E_{i+1,i+2} + E_{i,i+2} + E_{i+1,i+1}) + \\ &\quad (E_{i,i+1} + E_{i+1,i+2} + E_{i,i+2} + E_{i+1,i+1})b_{i,i+2}E_{i,i+2}, \end{aligned}$$

which means that $b_{i,i+2}E_{i,i+2} = 0$, that is, $\delta(E_{i,i+2}) = 0$.

By induction we assume that $\delta(E_{i,i+m}) = 0$ for $m = 2, 3, \dots, k-1$. Since

$$(E_{i,i+1} + E_{i+1,i+k} + E_{i,i+k} + E_{i+1,i+1})^2 = E_{i,i+1} + E_{i+1,i+k} + E_{i,i+k} + E_{i+1,i+1},$$

similarly to the case $k = 2$, we get $b_{i,i+k}E_{i,i+k} = 0$, so $\delta(E_{i,i+k}) = 0$. The proof is completed. \square

In this section, our main result is the following theorem.

Theorem 2.1 *Let R be a commutative ring with identity, $T_n(R)$ the R -algebra of all upper triangular n by n matrices over R . Then every local Jordan derivation δ of $T_n(R)$ is inner.*

Proof Let δ be a local Jordan derivation of $T_n(R)$. By Lemmas 2.2–2.5, we know that there exist some $M \in T_n(R)$ and $D \in T_n(R)$ such that $(\delta + \text{ad } M + \text{ad } D)(E_{ij}) = 0$ for $1 \leq i \leq j \leq n$. That is to say $\delta = -\text{ad } M - \text{ad } D$. So δ is an inner derivation of $T_n(R)$. \square

Remark Since each Jordan derivation of $T_n(R)$ is a local Jordan derivation, by Theorem 2.1, we know that every Jordan derivation of $T_n(R)$ is inner.

3. Local Jordan automorphisms

Tang, Cao and Zhang [14] gave an explicit description of Jordan isomorphisms of $T_n(R)$. For convenience of the proof of the main result in this section, we give another description of the Theorem 4.1 in [14] by the following lemma. Before giving this lemma, let us introduce two types of Jordan automorphisms of $T_n(R)$. In this section, 2 is a unit in R .

(A) Inner automorphisms

Let $T \in T_n^*(R)$. The mapping $\theta_T : A \mapsto TAT^{-1}$ is called an inner automorphism, which is an R -algebra automorphism. It is also a Jordan automorphism of $T_n(R)$.

(B) Graph automorphisms

Let $\varepsilon \in R$ be an idempotent element, $E_0 = \sum_{i=1}^n E_{i,n+1-i} \in T_n(R)$. The mapping $w_\varepsilon : A \mapsto \varepsilon A + (1 - \varepsilon)E_0 A^t E_0$ (t denotes the transpose of matrix) is a Jordan automorphism of $T_n(R)$. We call w_ε a graph automorphism. In general, graph automorphism is not an R -algebra automorphism.

Lemma 3.1 (the main theorem of [4]) *Let ψ be a Jordan automorphism of $T_n(R)$. Then there are inner automorphism θ and graph automorphism w_ε , respectively, of $T_n(R)$ such that $\psi = \theta \cdot w_\varepsilon$.*

Lemma 3.2 *Let \mathcal{A} be an algebra over R and φ a local Jordan automorphism of \mathcal{A} . Then*

- (1) $\varphi(E) = \varphi(E)^2$ for every idempotent E in \mathcal{A} ;
- (2) $\varphi(A)^2 = 0$ for every square-zero element A in \mathcal{A} ;
- (3) $\varphi(I) = I$, where I is the identity of \mathcal{A} .

Proof (1) Let E be an idempotent element in \mathcal{A} . Then

$$\varphi(E) = \varphi_E(E) = \varphi_E(E^2) = \varphi_E(E)^2 = \varphi(E)^2,$$

where φ_E is a Jordan automorphism that agrees with φ at E .

(2) Let $A \in \mathcal{A}$ such that $A^2 = 0$ and φ a local Jordan automorphism of \mathcal{A} . Then there is a Jordan automorphism φ_A , corresponding to A , such that

$$\varphi(A)^2 = \varphi_A(A)^2 = \varphi_A(A^2) = \varphi_A(0) = 0.$$

(3) Suppose $\varphi(I) = \varphi_I(I) = A$, $\varphi_I(B) = I$, where φ_I is a Jordan automorphism depending on I and $A, B \in \mathcal{A}$. Then

$$2\varphi_I(B) = \varphi_I(IB + BI) = \varphi_I(I)\varphi_I(B) + \varphi_I(B)\varphi_I(I) = 2A.$$

So $I = \varphi_I(B) = A$. That is to say $\varphi(I) = I$. \square

We will prove our main result in this section via the following lemmas.

Lemma 3.3 *Let φ be a local Jordan automorphism of $T_n(R)$ satisfying $\varphi(E_{11}) = E_{11}$. Then for any $A \in \mathfrak{S}_1$, $\varphi(A) \in \mathfrak{S}_1$.*

Proof For $2 \leq i \leq n$, since $(E_{11} + E_{ii})^2 = E_{11} + E_{ii}$, by Lemma 3.2, we have $[\varphi(E_{11} + E_{ii})]^2 = \varphi(E_{11} + E_{ii})$, which follows that $\varphi(E_{ii})E_{11} + E_{11}\varphi(E_{ii}) = 0$. That is to say $\varphi(E_{ii}) \in \mathfrak{S}_1$.

For $2 \leq i < j \leq n$, the idempotence of $E_{11} + E_{ii} + E_{ij}$ shows that the image of it under φ is also idempotent, then we have $\varphi(E_{ii} + E_{ij}) \in \mathfrak{S}_1$. So $\varphi(E_{ij}) \in \mathfrak{S}_1$. Since the set $\{E_{ij} \mid 2 \leq i \leq j \leq n\}$ is a basis of \mathfrak{S}_1 , we have for any $A \in \mathfrak{S}_1$, $\varphi(A) \in \mathfrak{S}_1$. \square

Lemma 3.4 *Let φ be a local Jordan automorphism of $T_n(R)$. If for any $A \in \mathfrak{S}_1$, $\varphi(A) \in \mathfrak{S}_1$ and $\varphi(E_{11}) = E_{11}$, then there exists an inner automorphism θ_X such that $\theta_X^{-1} \cdot \varphi(E_{11}) = E_{11}$ and $\theta_X^{-1} \cdot \varphi(E_{nn}) = E_{nn}$.*

Proof Since φ is a local Jordan automorphism, we have $\varphi(E_{nn}) = \varphi_{E_{nn}}(E_{nn})$, where $\varphi_{E_{nn}}$ is a Jordan automorphism depending on E_{nn} . By Lemma 3.1, we know there exist an idempotent

$\varepsilon \in R$ and $U = (u_{ij}) \in T_n^*(R)$ such that $\varphi(E_{nn}) = \varphi_{E_{nn}}(E_{nn}) = \theta_U \cdot w_\varepsilon(E_{nn}) \in \mathfrak{S}_1$, which follows $\varepsilon = 1, u_{1n} = 0$. Let

$$X = I + \sum_{i=2}^{n-1} u_{in} E_{in} + (u_{nn} - 1) E_{nn}.$$

Then $\theta_X(E_{11}) = E_{11} = \varphi(E_{11})$, $\theta_X(E_{nn}) = \varphi(E_{nn})$. So $\theta_X^{-1} \cdot \varphi(E_{11}) = E_{11}$ and $\theta_X^{-1} \cdot \varphi(E_{nn}) = E_{nn}$. \square

Lemma 3.5 *Let φ be a local Jordan automorphism of $T_n(R)$ satisfying $\varphi(E_{11}) = E_{11}$ and $\varphi(E_{nn}) = E_{nn}$. Then for any $A \in \mathfrak{S}_2$, $\varphi(A) \in \mathfrak{S}_2$.*

Proof For $2 \leq i \leq j \leq n-1$, by Lemma 3.3, we have that $\varphi(E_{ij}) \in \mathfrak{S}_1$.

For $2 \leq i \leq n-1$, by operating φ on the two sides of $(E_{ii} + E_{nn})^2 = E_{ii} + E_{nn}$, we get $\varphi(E_{ii})E_{nn} + E_{nn}\varphi(E_{ii}) = 0$, which means that $\varphi(E_{ii}) \in \mathfrak{S}_2$.

For $2 \leq i < j \leq n-1$, by applying φ on the two sides of $(E_{ii} + E_{ij} + E_{nn})^2 = E_{ii} + E_{ij} + E_{nn}$, we have $\varphi(E_{ii} + E_{ij})E_{nn} + E_{nn}\varphi(E_{ii} + E_{ij}) = 0$. This implies that $\varphi(E_{ii} + E_{ij}) \in \mathfrak{S}_2$, which follows $\varphi(E_{ij}) \in \mathfrak{S}_2$. Since the set $\{E_{ij} \mid 2 \leq i \leq j \leq n-1\}$ is a basis of \mathfrak{S}_2 , we have for any $A \in \mathfrak{S}_2$, $\varphi(A) \in \mathfrak{S}_2$. \square

Lemma 3.6 *Let φ be a local Jordan automorphism of $T_n(R)$. If for any $A \in \mathfrak{S}_2$, $\varphi(A) \in \mathfrak{S}_2$, then φ restricted to \mathfrak{S}_2 is a local Jordan automorphism of \mathfrak{S}_2 .*

Proof Since for any $A \in \mathfrak{S}_2$, we have $\varphi(A) \in \mathfrak{S}_2$, φ restricted to \mathfrak{S}_2 is a linear mapping of \mathfrak{S}_2 . For any $A \in \mathfrak{S}_2$, by the definition of φ , we know there exists a Jordan automorphism φ_A of $T_n(R)$ corresponding to A such that $\varphi(A) = \varphi_A(A)$. By Lemma 3.1, we know that there exist an idempotent $\varepsilon \in R$ and $T \in T_n^*(R)$ such that

$$\varphi(A) = \varphi_A(A) = \theta_T \cdot w_\varepsilon(A).$$

Suppose $T = (t_{ij})_{n \times n}$ and set $T_1 = (d_{ij})_{n \times n}$, where $d_{11} = t_{11}, d_{nn} = t_{nn}, d_{ij} = t_{ij}$ for $2 \leq i \leq j \leq n-1$ and $d_{1n} = d_{1j} = d_{jn} = 0$ for $2 \leq j \leq n-1$. By calculating, we have

$$\varphi(A) = \theta_T \cdot w_\varepsilon(A) = \theta_{T_1} \cdot w_\varepsilon(A).$$

For any $B \in \mathfrak{S}_2$, we have $\theta_{T_1} \cdot w_\varepsilon(B) \in \mathfrak{S}_2$. That is to say $\theta_{T_1} \cdot w_\varepsilon$ restricted to \mathfrak{S}_2 is a Jordan automorphism of \mathfrak{S}_2 . By the definition of local Jordan automorphism, we know that φ restricted to \mathfrak{S}_2 is a local Jordan automorphism of \mathfrak{S}_2 . \square

Lemma 3.7 *Let φ be a local Jordan automorphism of $T_n(R)$. If for any $A \in \mathfrak{S}_2$, we have $\varphi(A) = A$, $\varphi(E_{11}) = E_{11}$, and $\varphi(E_{nn}) = E_{nn}$, then there exists an inner automorphism θ_Z such that $\theta_Z \cdot \varphi$ is the identity mapping.*

Proof Since $E_{11} + E_{12}$ and $E_{22} + E_{12}$ are idempotents, by Lemma 3.2, so are $\varphi(E_{11} + E_{12}) = E_{11} + \varphi(E_{12})$ and $\varphi(E_{22} + E_{12}) = E_{22} + \varphi(E_{12})$. It follows that $\varphi(E_{12}) = aE_{12}$ for some $a \in R$. Similarly, we can get $\varphi(E_{n-1,n}) = bE_{n-1,n}$ for some $b \in R$. Since φ is a local Jordan

automorphism, we have $a, b \in R^*$. Let

$$Z = \begin{pmatrix} a^{-1} & & \\ & I_{n-2} & \\ & & b \end{pmatrix}.$$

Then $\theta_Z \cdot \varphi(E_{11}) = E_{11}$, $\theta_Z \cdot \varphi(E_{12}) = E_{12}$, $\theta_Z \cdot \varphi(E_{n-1,n}) = E_{n-1,n}$, $\theta_Z \cdot \varphi(E_{nn}) = E_{nn}$, $\theta_Z \cdot \varphi(A) = A$ for any $A \in \mathfrak{S}_2$. Denote $\theta_Z \cdot \varphi$ by φ_1 .

For $3 \leq i \leq n$, by applying φ_1 on the two sides of $(E_{11} + E_{1i})^2 = E_{11} + E_{1i}$ and $(E_{1i} + E_{ii})^2 = E_{1i} + E_{ii}$, we have $\varphi_1(E_{1i}) = \varphi_1(E_{1i})E_{11} + E_{11}\varphi_1(E_{1i})$ and $\varphi_1(E_{1i}) = E_{ii}\varphi_1(E_{1i}) + \varphi_1(E_{1i})E_{ii}$. So we may assume that $\varphi_1(E_{1i}) = a_{1i}E_{1i}$ for certain $a_{1i} \in R$. Consider the action of φ on $E_{22} + E_{12} + E_{2i} + E_{1i}$. Since it is an idempotent element, so is $\phi(E_{22} + E_{12} + E_{2i} + E_{1i})$. It follows that $a_{1i} = 1$, that is, $\varphi_1(E_{1i}) = E_{1i}$ for $3 \leq i \leq n$.

For $2 \leq k \leq n-2$, similar to the above, by applying φ_1 on the two sides of $(E_{kn} + E_{nn})^2 = E_{kn} + E_{nn}$, and $(E_{kk} + E_{kn})^2 = E_{kk} + E_{kn}$, we get $\varphi_1(E_{kn}) = a_{kn}E_{kn}$ for some $a_{kn} \in R$. The idempotence of $E_{k,n-1} + E_{n,n-1} + E_{kn} + E_{nn}$ shows that the image of it under φ is also idempotent. That is

$$\begin{aligned} E_{k,n-1} + E_{n,n-1} + a_{kn}E_{kn} + E_{nn} &= (E_{k,n-1} + E_{n,n-1} + a_{kn}E_{kn} + E_{nn})^2 \\ &= a_{kn}E_{k,n-1} + E_{n,n-1} + a_{kn}E_{kn} + E_{nn}. \end{aligned}$$

So $a_{kn} = 1$, which implies that $\varphi_1(E_{kn}) = E_{kn}$ for $2 \leq k \leq n-2$. So for any $A \in T_n(R)$, $\varphi_1(A) = A$. That is to say $\theta_Z \cdot \varphi$ is the identity mapping. \square

Now we state the main result in this section.

Theorem 3.1 *Let $T_n(R)$ be the R -algebra of all upper triangular n by n matrices over R . Then every local Jordan automorphism φ of $T_n(R)$ is a Jordan automorphism.*

Proof We will prove this theorem by induction on n . The result is trivial with $n = 1$, since the only Jordan automorphism of R itself is the identity mapping.

When $n = 2$, since φ is a local Jordan automorphism, there exists a Jordan automorphism $\varphi_{E_{11}}$, depending on E_{11} , such that $\varphi(E_{11}) = \varphi_{E_{11}}(E_{11})$. So $\varphi_{E_{11}}^{-1} \cdot \varphi(E_{11}) = E_{11}$. Denote $\varphi_{E_{11}}^{-1} \cdot \varphi$ by φ_1 . From $(E_{11} + E_{12})^2 = E_{11} + E_{12}$ and $(E_{11} + E_{22})^2 = E_{11} + E_{22}$, by Lemma 3.2, we get $\varphi_1(E_{12}) = bE_{12}$ and $\varphi_1(E_{22}) = cE_{22}$ for some $b, c \in R$. Clearly, φ_1 is also a local Jordan automorphism, so $b, c \in R^*$. Again by Lemma 3.2, we have $c^2 = c$, thus $c = 1$. Let $X = b^{-1}E_{11} + E_{22}$. Then $\theta_X \cdot \varphi_1(E_{ij}) = E_{ij}$ for $1 \leq i \leq j \leq 2$. So $\varphi = \varphi_{E_{11}} \cdot \theta_X^{-1}$. That is to say φ is a Jordan automorphism of $T_2(R)$.

When $n = 3$, similar to the case $n = 2$, we have $\varphi_{E_{11}}^{-1} \cdot \varphi(E_{11}) = E_{11}$. Denote $\varphi_{E_{11}}^{-1} \cdot \varphi$ by φ_1 . By Lemmas 3.3 and 3.4, we know there exists some $X \in T_3^*(R)$ such that $\theta_X^{-1} \cdot \varphi_1(E_{11}) = E_{11}$, and $\theta_X^{-1} \cdot \varphi_1(E_{33}) = E_{33}$. Denote $\theta_X^{-1} \cdot \varphi_1$ by φ_2 . By Lemma 3.2, we have $\varphi_2(E_{22}) = E_{22}$. By applying φ_2 on the two sides of $(E_{11} + E_{12})^2 = E_{11} + E_{12}$ and $(E_{12} + E_{22})^2 = E_{12} + E_{22}$, we have $\varphi_2(E_{12}) = aE_{12}$ with some $a \in R$. Similarly, we can get $\varphi_2(E_{23}) = bE_{23}$ and $\varphi_2(E_{13}) = cE_{13}$ for certain $b, c \in R$. Since φ_2 is also a local Jordan automorphism, we have $a, b \in R^*$. Let

$Y = a^{-1}E_{11} + E_{22} + bE_{33}$. Then $\theta_Y \cdot \varphi_2(E_{ii}) = E_{ii}$ for $1 \leq i \leq 3$ and $\theta_Y \cdot \varphi_2(E_{12}) = E_{12}$, $\theta_Y \cdot \varphi_2(E_{23}) = E_{23}$, $\theta_Y \cdot \varphi_2(E_{13}) = a^{-1}b^{-1}cE_{13}$. By operating $\theta_Y \cdot \varphi_2$ on the two sides of $(E_{12} + E_{22} + E_{23} + E_{13})^2 = E_{12} + E_{22} + E_{23} + E_{13}$, we have $a^{-1}b^{-1}c = 1$. So $\varphi = \varphi_{E_{11}} \cdot \theta_X \cdot \theta_Y^{-1}$, which implies φ is a Jordan automorphism.

By induction we assume that the theorem holds for matrices of size less than n ($n \geq 4$). Let φ be a local Jordan automorphism of $T_n(R)$. Then $\varphi(E_{11}) = \varphi_{E_{11}}(E_{11})$, where $\varphi_{E_{11}}$ is a Jordan automorphism depending on E_{11} . So $\varphi_{E_{11}}^{-1} \cdot \varphi(E_{11}) = E_{11}$. Obviously, $\varphi_{E_{11}}^{-1} \cdot \varphi$ is also a local Jordan automorphism of $T_n(R)$. Denote $\varphi_{E_{11}}^{-1} \cdot \varphi$ by φ_1 . By Lemmas 3.3 and 3.4, we know that there exists some $X \in T_n^*(R)$ such that $\theta_X^{-1} \cdot \varphi_1(E_{11}) = E_{11}$ and $\theta_X^{-1} \cdot \varphi_1(E_{nn}) = E_{nn}$. Denote $\theta_X^{-1} \cdot \varphi_1$ by φ_2 . By Lemmas 3.5 and 3.6, we know that φ_2 restricted to \mathfrak{S}_2 is a local Jordan automorphism of \mathfrak{S}_2 . Inductively, φ_2 restricted to \mathfrak{S}_2 is a Jordan automorphism of \mathfrak{S}_2 ; say there exist $S_0 \in \mathfrak{S}_2$ and $\varepsilon = \varepsilon^2 \in R$ such that $\varphi_2(A) = \theta_{E_{11}+S_0+E_{nn}} \cdot w_\varepsilon(A)$ for all $A \in \mathfrak{S}_2$. That is to say $w_\varepsilon^{-1} \cdot \theta_{E_{11}+S_0+E_{nn}}^{-1} \cdot \varphi_2(A) = A$ for all $A \in \mathfrak{S}_2$. Denote $w_\varepsilon^{-1} \cdot \theta_{E_{11}+S_0+E_{nn}}^{-1} \cdot \varphi_2$ by φ_3 . Considering the action of φ_3 on $E_{11} + E_{22}$, we have

$$\varphi_3(E_{11} + E_{22}) = \varphi_3(E_{11}) + E_{22} = \varepsilon E_{11} + (1 - \varepsilon)E_{nn} + E_{22}.$$

But on the other hand, by the definition of local automorphisms, the action of φ_3 on $E_{11} + E_{22}$ agrees with that of a Jordan automorphism on it. So there exist an idempotent $\eta \in R$ and $U \in T_n^*(R)$ such that

$$\varphi_3(E_{11} + E_{22}) = \theta_U \cdot w_\eta(E_{11} + E_{22}) \equiv \eta E_{11} + \eta E_{22} + (1 - \eta)E_{nn} + (1 - \eta)E_{n-1, n-1} \bmod \mathfrak{n},$$

which forces $\varepsilon = 1$. So $\varphi_3(E_{11}) = E_{11}$. Similarly, we can get $\varphi_3(E_{nn}) = E_{nn}$. By Lemma 3.7, we know there exists some $Z \in T_n^*(R)$ such that $\theta_Z \cdot \varphi_3$ is the identity mapping. That is to say φ is a Jordan automorphism. \square

4. The applications

We now use the result on the local Jordan derivations and local Jordan automorphisms of $T_n(R)$ to discuss the local derivations and local automorphisms of $T_n(R)$.

Theorem 4.1 *Let R be a commutative ring with identity 1 and unit 2. Then every local derivation η of $T_n(R)$ is inner.*

Proof Let η be a local derivation of $T_n(R)$. Since each derivation is a Jordan derivation, we know η is a local Jordan derivation. By Theorem 2.1, we know that η is an inner derivation of $T_n(R)$. \square

Theorem 4.2 *Let R be a commutative ring with identity 1 and unit 2. Then every local automorphism ϕ of $T_n(R)$ is inner.*

Proof Let ϕ be a local automorphism of $T_n(R)$. It is clear that every automorphism is a Jordan automorphism, so ϕ is a local Jordan automorphism of $T_n(R)$. By Theorem 3.1 and Lemma 3.1, there exist an inner automorphism θ and a graph automorphism w_ε such that $\phi = \theta \cdot w_\varepsilon$.

Considering the action of ϕ on E_{11} , we have

$$\phi(E_{11}) = \theta \cdot w_\varepsilon(E_{11}) \equiv \varepsilon E_{11} + (1 - \varepsilon)E_{nn} \pmod{\mathbf{n}}.$$

On the other hand, since ϕ is a local automorphism, there is an automorphism $\phi_{E_{11}}$, depending on E_{11} , such that $\phi(E_{11}) = \phi_{E_{11}}(E_{11})$. By [9], we know there exists some $T \in T_n^*(R)$ such that

$$\phi(E_{11}) = \phi_{E_{11}}(E_{11}) = \theta_T(E_{11}) \equiv E_{11} \pmod{\mathbf{n}},$$

so $\varepsilon = 1$. That is to say ϕ is an inner automorphism. \square

References

- [1] LARSON D R. *Reflexivity, algebraic reflexivity, and linear interpolation* [J]. Amer. J. Math., 1988, **110**(2): 283–299.
- [2] LARSON D.R, SOUROUR A.R. *Local derivations and local automorphisms of $B(H)$* [J]. Proc. Sympos. Pure Math., 1990, **51**(2): 187–194.
- [3] KADISON R V. *Local derivations* [J]. J. Algebra, 1990, **130**(2): 494–509.
- [4] JOHNSON B E. *Local derivations on C^* -algebras are derivations* [J]. Trans. Amer. Math. Soc., 2001, **353**(1): 313–325.
- [5] CRIST R L. *Local derivations on operator algebras* [J]. J. Funct. Anal., 1996, **135**(1): 76–92.
- [6] ZHANG Jianhua, JI Guoxing, CAO Huaixin. *Local derivations of nest subalgebras of von Neumann algebras* [J]. Linear Algebra Appl., 2004, **392**:61–69.
- [7] ZHANG Jianhua, YANG Aili, PAN Fangfang. *Local automorphisms of nest subalgebras of factor Von Neumann algebras* [J]. Linear Algebra Appl., 2005, **402**: 335–344.
- [8] BREŠAR M, ŠEMRL P. *Mappings which preserve idempotents, local automorphisms, and local derivations* [J]. Canad. J. Math., 1993, **45**(3): 483–496.
- [9] CRIST R. *Local automorphisms* [J]. Pro. Amer. Math. Soc., 2000, **128**(5): 1409–1414.
- [10] BENKOVIČ D. *Jordan derivations and antiderivations on triangular matrices* [J]. Linear Algebra Appl., 2005, **397**: 235–244.
- [11] BEIDAR K I, BREŠAR M, CHEBOTAR M.A. *Jordan isomorphisms of triangular matrix algebra over a connected commutative ring* [J]. Linear Algebra Appl., 2000, **312**(1-3): 197–201.
- [12] CAO Youan. *Automorphisms of certain Lie algebras of upper triangular matrices over a commutative ring* [J]. J. Algebra, 1997, **189**(2): 506–513.
- [13] KEZLAN T P. *A note on Algebra automorphisms of triangular matrices over commutative rings* [J]. Linear Algebra Appl., 1990, **135**:181–184.
- [14] TANG Xiaomin, CAO Chongguang, ZHANG Xian. *Modular automorphisms preserving idempotence and Jordan isomorphisms of triangular matrices over commutative rings* [J]. Linear Algebra Appl., 2001, **338**(1-3): 145–152.
- [15] ZHANG Jianhua, YU Weiyan. *Jordan derivations of triangular algebras* [J]. Linear Algebra Appl., 2006, **419**(1): 251–255.