Journal of Mathematical Research & Exposition
May, 2010, Vol. 30, No. 3, pp. 527–535
DOI:10.3770/j.issn:1000-341X.2010.03.017
Http://jmre.dlut.edu.cn

Semi-Cover-Avoiding Properties and the Structure of Finite Groups

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Abstract In this paper, the so-called π-cover-avoiding properties of subgroups are defined and investigated. In terms of this property, we characterize the π-solvability of finite groups. Some other new results are also obtained based on the assumption that some subgroups have the semi cover-avoiding properties in a finite group.

Keywords maximal subgroup; semi π-cover-avoiding property; π-solvable group.

Document code A
MR(2000) Subject Classification 20D10; 20D20
Chinese Library Classification O152.1

1. Introduction

All groups considered in this article are finite.

The relationship between the properties of maximal subgroups of a group G and the structure of G has been studied extensively. In this aspect, there have been two important results. One states that a group G is nilpotent if and only if every maximal subgroup of G is normal in G. Another due to Huppert asserts that a group G is supersolvable if and only if every maximal subgroup of G has prime index in G. In order to characterize the solvability of groups analogously, many authors have investigated various properties of subgroups from different angles. For example, Wang [1] defined the c-normality of subgroups and proved that a group G is solvable if and only if every maximal subgroup of G is c-normal, whereas Guo and Shum [2] also described the solvable groups in terms of cover-avoidance properties. The cover-avoidance property has been introduced for a long time, and many authors such as Gaschütz [3], Gillam [4] and Tomkinson [5] have studied it. However, the c-normality and the cover-avoidance property do not cover each other (see Examples 4.1 and 4.2 in [6]). Recently, Fan et al. [6] introduced the semi-cover-avoiding properties which cover not only the c-normality but also the cover-avoidance

Received June 30, 2008; Accepted June 30, 2009
Supported by the National Natural Science Foundation of China (Grant No. 10771132), the Science and Technology Foundation of Shanxi Province for Colleges (Grant No. 20081022) and the Team Innovation Research Foundation of Shanxi University of Finance and Economics.
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property, and they proved that a group G is solvable if and only if either every maximal subgroup of G is semi-cover-avoiding in G, or every Sylow subgroup of G is semi-cover-avoiding in G. Furthermore, the theorems for a group to be supersolvable obtained by Srinivasan [7] and Buckley [8] are also generalized. The further research about the influence of semi-cover-avoiding properties on solvability and p-nilpotency has been made in [9] and [10].

In this paper, in order to characterize the π-solvability of groups, we first define and investigate the semi π-cover-avoiding properties, and then show that a group G is π-solvable if and only if every maximal subgroup of G is semi π-cover-avoiding in G, or every Sylow subgroup of G is semi π-cover-avoiding in G. More attention is paid to minimizing the number of the maximal subgroups needed to characterize the structure of G. Also we prove that a group with every 2-maximal subgroup being semi π-cover-avoiding is π-solvable. Finally, we study groups with the maximal or 2-maximal subgroups of its Sylow subgroups having semi-cover-avoiding properties. Some known results are generalized.

For a group G, $\pi(G)$ denotes the set of primes dividing the order of G; $M < \cdot\, G$ denotes M is a maximal subgroup of G. π always denotes a set of some primes. The other terminology and notations employed agree with standard usage.

2. Basic definitions and preliminaries

Let G be a group. If K and H are two normal subgroups of G with $K \leq H$, then we call H/K a normal factor of G. A normal factor H/K of G is called a π-normal factor if there exists a prime $p \in \pi$ such that $p||H/K|$. Clearly, each normal factor of G is either a π-normal factor or a π'-group. It should be noted that a π-normal factor of G is not necessarily a π-group. Sometimes, without causing confusion from the context, we also say "a π-factor" that actually means "a π-normal factor". When $\pi = \{p\}$, a π-normal factor of G is briefly called a p-normal factor. We say that a chief factor of G is a π- chief factor if it is a π-normal factor.

Let H/K be a normal factor of a group G and L a subgroup of G. We say that L covers H/K if $LH = LK$, while we say that L avoids H/K if $L \cap H = L \cap K$. It is easy to see that L covers H/K if and only if $LK/K \geq H/K$, while L avoids H/K if and only if $LK/K \cap H/K = 1$. Moreover, if L covers (or avoids) H/K, then L covers (or avoids) any normal factor H_1/K_1 of G with $K \leq K_1 < H_1 \leq H$.

Definition 2.1 *Let L be a subgroup of a group G.*

1) ([6, Definition 2.1]) If there is a chief series of G such that L covers or avoids every chief factor of this series, then L is called a semi-cover-avoiding subgroup of G, or say that L has the semi-cover-avoidance properties in G.

2) If there is a chief series of G such that L covers or avoids every π-chief factor of this series, then L is called a semi π-cover-avoiding subgroup of G, or say that L has the semi π-cover-avoidance properties in G.

Clearly, the semi π-cover-avoidance properties is a generalization of the semi-cover-avoidance properties. If $\pi = \pi(G)$, then a semi π-cover-avoiding subgroup of G is certainly a semi-cover-avoiding subgroup of G. By definition of cover-avoidance properties of subgroups, we can easily prove the following two lemmas:

Lemma 2.2 *Let L be a subgroup of a group G and N a normal subgroup of G with $N \le L$. Then L is semi π-cover-avoiding in G if and only if L/N is semi π-cover-avoiding in G/N.*

Lemma 2.3 *Let L and M be subgroups of a group G such that $L \le M$. If L is semi π-cover-avoiding in G, then L is also semi π-cover-avoiding in M.*

Lemma 2.4 *Suppose that L and M are subgroups of a group G with $L \le M$, and N is a normal subgroup of G such that $(|N|, |L|) = 1$.*

1) If L is a semi π-cover-avoiding subgroup of M, then L and LN both are semi π-cover-avoiding subgroups of MN. Especially, LN/N is a semi π-cover-avoiding subgroup of MN/N.

2) Conversely, if LN/N is a semi π-cover-avoiding subgroup of MN/N, then L and LN both are semi π-cover-avoiding subgroups of MN. Especially, L is a semi π-cover-avoiding subgroup of M.

Proof 1) Let $1 = M_0 < M_1 < \cdots < M_n = M$ be a chief series of M such that L covers or avoids every π-factor of this series. Consider the normal series

$$1 \le N = M_0N \le M_1N \le \cdots \le M_nN = MN \tag{1}$$

of MN. Suppose that $M_iN/M_{i-1}N$ is a π-factor, $1 \le i \le n$. Since $M_iN/M_{i-1}N$ is a homomorphic image of M_i/M_{i-1}, M_i/M_{i-1} is a π-factor, and so L covers or avoids M_i/M_{i-1}. If L covers M_i/M_{i-1}, then $LM_{i-1} \ge M_i$, and therefore $L(M_{i-1}N) \ge M_iN$, which means that both L and LN cover $M_iN/M_{i-1}N$. If L avoids M_i/M_{i-1}, then $L \cap M_i \le M_{i-1}$. We claim that both L and LN also avoid $M_iN/M_{i-1}N$. In fact, let $h \in L \cap M_iN$. Then $h = xy$ for some $x \in M_i$ and $y \in N$. Since $(|L|, |N|) = 1$, we may assume that h is a σ-element and y is a σ'-element, where σ is a set of some primes. Decompose x as $x = x_\sigma x_{\sigma'}$, where $x_\sigma, x_{\sigma'} \in \langle x \rangle$, and x_σ is a σ-element and $x_{\sigma'}$ is a σ'-element. Let $K = M_i \cap N$. Then $K \trianglelefteq M$. Clearly, $[M_i/K, (M \cap N)/K] = 1$. Since $y \in M \cap N$, we have $[x_{\sigma'}K, yK] = 1 = [x_\sigma K, yK]$, from which we deduce that $x_{\sigma'}yK$ is still a σ'-element, and therefore $o(hK) = o(x_\sigma K)o(x_{\sigma'}yK)$. Note that hK is a σ-element, so $o(x_{\sigma'}yK) = 1$ and $x_{\sigma'}y \in K \le M_i$. Thus $h = x_\sigma(x_{\sigma'}y) \in M_i$, and $h \in L \cap M_i \le M_{i-1}$. This shows that $L \cap M_iN \le M_{i-1}$ and $LN \cap M_iN = (L \cap M_iN)N \le M_{i-1}N$, and our claim is now proved. Moreover, since L avoids $N/1$ and LN covers $N/1$, L and LN both cover or avoid any π-factor of series (1). It follows that L and LN cover or avoid any π-factor of each chief series of MN obtained by refining the series (1). Hence L and LN are semi π-cover-avoiding subgroups of MN.

2) By Lemmas 2.2 and 2.3, we need only prove that L is a semi π-cover-avoiding subgroup of MN. Without loss of generality, we assume that $MN = G$. This means that LN/N is semi π-cover-avoiding in G/N. So G/N has a chief series $1 < G_1/N < G_2/N < \cdots < G_m/N = G/N$ such that LN/N covers or avoids every π-factor of this series. Since $(|L|, |N|) = 1$, it follows

that L covers or avoids every π-factor of the normal series

$$1 \leq N < G_1 < \cdots < G_m = G \tag{2}$$

of G. Then L covers or avoids every π-factor of each chief series of G obtained by refining the series (2). Hence L is semi π-cover-avoiding in G. The proof of the lemma is completed. □

Combining Lemmas 2.2 and 2.4, we have

Lemma 2.5 *Let L be a subgroup of a group G and N a normal subgroup of G such that either $L \geq N$ or $(|L|, |N|) = 1$. Then L is a semi π-cover-avoiding subgroup of G if and only if LN/N is a semi π-cover-avoiding subgroup of G/N. Especially, L is a semi-cover-avoiding subgroup of G if and only if LN/N is a semi-cover-avoiding subgroup of G/N.*

3. Main results

Let G be a group and p a prime. For convenience of statement, we give the following families of maximal subgroups of G:

$\mathcal{F}_n(G) = \{M \mid M <\cdot\, G \text{ and } M \text{ is non-nilpotent}\}$,

$\mathcal{F}_c(G) = \{M \mid M <\cdot\, G \text{ and } |G : M| \text{ is composite }\}$,

$\mathcal{F}^p(G) = \{M \mid M <\cdot\, G \text{ and } M \geq N_G(P) \text{ for a Sylow } p\text{-subgroup } P \text{ of } G\}$,

$\mathcal{F}^\pi(G) = \bigcup_{p\in\pi} \mathcal{F}^p(G)$,

$\mathcal{F}_p(G) = \{M \mid M <\cdot\, G \text{ and } M \geq P \text{ for some Sylow } p\text{-subgroup } P \text{ of } G\}$,

$\mathcal{F}_{pcn}(G) = \mathcal{F}_p(G) \cap \mathcal{F}_c(G) \cap \mathcal{F}_n(G)$.

Theorem 3.1 *A group G is π-solvable if and only if M is a semi π-cover-avoiding subgroup of G for any $M \in \mathcal{F}^\pi(G)$.*

Proof " $\Rightarrow$ ". Let M be a maximal subgroup of G and H/K a π-chief factor of G. Since any chief factor of G is either a π'-group or a solvable group, we have that H/K is an elementary abelian p-group for some prime $p \in \pi$. If $MK/K \not\geq H/K$, then $K \leq M$, and $M/K \cap H/K \trianglelefteq G/K$. It follows that $M/K \cap H/K = 1$. Hence M is a semi π-cover-avoiding subgroup of G.

" $\Leftarrow$ ". Assume the theorem is false and let G be a counterexample with the smallest order. Let N be a minimal normal subgroup of G. By Lemma 2.2, the hypotheses of the theorem are inherited by G/N, and therefore G/N is π-solvable. Since the class of π-solvable groups is a formation, N is a unique minimal normal subgroup of G. Also N is neither a π'-group nor a solvable group, and so $p||N|$ for some $p \in \pi$. Choose $P \in \mathrm{Syl}_p(N)$ and $P^* \in \mathrm{Syl}_p(G)$ such that $P = N \cap P^*$. Then $P < N$ and there exists $M <\cdot\, G$ such that $N_G(P^*) \leq N_G(P) \leq M$, and by Frattini argument, $G = NM$. It follows that $M \in \mathcal{F}^\pi(G)$ and M covers or avoids $N/1$. However, since $1 \neq P \leq N \cap M$, we have $M \geq N$ and hence $G = NM \leq M$, a contradiction. □

Corollary 3.2 *A group G is π-solvable if and only if M is a semi π-cover-avoiding subgroup of G for any $M <\cdot\, G$. Especially, G is solvable if and only if for any $M <\cdot\, G$, M is semi cover-avoiding in G.*

Recall that a group of odd order is solvable, hence a group is solvable if and only if it is 2-solvable. By Theorem 3.1, we have:

Corollary 3.3 *A group G is solvable if and only if M is a semi 2-cover-avoiding subgroup of G for any $M \in \mathcal{F}^2(G)$.*

Lemma 3.4 ([11, IX, Lemma 1.11]) *Let P be a Sylow p-subgroup of a group G, where p is the largest prime dividing the order of G. Then either $P \trianglelefteq G$ or the maximal subgroups of G containing $N_G(P)$ have composite indices in G.*

Theorem 3.5 *A group G is π-solvable if and only if M is a semi π-cover-avoiding subgroup of G for any $M \in \mathcal{F}_{pcn}(G)$, where p is the largest prime dividing the order of G.*

Proof We only prove the sufficiency. Assume the theorem is false and let G be a counterexample of minimal order. Choose $N \trianglelefteq G$ such that $\bar{G} = G/N$ is not π-solvable and N has the largest possible order. Then $\bar{G}$ has a unique minimal normal subgroup $\bar{U}$, and $\bar{U}$ is not π-solvable, especially $\bar{U}$ is not a π'-group. Let $\bar{Q} \in \mathrm{Syl}_q(\bar{U})$, where q is the largest prime dividing $|\bar{U}|$. Clearly, $2 < q \leq p$. The minimality of $\bar{U}$ implies that there exists a maximal subgroup $\bar{M} = M/N$ of $\bar{G}$ such that $N_{\bar{G}}(\bar{Q}) \leq N_{\bar{G}}(Z(J(\bar{Q}))) \leq \bar{M}$, where $J(\bar{Q})$ is the Thompson subgroup of $\bar{Q}$, and by Frattini argument, $\bar{G} = \bar{U}\bar{M}$. If $\bar{M}$ is nilpotent, then $N_{\bar{U}}(Z(J(\bar{Q})))$ is nilpotent, and by Glauberman-Thompson Theorem, $\bar{U}$ is q-nilpotent, which implies that $\bar{Q} = \bar{U}$, a contradiction. Thus $\bar{M}$ is non-nilpotent. Also since $|G : M| = |\bar{G} : \bar{M}| = |\bar{U} : \bar{U} \cap \bar{M}|$ and $N_{\bar{U}}(\bar{Q}) \leq \bar{M} \cap \bar{U}$, we have $q \nmid |\bar{U} : \bar{U} \cap \bar{M}|$, and by Lemma 3.4, $|G : M|$ is composite. Hence $r < q$ for any prime r dividing $|G : M|$ and thus $p \nmid |G : M|$. These arguments show that $M \in \mathcal{F}_{pcn}(G)$. By Lemma 2.2, $\bar{M}$ is semi π-cover-avoiding in $\bar{G}$, and so $\bar{M}$ covers or avoids $\bar{U}$. Since $1 \neq \bar{Q} \leq \bar{U} \cap \bar{M}$, it follows that $\bar{M} \geq \bar{U}$ and therefore $\bar{G} = \bar{U}\bar{M} \leq \bar{M}$, the final contradiction. □

Corollary 3.6 *A group G is solvable if and only if M is a semi-cover-avoiding subgroup of G for any $M \in \mathcal{F}_{pcn}(G)$, where p is the largest prime dividing the order of G.*

Theorem 3.7 *A group G is π-separable if and only if there exists a Hall π-subgroup H of G such that H is semi π-cover-avoiding in G.*

Proof " $\Rightarrow$ ". By [12, Theorem 6.8], G has a Hall π-subgroup H. Let L/K be any π-chief factor of G. Clearly, L/K is a π-group. Since HK/K is a Hall π-subgroup of G/K, it follows that $HK/K \geq L/K$, and therefore H is semi π-cover-avoiding in G.

" $\Leftarrow$ ". Conversely, let H be a Hall π-subgroup of G such that H is semi π-cover-avoiding in G. Then there exists a chief series

$$1 = G_0 < G_1 < \cdots < G_n = G \tag{3}$$

of G such that H covers or avoids every π-factor of this series. Let G_i/G_{i-1} be a π-factor. If H covers G_i/G_{i-1}, then G_i/G_{i-1} is a π-group. If H avoids G_i/G_{i-1}, then $HG_{i-1}/G_{i-1} \cap G_i/G_{i-1} = 1$. It follows that G_i/G_{i-1} is a π'-group, a contradiction. Hence every chief factor of series (3) is either a π-group or a π'-group, and so G is π-separable. $\square$

Theorem 3.8 *A group G is π-solvable if there exists a solvable Hall π-subgroup H of G such that H is semi π-cover-avoiding in G.*

Proof If G has a solvable Hall π-subgroup H such that H is semi π-cover-avoiding in G, then there exists a chief series

$$1 = G_0 < G_1 < \cdots < G_n = G \tag{4}$$

of G such that H covers or avoids every π-factor of this series. From the proof of Theorem 3.7, we see that each π-factor of series (4) is covered by H. It follows that every chief factor of series (4) is either a π'-group or a solvable group, and so G is π-solvable. $\square$

Theorem 3.9 *A group G is π-solvable if and only if every Sylow subgroup of G is semi π-cover-avoiding in G.*

Proof "$\Rightarrow$". Let G be a π-solvable group and P a Sylow subgroup of G. If L/K is a π-chief factor of G, then L/K is an elementary abelian q-group for some $q \in \pi$. Since PK/K is a Sylow subgroup of G/K, it follows that either $PK/K \geq L/K$ or $PK/K \cap L/K = 1$. Thus P is semi π-cover-avoiding in G.

"$\Leftarrow$". Conversely, if for any $p \in \pi$, there exists a Sylow p-subgroup P of G such that P is semi π-cover-avoiding in G, then clearly P is semi p-cover-avoiding in G. It follows from Theorem 3.8 that G is p-solvable. Consequently G is π-solvable. $\square$

Corollary 3.10 *A group G is solvable if and only if every Sylow subgroup of G is semi-cover-avoiding in G.*

A subgroup L of a group G is called a 2-maximal subgroup of G if there exists a maximal subgroup M of G such that L is maximal in M.

Theorem 3.11 *Let G be a group. If every 2-maximal subgroup of G is semi π-cover-avoiding in G, then G is π-solvable.*

Proof Assume that the theorem is not true and let G be a counterexample of minimal order. First, we claim that G is not simple. Otherwise, let M be a maximal subgroup of G. Since G is not solvable, $|G|$ is not a prime, and so $M \neq 1$. Let L be a maximal subgroup of M. Then L is semi π-cover-avoiding in G. Noticing that G is not a π'-group and $1 < G$ is the unique chief series of G, we have that $L \cap G = 1$. This implies that $L = 1$ and M is of prime order. By [13, IV, Theorem 7.4], G is solvable, a contradiction. Hence our claim holds.

Now let N be a minimal normal subgroup of G. Then $N < G$. If G/N is of prime order, then G/N is solvable. If G/N is not of prime order, then any maximal subgroup of G/N is non-trivial. It follows from Lemma 2.2 that the hypotheses of the theorem are inherited by G/N, and therefore G/N is π-solvable. Hence in any case, G/N is π-solvable. From which we obtain that N is not π-solvable, especially N is not a π'-group. Also it is easy to see that N is a unique minimal normal subgroup of G. Let P be a Sylow subgroup of N. Then $1 < P < N$ and by

Frattini argument we have $G = NM$, where M is a maximal subgroup of G containing $N_G(P)$. If $N \cap M = M$, then $M \leq N$ and $G = NM \leq N$, a contradiction. So $P \leq N \cap M < M$. Let L be a maximal subgroup of M such that $L \geq N \cap M$. Then L is semi π-cover-avoiding in G, especially L covers or avoids the π-factor $N/1$. Since $P \leq L \cap N$, we have $L \geq N$. Consequently $N \leq M$ and therefore $G = NM \leq M$, the final contradiction. The proof is completed. □

The remainder of this section is devoted to the investigation of groups with the maximal subgroups or 2-maximal subgroups of its Sylow subgroups having semi-cover-avoiding property.

Lemma 3.12 ([10, Theorem 3.2]) *Let p be the smallest prime dividing the order of a group G and $P \in \mathrm{Syl}_p(G)$. If P is cyclic or every maximal subgroup of P is semi-cover-avoiding in G, then G is p-nilpotent.*

Lemma 3.13 ([14, Lemma 1]) *Let $\mathcal{U}$ be a saturated formation containing the class of supersolvable groups. If N is a cyclic normal subgroup of a group G such that $G/N \in \mathcal{U}$, then $G \in \mathcal{U}$.*

Theorem 3.14 *Suppose that $\mathcal{U}$ is a saturated formation containing the class of supersolvable groups. Let H be a normal subgroup of a group G such that $G/H \in \mathcal{U}$. If for any Sylow subgroup S of H, either S is cyclic or every maximal subgroup of S is semi-cover-avoiding in G, then $G \in \mathcal{U}$.*

Proof Assume the theorem is false and let G be a counterexample with minimal order. By Lemmas 2.3 and 3.12, H is a group with Sylow tower of supersolvable type. Then $P \trianglelefteq G$ for the Sylow p-subgroup P of H, where p is the largest prime dividing $|H|$. Let N be a minimal normal subgroup of G contained in P. By Lemma 2.5, G/N satisfies the hypotheses of the theorem with respect to H/N so that $G/N \in \mathcal{U}$. Since $\mathcal{U}$ is a formation, N is a unique minimal normal subgroup of G contained in P. Furthermore, by Lemma 3.13, N is not cyclic, and P is not cyclic either. Note that $\Phi(P) \leq \Phi(G)$, so if $\Phi(P) \neq 1$, then $N \leq \Phi(P)$, and therefore $G/\Phi(G) \in \mathcal{U}$, a contradiction. Hence $\Phi(P) = 1$.

Now let P_1 be any maximal subgroup of P. We claim that there must be $P_1 \geq N$ or $P_1 \cap N = 1$. In fact, by the assumption, G has a chief series $1 = G_0 < G_1 < \cdots < G_n = G$ such that P_1 covers or avoids every chief factor of this series. Choose the integer k such that $P \cap G_k = 1$ but $P \cap G_{k+1} \neq 1$. Clearly, $N \leq P \cap G_{k+1}$. If P_1 avoids G_{k+1}/G_k, then $P_1 \cap G_{k+1} \leq G_k \cap P = 1$ and so $P_1 \cap N = 1$. If P_1 covers G_{k+1}/G_k, then $P_1 G_k \geq G_{k+1}$, and therefore $G_{k+1} = (G_{k+1} \cap P_1)G_k \leq (G_{k+1} \cap P)G_k \leq G_{k+1}$. It follows that $G_{k+1} \cap P_1 = G_{k+1} \cap P$ and hance $P_1 \geq N$.

Finally, since $\Phi(P) = 1$, there exists a maximal subgroup P_1 of P such that $N \not\leq P_1$. It follows from the above result that $P_1 \cap N = 1$ and consequently $|N| = p$, a contradiction. □

Corollary 3.15 *Let G be a group. If for any Sylow subgroup S of G, either S is cyclic or every maximal subgroup of S is semi-cover-avoiding in G, then G is supersolvable.*

Lemma 3.16 ([2, Lemma 3.12]) *Let p be the smallest prime dividing the order of a group G.*

If G is A_4-free and $p^3 \nmid |G|$, then G is p-nilpotent.

Lemma 3.17 *Let G be a group of even order and $P \in \mathrm{Syl}_2(G)$. If G is A_4-free and P has a 2-maximal subgroup P_1 such that P_1 is semi-cover-avoiding in G, then G is solvable.*

Proof If $|P| \leq 2$, then G is 2-nilpotent and so it is solvable. Now we assume that $|P| \geq 4$. By assumption, G has a chief series such that P_1 covers or avoids every factor of the series. Let H/K be any factor of this series. If P_1 covers H/K, then H/K is a 2-group. If P_1 avoids H/K, then $P_1K/K \cap H/K = 1$. It follows that $2^3 \nmid |H/K|$ since $|PK/K : P_1K/K| \leq 4$. By Lemma 3.16, H/K is 2-nilpotent and it is solvable. Hence G has a chief series in which every factor is solvable, which implies that G is solvable. The proof is completed. □

Theorem 3.18 *Let p be the smallest prime dividing the order of a group G and $P \in \mathrm{Syl}_p(G)$. If G is A_4-free and every normal 2-maximal subgroup of P is semi-cover-avoiding in G, then G is p-nilpotent.*

Proof Since a group of odd order is solvable, it follows from Lemma 3.17 that G is solvable. Now assume that the theorem is false and let G be a counterexample with minimal order. By Lemma 2.5, $G/O_{p'}(G)$ satisfies the hypotheses of the theorem, and the minimality of G implies that $O_{p'}(G) = 1$. Let N be a minimal normal subgroup of G. Then N is a p-group and $N \leq P$. It is easy to see that G/N also satisfies the hypotheses of the theorem and accordingly, G/N is p-nilpotent. Noticing that the class of p-nilpotent groups is a saturated formation, we have N is a unique minimal normal subgroup of G. Let T/N be the normal p-complement of G/N. If $|N| \leq p^2$, then by Lemma 3.16, T is p-nilpotent and therefore G is p-nilpotent, a contradiction. So $|N| \geq p^3$. Let P_1 be any normal 2-maximal subgroup of P. Since P_1 is semi-cover-avoiding in G, P_1 covers or avoids $N/1$. If P_1 avoids N, then $|P_1N| = |P_1||N| > |P|$, a contradiction. Hence P_1 covers N and $P_1 \geq N$. It follows that $N \leq \Phi(P)$ since every maximal subgroup of P contains a normal 2-maximal subgroup of P. Consequently by [13, III, Lemma 3.3], $N \leq \Phi(G)$, and $G/\Phi(G)$ is p-nilpotent, the final contradiction. □

Corollary 3.19 *Let G be a group. If G is A_4-free and every normal 2-maximal subgroup of each Sylow subgroup of G is semi-cover-avoiding in G, then G is a group with Sylow tower of supersolvable type.*

Remark 3.20 Lemma 3.17, Theorem 3.18 and Corollary 3.19 are not true if we remove the condition "G is A_4-free". For example, see A_5.

Remark 3.21 A group satisfying the conditions of Corollary 3.19 is not necessarily a supersolvable group. For example, see Frobenius groups of order 36.

Lemma 3.22 ([2, Lemma 3.16]) *Let $\mathcal{T}$ be the class of groups with Sylow tower of supersolvable type and H a normal subgroup of a group G such that $G/H \in \mathcal{T}$. If G is A_4-free, and H is a q-group for some prime q with $|H| \leq q^2$, then $G \in \mathcal{T}$.*

Theorem 3.23 *Suppose that $\mathcal{T}$ is the class of groups with Sylow tower of supersolvable type. Let H be a normal subgroup of a group G such that $G/H \in \mathcal{T}$. If G is A_4-free, and every normal 2-maximal subgroup of each Sylow subgroup of H is semi-cover-avoiding in G, then $G \in \mathcal{T}$.*

Proof Assume the theorem is false and let G be a counterexample of minimal order. According to Corollary 3.19, $H \in \mathcal{T}$, and $P \trianglelefteq G$ for $P \in \mathrm{Syl}_p(H)$, where p is the largest prime dividing the order of H. By Lemma 2.5, G/P satisfies the hypotheses of the theorem with respect to H/P, and so $G/P \in \mathcal{T}$. It follows that $G/O_{p'}(G)$ satisfies the hypotheses of the theorem with respect to $PO_{p'}(G)/O_{p'}(G)$. If $O_{p'}(G) \neq 1$, then $G/O_{p'}(G) \in \mathcal{T}$, and therefore $G \in \mathcal{T}$ since $G = G/(P \cap O_{p'}(G)) \lesssim G/P \times G/O_{p'}(G)$, a contradiction. Hence $O_{p'}(G) = 1$. Let N be a minimal normal subgroup of G. Then N is a p-group. We claim that $N \leq P$. Otherwise, if $N \nleq P$, then $P \notin \mathrm{Syl}_p(G)$. If p is the largest in $\pi(G)$, then from $G/P \in \mathcal{T}$, we have $G \in \mathcal{T}$, a contradiction. Let q be the largest prime in $\pi(G)$ and $M/P \in \mathrm{Syl}_q(G/P)$. Then $q > p$, $M/P \trianglelefteq G/P$ and $M < G$. Clearly, $M/P \in \mathcal{T}$ and by Lemma 2.3, M satisfies the conditions of the theorem with respect to P, and so $M \in \mathcal{T}$. Let $Q \in \mathrm{Syl}_q(M)$. Then $Q \trianglelefteq G$ and $1 \neq Q \leq O_{p'}(G)$, a contradiction. Hence our claim holds. By Lemma 2.5 once more, G/N satisfies the hypotheses of the theorem with respect to P/N and $G/N \in \mathcal{T}$. Since $\mathcal{T}$ is saturated, N is a unique minimal normal subgroup of G. Also by Lemma 3.22, $|N| \geq p^3$. If $\Phi(P) \neq 1$, then $N \leq \Phi(P)$, and therefore $G/\Phi(G) \in \mathcal{T}$, a contradiction. Thus $\Phi(P) = 1$, and P is elementary abelian. Choose $K \leq P$ such that $P = N \times K$. Let P_1 be a 2-maximal subgroup of P such that $P_1 \geq K$. Then P_1 is semi-cover-avoiding in G, especially P_1 covers or avoids $N/1$. Clearly, $P_1 \ngeq N$, and so $P_1 \cap N = 1$, from which we deduce that $|N| \leq p^2$, the final contradiction. □

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