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Uniqueness Theorem of Algebroidal Functions in an Angular Domain

Hui Fang LIU[1,2,*], **Dao Chun SUN**[2]

1. *Institute of Mathematics and Informatics, Jiangxi Normal University, Jiangxi* 330027*, P. R. China;*
2. *School of Mathematics, South China Normal University, Guangdong* 510631*, P. R. China*

Abstract Let $W(z)$ and $M(z)$ be v-valued and k-valued algebroidal functions respectively, $\triangle(\theta)$ be a b-cluster line of order ∞ (or $\rho(r)$) of $W(z)$ (or $M(z)$). It is shown that $W(z)\equiv M(z)$ provided $\overline{E}(a_j, W(z)) = \overline{E}(a_j, M(z))$ $(j = 1,\ldots,2v+2k+1)$ holds in the angular domain $\Omega(\theta-\delta,\theta+\delta)$, where b, a_j $(j=1,\ldots,2v+2k+1)$ are complex constants. The same results are obtained for the case that $\triangle(\theta)$ is a Borel direction of order ∞ (or $\rho(r)$) of $W(z)$ (or $M(z)$).

Keywords algebroidal function; order; uniqueness.

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1. Introduction

Suppose that $A_v(z),\ldots,A_0(z)$ are analytic functions without common zeros in the complex plane and the indecomposable equation

$$\psi(z,W) := A_v(z)W^v + A_{v-1}(z)W^{v-1} + \cdots + A_1(z)W + A_0(z) = 0 \tag{1}$$

defines a v-valued algebroidal function $W(z)$ in the complex plane(if $A_v(z)\equiv 1$, then $W(z)$ is called a v-valued integral algebroidal function), where $A_0(z)\not\equiv 0$, otherwise $W(z)$ is a reducible algebroidal function; where $A_v(z)\not\equiv 0$, otherwise $W(z)$ is $v-1$ valued or less. In particular, $W(z)$ is exactly a meromorphic function when $v=1$.

In this paper, we adopt the standard notations and definitions as explained in [1]. The notations $\Delta(\theta)$ and $\Omega(\alpha,\beta)$ denote a ray $\arg z=\theta$ and a set $\{z|\alpha<\arg z<\beta\}$ in the complex plane, respectively. For an arbitrary complex number a, $n(r,\Omega(\alpha,\beta),W(z)=a)$ denotes the number of roots of the equation $W(z)-a=0$ in the domain $\Omega(\alpha,\beta)\bigcap |z|<r$, multiple roots being counted multiply, and $\overline{E}(a,W(z))$ denotes the set of zeros of $W(z)-a$ (namely the set of points z satisfying the equation $\psi(z,a)=0$), with every zero counted only once.

We also need the following definitions.

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* Corresponding author
E-mail address: liuhuifang73@sina.com (H. F. LIU)

Definition 1 *Suppose that $W(z)$ is a v-valued algebroidal function defined by (1), a is an arbitrary complex number, and $\Delta(\theta)$ is a ray in the complex plane. If*

$$\overline{\lim_{r\to\infty}}\frac{\log n(r,\Omega(\theta-\varepsilon,\theta+\varepsilon),W=a)}{\log r}=\infty$$

holds for any given $\varepsilon>0$, then we call $\Delta(\theta)$ a a-cluster line of order ∞ of $W(z)$.

Definition 2 *Suppose that $W(z)$ is a v-valued algebroidal function defined by (1), $\Delta(\theta)$ is a ray in the complex plane. If for any complex number a (at most $2v$ number of exceptional values), $\Delta(\theta)$ is always an a-cluster line of order ∞ of $W(z)$, then we call $\Delta(\theta)$ a Borel direction of order ∞ of $W(z)$.*

Suppose that $W(z)$ is a v-valued algebroidal function of infinite order defined by (1). According to the result of Hiong in [2], there exists a function $U(r)=r^{\rho(r)}$ satisfying the following conditions:

(i) $\rho(r)$ is continuous and nondecreasing for $r\geq r_0$ $(r_0>0)$, and tends to $+\infty$ as $r\to+\infty$.

(ii) $\lim_{r\to\infty}\frac{\log U(R)}{\log U(r)}=1$, $R=r+\frac{r}{\log U(r)}$.

(iii) $\overline{\lim\limits_{r\to\infty}}\frac{\log T(r,\ W)}{\log U(r)}=1$.

We say that the function $U(r)$ is the type function of $W(z)$.

Definition 3 *Suppose that $W(z)$ is a v-valued algebroidal function defined by (1) and $U(r)=r^{\rho(r)}$ is its type function. Let a be an arbitrary complex number and $\Delta(\theta)$ be a ray in the complex plane. If*

$$\overline{\lim_{r\to\infty}}\frac{\log n(r,\Omega(\theta-\varepsilon,\theta+\varepsilon),W=a)}{\log U(r)}=1$$

holds for any given $\varepsilon>0$, then we call $\Delta(\theta)$ an a-cluster line of order $\rho(r)$ of $W(z)$.

Definition 4 *Suppose that $W(z)$ is a v-valued algebroidal function defined by (1), $\Delta(\theta)$ is a ray in the complex plane. If for any complex number a (at most $2v$ number of exceptional values), $\Delta(\theta)$ is always an a-cluster line of order $\rho(r)$ of $W(z)$, then we call $\Delta(\theta)$ a Borel direction of order $\rho(r)$ of $W(z)$.*

The uniqueness theory of algebroidal functions in the complex plane is an interesting problem. Many authors, such as Valiron [3], He [4, 5] and Yi [6], have ever done a lot of works. But owing to the complexity of its branch points, the results in uniqueness theory are fewer than those in meromorphic functions, as found in [7]. The uniqueness theory of algebroidal functions was firstly investigated by Valiron, who proved the following result in [3].

Theorem A *Let $W(z)$ and $M(z)$ be two v-valued algebroidal functions. If there are $4v+1$ number of $a_j\in C$ such that $E(a_j,W(z))=E(a_j,M(z))$ $(j=1,\ldots,4v+1)$, then $W(z)\equiv M(z)$.*

In [4], He obtained an improvement as follows.

Theorem B *Let $W(z)$ and $M(z)$ be v-valued and k-valued algebroidal functions, respectively, and let $k\leq v$. If there are $4v+1$ number of $a_j\in C$ such that $\overline{E}(a_j,W(z))=\overline{E}(a_j,M(z))$ $(j=1,\ldots,4v+1)$, then $W(z)\equiv M(z)$.*

It is natural to ask whether the conditions of Theorem B can be weakened. In this paper, we obtain the following results which answer the question.

Theorem 1 *Let $W(z)$ and $M(z)$ be v-valued and k-valued irreducible algebroidal functions in the complex plane, respectively, $a_1, \ldots, a_{2v+2k+1}$ be distinct complex numbers. If $\triangle(\theta)$ is a b- cluster line of order ∞ of $W(z)$ (or $M(z)$), such that $\overline{E}(a_j, W(z)) = \overline{E}(a_j, M(z))$ $(j = 1, \ldots, 2v+2k+1)$ holds in an angular domain $\Omega(\theta - \delta, \theta + \delta)$, where b is an arbitrary complex number and δ is a sufficiently small positive number, then $W(z) \equiv M(z)$.*

From Theorem 1 we can get the following two corollaries.

Corollary 1 *Suppose that $W(z)$, $M(z)$ and $a_1, \ldots, a_{2v+2k+1}$ satisfy the hypothesis of Theorem 1. If $\triangle(\theta)$ is a Borel direction of order ∞ of $W(z)$ (or $M(z)$), such that $\overline{E}(a_j, W(z)) = \overline{E}(a_j, M(z))$ $(j = 1, \ldots, 2v+2k+1)$ holds in an angular domain $\Omega(\theta - \delta, \theta + \delta)$, where δ is a sufficiently small positive number, then $W(z) \equiv M(z)$.*

Corollary 2 *Let $W(z)$ and $M(z)$ be two v-valued irreducible algebroidal functions, $a_1, \ldots, a_{4v+1}$ be distinct complex numbers. If $\triangle(\theta)$ is a b-cluster line (or a Borel direction) of order ∞ of $W(z)$ (or $M(z)$), such that $\overline{E}(a_j, W(z)) = \overline{E}(a_j, M(z))$ $(j = 1, \ldots, 4v+1)$ holds in an angular domain $\Omega(\theta - \delta, \theta + \delta)$, where b is an arbitrary complex number and δ is a sufficiently small positive number, then $W(z) \equiv M(z)$.*

Theorem 2 *Suppose that $W(z)$, $M(z)$ and $a_1, \ldots, a_{2v+2k+1}$ satisfy the hypothesis of Theorem 1. If $\triangle(\theta)$ is a b-cluster line of order $\rho(r)$ of $W(z)$ (or $M(z)$), such that $\overline{E}(a_j, W(z)) = \overline{E}(a_j, M(z))$ $(j = 1, \ldots, 2v+2k+1)$ holds in an angular domain $\Omega(\theta - \delta, \theta + \delta)$, where b is an arbitrary complex number and δ is a sufficiently small positive number, then $W(z) \equiv M(z)$.*

From Theorem 2 we also can get the following two corollaries.

Corollary 3 *Suppose that $W(z)$, $M(z)$ and $a_1, \ldots, a_{2v+2k+1}$ satisfy the hypothesis of Theorem 1. If $\triangle(\theta)$ is a Borel direction of order $\rho(r)$ of $W(z)$ (or $M(z)$), such that $\overline{E}(a_j, W(z)) = \overline{E}(a_j, M(z))$ $(j = 1, \ldots, 2v+2k+1)$ holds in an angular domain $\Omega(\theta - \delta, \theta + \delta)$, where δ is a sufficiently small positive number, then $W(z) \equiv M(z)$.*

Corollary 4 *Let $W(z)$ and $M(z)$ be two v-valued irreducible algebroidal functions, $a_1, \ldots, a_{4v+1}$ be distinct complex numbers. If $\triangle(\theta)$ is a b-cluster line (or a Borel direction) of order $\rho(r)$ of $W(z)$ (or $M(z)$), such that $\overline{E}(a_j, W(z)) = \overline{E}(a_j, M(z))$ $(j = 1, \ldots, 4v+1)$ holds in an angular domain $\Omega(\theta - \delta, \theta + \delta)$, where b is an arbitrary complex number and δ is a sufficiently small positive number, then $W(z) \equiv M(z)$.*

2. Some Lemmas

Suppose $W(z)$ is a v-valued irreducible algebroidal function defined by (1). Its single valued domain is the connected Riemman surface $\widetilde{T_z}$. The point on the connected Riemman surface $\widetilde{T_z}$ is the regular function element $\widetilde{b} = \{\omega_{b,j}(z), B(b,r)\}$ $(\omega_{b,j}(z)$ denotes an analytic function in the disc $B(b,r)=\{z||z-b|<r\}$). There exists a path $\gamma\subset\widetilde{T_z}$ for any two regular function elements $(\omega_{b,j}(z),B(b,r))$ and $(\omega_{a,t}(z),B(a,r))$ to extend analytically to each other. We often write $W=W(z)=\{\omega_j(z),b\}_{j=1}^{v}$.

In [8] and [9], Sun and Gao have given the definitions of the addition, the negative element of two algebroidal functions, respectively.

Definition 5 *Let $W(z)=\{\omega_j(z),b\}_{j=1}^{v}$ and $M(z)=\{m_t(z),b\}_{t=1}^{k}$ be v-valued and k-valued algebroidal functions, respectively. The sum of $W(z)$ and $M(z)$ is defined as*

$$\begin{aligned}(W+M)(z)&=\{((\omega+m)_j(z),b)\}_{j=1}^{vk}\\&=\{(\omega_j(z)+m_t(z),b);j=1,\dots,v;t=1,\dots,k\}.\end{aligned}$$

The negative element of $W(z)$ is the algebroidal function $-W(z)=\{-\omega_j(z),b\}_{j=1}^{v}$ defined by the equation

$$\begin{aligned}\psi^{-}(z,W)&=A_v(z)(W+\omega_1(z))(W+\omega_2(z))\cdots(W+\omega_v(z))\\&=A_v(z)W^v-A_{v-1}(z)W^{v-1}+\cdots+(-1)^vA_0(z)=0.\end{aligned}$$

They also obtained the following result.

Lemma 1 *The sum $(W+M)(z)$ of the v-valued algebroidal function $W(z)$ and the k-valued algebroidal function $M(z)$ is a vk-valued algebroidal function. The negative element $-W(z)$ of the v-valued algebroidal function $W(z)$ is also a v-valued algebroidal function.*

From the above results, we can obtain the definition of the subtraction of two algebroidal functions as follows.

Definition 6 *Let $W(z)=\{\omega_j(z),b\}_{j=1}^{v}$ and $M(z)=\{m_t(z),b\}_{t=1}^{k}$ be v-valued and k-valued algebroidal functions, respectively. The difference subtracting $M(z)$ from $W(z)$ is defined as*

$$\begin{aligned}(W-M)(z)&=\{((\omega-m)_j(z),b)\}_{j=1}^{vk}\\&=\{(\omega_j(z)-m_t(z),b);j=1,\dots,v;t=1,\dots,k\}.\end{aligned}$$

We also can obtain the following result.

Lemma 2 *Let $W(z)$ and $M(z)$ be v-valued and k-valued algebroidal functions, respectively. The difference $(W-M)(z)$ is a vk-valued algebroidal function.*

Lemma 3 *Let $W(z)$ and $M(z)$ be v-valued and k-valued algebroidal functions, respectively. If 0 is not the pole of $W(z)$ and $M(z)$. Then*

$$T(r,W\pm M)\le T(r,W)+T(r,M)+\log 2.$$

Remark This Lemma is Lemma 1 in [8].

Lemma 4 *Let $W(z)$ and $M(z)$ be v-valued and k-valued algebroidal functions in the disc $|z|<1$, respectively. Suppose that $W(z)$ (or $M(z)$) is of infinite order, and $a_1,\dots,a_{2v+2k+1}$ are distinct complex numbers. If $\overline{E}(a_j,W(z))=\overline{E}(a_j,M(z))$ $(j=1,\dots,2v+2k+1)$ holds in the* *disc* $|z|<1$, *then* $W(z)\equiv M(z)$.

Proof Firstly, we may assume that a_j $(j=1,\ldots,2v+2k+1)$ are all finite. By the second fundamental Theorem of algebroidal functions in the disc $|z|<1$, as found in [10], we get

$$(2k+1)T(r,W)<\sum_{j=1}^{2v+2k+1}\overline{N}(r,W=a_j)+S(r,W), \tag{2}$$

where $S(r,W)=O\{\log T(r,W)\}+O\{\log\frac{1}{1-r}\}$, possibly outside an exceptional set $E\subset[0,1)$ such that $\int_E\frac{dr}{1-r}<\infty$;

$$(2v+1)T(r,M)<\sum_{j=1}^{2v+2k+1}\overline{N}(r,M=a_j)+S(r,M), \tag{3}$$

where $S(r,M)=O\{\log T(r,M)\}+O\{\log\frac{1}{1-r}\}$, possibly outside an exceptional set $F\subset[0,1)$ such that $\int_F\frac{dr}{1-r}<\infty$.

Now assume, to the contrary, that $W(z)\not\equiv M(z)$. Then by the assumption that $\overline{E}(a_j,W(z))=\overline{E}(a_j,M(z))$, we have

$$\sum_{j=1}^{2v+2k+1}\overline{n}(r,W=a_j)=\sum_{j=1}^{2v+2k+1}\overline{n}(r,M=a_j)\le\overline{n}(r,\frac{1}{W(z)-M(z)}).$$

Hence we obtain

$$\sum_{j=1}^{2v+2k+1}\overline{N}(r,W=a_j)\le k\overline{N}(r,\frac{1}{W(z)-M(z)}),$$

$$\sum_{j=1}^{2v+2k+1}\overline{N}(r,M=a_j)\le v\overline{N}(r,\frac{1}{W(z)-M(z)}).$$

Then using the first fundamental Theorem of algebroidal functions in the disc $|z|<1$ and Lemma 3, we have

$$\sum_{j=1}^{2v+2k+1}\overline{N}(r,W=a_j)\le kT(r,\frac{1}{W(z)-M(z)})\le k\{T(r,W)+T(r,M)\}+O(1).$$

By the same reasoning, we have

$$\sum_{j=1}^{2v+2k+1}\overline{N}(r,M=a_j)\le v\{T(r,W)+T(r,M)\}+O(1).$$

Therefore we obtain

$$v\sum_{j=1}^{2v+2k+1}\overline{N}(r,W=a_j)+k\sum_{j=1}^{2v+2k+1}\overline{N}(r,M=a_j)\le 2vk\{T(r,W)+T(r,M)\}+O(1). \tag{4}$$

From (2), (3) and (4), we obtain

$$vT(r,W)+kT(r,M)\le S(r,W)+S(r,M). \tag{5}$$

Since $W(z)$ (or $M(z)$) is an algebroidal function of infinite order, (5) is a contradiction. So $W(z)\equiv M(z)$.

Now assume that one of the a_j $(j = 1, \dots, 2v+2k+1)$ is infinity. Without loss of generality, we assume $a_{2v+2k+1} = \infty$. Take any finite value c such that $c \neq a_j$ $(j = 1, \dots, 2v+2k)$. Set

$$F(z) = \frac{1}{W(z)-c}, \quad G(z) = \frac{1}{M(z)-c}.$$

And put

$$b_j = \frac{1}{a_j - c} \ (j = 1, \dots, 2v+2k), \quad b_{2v+2k+1} = 0.$$

Then we have $\overline{E}(b_j, F(z)) = \overline{E}(b_j, G(z))$. From the above proof, we have $F(z) \equiv G(z)$. Hence $W(z) \equiv M(z)$.

3. Proof of Theorems

Proof of Theorem 1 Without loss of generality, we may assume that $\triangle(\theta)$ is a b-cluster line of order ∞ of algebroidal function $W(z)$ and $\theta = 0$. Set

$$u(z) = \frac{z^{\frac{\pi}{2\delta}} - 1}{z^{\frac{\pi}{2\delta}} + 1}, \ z(u) = \Big(\frac{1+u}{1-u}\Big)^{\frac{2\delta}{\pi}}.$$

Then they map $\Omega(-\delta, \delta)$ conformally on $|u| < 1$. Setting $z = pe^{i\varphi}$ gives

$$\begin{aligned}
|u(z)| &= \Big|\frac{p^{\frac{\pi}{2\delta}} \cos \frac{\pi\varphi}{2\delta} - 1 + ip^{\frac{\pi}{2\delta}} \sin \frac{\pi\varphi}{2\delta}}{p^{\frac{\pi}{2\delta}} \cos \frac{\pi\varphi}{2\delta} + 1 + ip^{\frac{\pi}{2\delta}} \sin \frac{\pi\varphi}{2\delta}}\Big| \\
&= \sqrt{\frac{p^{\frac{\pi}{\delta}} + 1 - 2p^{\frac{\pi}{2\delta}} \cos \frac{\pi\varphi}{2\delta}}{p^{\frac{\pi}{\delta}} + 1 + 2p^{\frac{\pi}{2\delta}} \cos \frac{\pi\varphi}{2\delta}}} \\
&= 1 - 2p^{-\frac{\pi}{2\delta}}(1 + o(1)) \cos \frac{\pi\varphi}{2\delta}.
\end{aligned}$$

Set $0 < \delta' < \delta, 0 < \eta < 2\cos\frac{\pi\delta'}{2\delta}$. Then when r is sufficiently large, we have

$$y = 1 - \eta r^{-\frac{\pi}{2\delta}} > \max\{|u(pe^{i\varphi})| | 0 < p \leq r, |\varphi| \leq \delta'\}.$$

Since $\Delta(0)$ is a b-cluster line of order ∞ of algebroidal function $W(z)$, there is a sequence of points $\{r_n\}$ such that for any given large $L(> 0)$,

$$n(r_n, \Omega(-\delta', \delta'), W(z) = b) > {r_n}^L.$$

Set $y_n = 1 - \eta r_n^{-\frac{\pi}{2\delta}}$, then

$$n(y_n, W(z(u)) = b) > n(r_n, \Omega(-\delta', \delta'), W(z) = b) > {r_n}^L.$$

Set $r'_n = 2r_n, y'_n = 1 - \eta r_n'^{-\frac{\pi}{2\delta}}$, then

$$y'_n = 1 - \frac{1 - y_n}{2^{\frac{\pi}{2\delta}}} = y_n + (1 - y_n)(1 - 2^{-\frac{\pi}{2\delta}}).$$

Hence

$$\begin{aligned}
T(y'_n, W(z(u))) &> N(y'_n, W(z(u)) = b) - B \\
&> \frac{1}{v} \int_{y_n}^{y'_n} \frac{n(y, W(z(u)) = b)}{y} \mathrm{d}y - B \\
&> \frac{1}{v} n(y_n, W(z(u)) = b) \log \frac{y'_n}{y_n} - B
\end{aligned}$$

$$> A(1-y_n){r_n}^L$$
$$> A(\frac{1}{1-y_n'})^{\frac{2\delta}{\pi}L-1},$$

where A and B are positive numbers, which may be different at different place. So we have

$$\overline{\lim_{y_n'\to 1}}\frac{\log T(y_n', W(z(u))}{\log\frac{1}{1-y_n'}} \geq \frac{2\delta}{\pi}L-1.$$

Therefore $W(z(u))$ is an algebroidal function of infinite order in the disc $|u|<1$.

Since $\overline{E}(a_j, W(z)) = \overline{E}(a_j, M(z))$ holds in the angular domain $\Omega(-\delta,\delta)$, we get $\overline{E}(a_j, W(z(u)))$ $= \overline{E}(a_j, M(z(u)))$ in $|u|<1$. So by Lemma 4, we obtain that $W(z(u)) \equiv M(z(u))$ holds in $|u|<1$, which leads to $W(z)\equiv M(z)$ in $\Omega(-\delta,\delta)$. Then by the uniqueness Theorem of analytic functions, we get $W(z)\equiv M(z)$ in the complex plane.

Proof of Theorem 2 Similar to the proof of Theorem 1, we may assume that $\triangle(\theta)$ is a b-cluster line of order $\rho(r)$ of algebroidal function $W(z)$ and $\theta=0$. Set

$$u(z)=\frac{z^{\frac{\pi}{2\delta}}-1}{z^{\frac{\pi}{2\delta}}+1},\ z(u)=\Big(\frac{1+u}{1-u}\Big)^{\frac{2\delta}{\pi}}.$$

Since $\Delta(0)$ is a b-cluster line of order $\rho(r)$ of algebroidal function $W(z)$, there is a sequence of points $\{r_n\}$ such that for any given $\varepsilon>0$

$$n(r_n, \Omega(-\delta',\delta'), W(z)=b) > U^{1-\varepsilon}(r_n).$$

Set $y_n = 1-\eta r_n^{-\frac{\pi}{2\delta}}$, then

$$n(y_n, W(z(u))=b) > n(r_n, \Omega(-\delta',\delta'), W(z)=b) > U^{1-\varepsilon}(r_n).$$

Set $r_n'=2r_n, y_n'=1-\eta {r_n'}^{-\frac{\pi}{2\delta}}$, then

$$y_n' = 1-\frac{1-y_n}{2^{\frac{\pi}{2\delta}}} = y_n+(1-y_n)(1-2^{-\frac{\pi}{2\delta}}).$$

Hence

$$\begin{aligned}
T(y_n', W(z(u))) &> N(y_n', W(z(u))=b)-B\\
&> \frac{1}{v}\int_{y_n}^{y_n'}\frac{n(y,W(z(u))=b)}{y}\mathrm{d}y - B\\
&> \frac{1}{v}n(y_n, W(z(u))=b)\log\frac{y_n'}{y_n} - B > A(1-y_n)U^{1-\varepsilon}(r_n)\\
&= A\eta r_n^{-\frac{\pi}{2\delta}}U^{1-\varepsilon}(r_n),
\end{aligned}$$

where A and B are positive numbers, which may be different at different place. So we have

$$\overline{\lim_{y_n'\to 1}}\frac{\log T(y_n', W(z(u))}{\log\frac{1}{1-y_n'}} \geq \overline{\lim_{y_n'\to 1}}\frac{\log A\eta r_n^{-\frac{\pi}{2\delta}}U^{1-\varepsilon}(r_n)}{\log\frac{(2r_n)^{\frac{\pi}{2\delta}}}{\eta}} = \infty.$$

Therefore $W(z(u))$ is an algebroidal function of infinite order in the disc $|u|<1$. Then by the similar reasoning as in the proof of Theorem 1, we get $W(z)\equiv M(z)$ in the complex plane.

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