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Wide Diameter of Generalized Petersen Graphs

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Abstract Generalized Petersen graphs are commonly used interconnection networks, and wide diameter is an important parameter to measure fault-tolerance and efficiency of parallel processing computer networks. In this paper, we show that the diameter and 3-wide diameter of generalized Petersen graph $P(m,a)$ are both $O(\frac{m}{2a})$, where $a \geq 3$.

Keywords Petersen graph; diameter; wide diameter.

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1. Introduction

Let x and y be two distinct vertices of a graph G. A set of internally disjoint (x,y)-paths in G is called an (x,y)-container in G, denoted by $C(G;x,y)$. The number of paths in $C(G;x,y)$ is called the width of $C(G;x,y)$. An (x,y)-container with width w is denoted by $C_w(G;x,y)$. The length of $C(G;x,y)$, denoted by $l(C(G;x,y))$, is the largest length of paths in $C(G;x,y)$.

Suppose that G is a w-connected graph, $w \geq 1$. By Menger's theorem there exists an (x,y)-container $C_w(G;x,y)$ for any pair of two distinct vertices x and y in G. The distance with width w, wide-distance or w-distance for short, from x to y, denoted by $d_w(G;x,y)$, is defined as the minimum length over all (x,y)-containers $C_w(G;x,y)$. The diameter with width w, wide-diameter or w-diameter for short, of G, denoted by $d_w(G)$, is defined as

$$d_w(G) = \max\{d_w(G;x,y) : x,y \in V(G), x \neq y\}.$$

It is desirous that an ideal interconnection network G should be one with connectivity $\kappa(G)$ as large as possible and diameter $d(G)$ as small as possible. The wide-diameter $d_w(G)$ combines connectivity $\kappa(G)$ and diameter $d(G)$, where $1 \leq w \leq \kappa(G)$. Hence $d_w(G)$ is a more suitable parameter than $d_w(G)$ to measure fault-tolerance and efficiency of parallel processing computer

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networks. Thus, determining the value of $d_w(G)$ is of significance for a given graph G and an integer w. However, Hsu [1] proved that this problem is NP-complete.

The concept of wide-diameter was proposed by Hsu and Lyuu [2], who used the term "wide-diameter" for the first time. Their study is based on the concept of the container stated above. Flandrin and Li [3] proposed this concept motivated by the property $P_{l,w}$ by Faudree, Jacobson, Ordman, Schelp, and Tuza. A graph G has the property $P_{l,w}$ implies that there exist w internally disjoint (x, y)-paths in G of length at most l for any pair of two distinct vertices x and y. It is clear that the graph G has the property that $P_{l,w}$ is equivalent to that $d_w(G) \leq l$.

Let m and a be integers such that $m \geq 3$, $1 \leq a < m$ and $m > 2a$. For such m, a, the generalized Petersen graph $P(m, a)$ is defined by

$$V(P(m,a)) = \{u_i, v_i : 0 \leq i \leq m-1\},$$
$$E(P(m,a)) = \{(u_i, u_{i+1}),\ (u_i, v_i),\ (v_i, v_{i+a}) : 0 \leq i \leq m-1\},$$

where and in the sequel the subscripts of vertices are computed under modulo m. When we draw the graph, we can order the vertices in U in one circle (called outer circle) and the vertices in V in one invisible circle (called inner circle).

Since $P(m, a)$'s form an important class of 3-regular and 3-connected graphs with $2m$ vertices and $3m$ edges, which are commonly used interconnection networks and have been studied by various researchers. It is desirable to determine $P(m, a)$'s wide diameter. Recently, Liaw and Chang [4] gave the wide diameter for many specific classes of networks. Krishnamoorth and Krishnamurthy [5] gave the fault diameter for generalized Petersen graph. Hou [6] gave the w-diameter for $P(m, 2)$. In this paper, we give the 3-wide diameter of $P(m, a)$ $(a \geq 3)$.

For two integers a and b with $a \leq b$, by $[a, b]$ we denote the set $\{a, a+1, \ldots, b\}$.

2. Statement of the main result

Lemma 1 *The diameter of $P(m, a)$ is $O(\frac{m}{2a})$.*

Proof The case of $a = 1$ is trivial, and the case of $a = 2$ has been proved in [6]. Now we consider the case of $a = 3$. Given an integer i $(\leq \frac{m}{2})$. When i is large enough, we consider the distances from u_0 to u_i, u_0 to v_i and v_0 to v_i. Since the vertices in inner circle jump in steps of 3, we will walk in the inner circle as much as possible. When $i \equiv 0 \pmod 3$, we enter in the inner circle from vertex v_0; when $i \equiv 1 \pmod 3$, we enter in the inner circle from vertex v_1; when $i \equiv 2 \pmod 3$, we enter in the inner circle from vertex v_2, and leave the circle from vertex v_i. For example,

$$u_0 - u_1 - u_2 - v_2 - v_5 - \cdots - v_{3k+2} - u_{3k+2},$$

$$v_0 - u_0 - u_1 - v_1 - v_4 - \cdots - v_{3k+1},$$

where k is an integer. So for any two distinct vertices x and y, $\mathrm{dist}(x, y) \leq \frac{m}{6} + \alpha(3)$, where $\alpha(3) = 4$. Given an integer $s < \frac{m}{6}$, the subscript of vertex that any path with length s starting from u_0 can reach belongs to $[-3s, 3s]$, where $\lfloor \frac{m}{2} \rfloor$ does not. So we have that the diameter of

$P(m,3)$ is $O(\frac{m}{6})$. When $a>3$, by the similar method, we yield $d(P(m,a))=O(\frac{m}{2a})$.

Theorem 2 *The 3-wide diameter of $P(m,a)$ is $O(\frac{m}{2a})$, where $a \geq 3$.*

Proof First we consider the case of $a=3$. Let x,y be any two distinct vertices of $P(m,3)$. We will exhibit the path strategies when x and y are in different cases.

Case 1 $x=u_0$ and $y=u_i$, $3 \leq i \leq \frac{m}{2}$. We consider three subcases:

Case 1.1 $i \equiv 0 \pmod 3$. We have three vertex-disjoint paths from x to y,

$$P_1: u_0 - v_0 - v_3 - \cdots - v_i - u_i,$$

$$P_2: u_0 - u_1 - v_1 - v_4 - \cdots - v_{1+i} - u_{1+i} - u_i$$

and

$$P_3: u_0 - u_{-1} - v_{-1} - v_2 - \cdots - v_{i-1} - u_{i-1} - u_i$$

with lengths $|P_1| = \frac{i}{3}+2, |P_2| = \frac{i}{3}+4$ and $|P_3| = \frac{i}{3}+4$. And the container has length $\frac{i}{3}+4$.

Case 1.2 $i \equiv 1 \pmod 3$. We have three vertex-disjoint paths from x to y,

$$P_1: u_0 - v_0 - v_3 - \cdots - v_{i-1} - u_{i-1},$$

$$P_2: u_0 - u_1 - v_1 - v_4 - \cdots - v_i - u_i$$

and

$$P_3: u_0 - u_{-1} - v_{-1} - v_2 - \cdots - v_{1+i} - u_{1+i} - u_i$$

with lengths $|P_1| = \frac{i-1}{3}+3$, $|P_2| = \frac{i-1}{3}+3$ and $|P_3| = \frac{i-1}{3}+5$. And the container has length $\frac{i-1}{3}+5$.

Case 1.3 $i \equiv 2 \pmod 3$. We have three vertex-disjoint paths from x to y,

$$P_1: u_0 - v_0 - v_3 - \cdots - v_{i+1} - u_{i+1},$$

$$P_2: u_0 - u_1 - v_1 - v_4 - \cdots - v_{i-1} - u_{i-1} - u_i$$

and

$$P_3: u_0 - u_{-1} - v_{-1} - v_2 - \cdots - v_i - u_i$$

with lengths $|P_1| = \frac{i-2}{3}+4, |P_2| = \frac{i-2}{3}+4$ and $|P_3| = \frac{i-2}{3}+4$. And the container has length $\frac{i-2}{3}+4$.

Case 2 $x=u_0$ and $y=v_i$, $3 \leq i \leq \frac{m}{2}$. We consider three subcases:

Case 2.1 $i \equiv 0 \pmod 3$. We have three vertex-disjoint paths from x to y,

$$P_1: u_0 - v_0 - v_3 - \cdots - v_{i-3} - v_i,$$

$$P_2: u_0 - u_1 - v_1 - v_4 - \cdots - v_{i-2} - u_{i-2} - u_{i-1} - u_i - v_i$$

and

$$P_3 : u_0 - u_{-1} - v_{-1} - v_2 - \cdots - v_{i+2} - u_{i+2} - u_{i+3} - v_{i+3} - v_i$$

with lengths $|P_1| = \frac{i}{3} + 1, |P_2| = \frac{i}{3} + 5$ and $|P_3| = \frac{i}{3} + 7$. And the container has length $\frac{i}{3} + 7$.

Case 2.2 $i \equiv 1 \pmod 3$. We have three vertex-disjoint paths from x to y,

$$P_1 : u_0 - v_0 - v_3 - \cdots - v_{i-1} - u_{i-1} - u_i - v_i,$$

$$P_2 : u_0 - u_1 - v_1 - v_4 - \cdots - v_{i-3} - v_i$$

and

$$P_3 : u_0 - u_{-1} - v_{-1} - v_2 - \cdots - v_{i+1} - u_{i+1} - u_{i+2} - u_{i+3} - v_i$$

with lengths $|P_1| = \frac{i-1}{3} + 4, |P_2| = \frac{i-1}{3} + 2$ and $|P_3| = \frac{i-1}{3} + 8$. And the container has length $\frac{i-1}{3} + 8$.

Case 2.3 $i \equiv 2 \pmod 3$. We have three vertex-disjoint paths from x to y,

$$P_1 : u_0 - v_0 - v_3 - \cdots - v_{i-2} - u_{i-2} - u_{i-1} - u_i - v_i,$$

$$P_2 : u_0 - u_1 - v_1 - v_4 - \cdots - v_{i+2} - u_{i+2} - u_{i+3} - v_{i+3} - v_i$$

and

$$P_3 : u_0 - u_{-1} - v_{-1} - v_2 - \cdots - v_{i-3} - v_i$$

with lengths $|P_1| = \frac{i-2}{3} + 5$, $|P_2| = \frac{i-2}{3} + 7$ and $|P_3| = \frac{i-2}{3} + 3$. And the container has length $\frac{i-2}{3} + 7$.

Case 3 $x = v_0$ and $y = v_i$, $3 \leq i \leq \frac{m}{2}$. We consider three subcases:

Case 3.1 $i \equiv 0 \pmod 3$. We have three vertex-disjoint paths from x to y,

$$P_1 : v_0 - v_3 - \cdots - v_{i-3} - v_i,$$

$$P_2 : v_0 - u_0 - u_1 - u_2 - v_2 - \cdots - v_{i+2} - u_{i+2} - u_{i+3} - v_{i+3} - v_i$$

and

$$P_3 : v_0 - v_{-3} - u_{-3} - u_{-2} - v_{-2} - \cdots - v_{i-2} - u_{i-2} - u_{i-1} - u_i - v_i$$

with lengths $|P_1| = \frac{i}{3}$, $|P_2| = \frac{i}{3} + 8$ and $|P_3| = \frac{i}{3} + 8$. And the container has length $\frac{i}{3} + 8$.

Case 3.2 $i \equiv 1 \pmod 3$. We have three vertex-disjoint paths from x to y,

$$P_1 : v_0 - v_3 - \cdots - v_{i-1} - u_{i-1} - u_i - v_i,$$

$$P_2 : v_0 - u_0 - u_1 - u_2 - v_2 - \cdots - v_{i+1} - u_{i+1} - u_{i+2} - u_{i+3} - v_{i+3} - v_i$$

and

$$P_3 : v_0 - v_{-3} - u_{-3} - u_{-2} - v_{-2} - \cdots - v_i$$

with lengths $|P_1| = \frac{i-1}{3} + 3, |P_2| = \frac{i-1}{3} + 9$ and $|P_3| = \frac{i-1}{3} + 5$. And the container has length $\frac{i-1}{3} + 9$.

Case 3.3 $i \equiv 2 \pmod 3$. We have three vertex-disjoint paths from x to y,

$$P_1 : v_0 - v_3 - \cdots - v_{i-2} - u_{i-2} - u_{i-1} - u_i - v_i,$$

$$P_2 : v_0 - u_0 - u_1 - v_1 - v_4 - \cdots - v_{i+2} - u_{i+2} - u_{i+3} - v_{i+3} - v_i$$

and

$$P_3 : v_0 - v_{-3} - u_{-3} - u_{-2} - u_{-1} - v_{-1} - v_2 - \cdots - v_{i-3} - v_i$$

with lengths $|P_1| = \frac{i-2}{3} + 4$, $|P_2| = \frac{i-2}{3} + 8$ and $|P_3| = \frac{i-2}{3} + 6$. And the container has length $\frac{i-2}{3} + 8$.

So we get that $d_3(P(m,3)) \le \lfloor \frac{m}{6} \rfloor + 9$. Since $d(P(m,3)) = O(\frac{m}{6})$, we have that $d_3(P(m,3)) = O(\frac{m}{6})$.

Now we consider the general case, let $a > 3$. When $0 \le k < a$, set $V_k = \{v_i : i \equiv k \pmod a, 0 \le i \le \frac{m}{2}\}$. When $i \equiv j \pmod a, 0 \le j < a, 0 \le i \le \frac{m}{2}$, by employing V_{j-1}, V_j, V_{j+1}-paths, we can construct $(u_0, u_i), (u_0, v_i), (v_0, v_i)$-containers, similar to the case of $a = 3$. For example, when $a = 4$, $i \equiv 0 \pmod 4$, we construct a (v_0, v_i)-container as follows:

$$P_1 : v_0 - v_4 - \cdots - v_{i-4} - v_i,$$

$$P_2 : v_0 - u_0 - u_1 - v_1 - v_5 - \cdots - v_{i+1} - u_{i+1} - u_i - v_i$$

and

$$P_3 : v_0 - v_{-4} - u_{-4} - u_{-5} - v_{-5} - v_{-1} - v_3 - \cdots - v_{i+3} - u_{i+3} - u_{i+4} - v_{i+4} - v_i.$$

And in the similar way, we yield that $d_3(P(m,a)) = O(\frac{m}{2a})$.

References

[1] HSU D F. *On container width and length in graphs, groups, and networks* [J]. IEICE Trans. Fund. Elect. Comm. Comput., 1994, **E77-A**: 668–680.

[2] HSU D F, LYUU Y D. *A graph-theoretical study of transmission delay and fault tolerance* [J]. International Journal of Mini and Microcomputers, 1994, **16**(1): 35–42.

[3] FLANDRIN E, LI H. *Mengerian properties, hamiltonicity, and claw-free graphs* [J]. Networks, 1994, **24**(2): 177–183.

[4] LIAW S C, CHANG G J. *Generalized diameters and Rabin numbers of networks* [J]. J. Comb. Optim., 1999, **2**(4): 371–384.

[5] KRISHNAMOORTH M S, KRISHNAMURTHY B. *Fault diameter of intersection networks* [J]. Comput. Math. Applic., 1987, **13**(5/6): 577–582.

[6] HOU Xinmin, WANG Tianming. *Wide diameters of generalized Petersen graphs* [J]. J. Math. Res. Exposition, 2004, **24**(2): 249–253.